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Calculus of Variations

Smooth solutions to a class of mixed type Monge–Ampère equations

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Abstract We prove the existence of C^∞ local solutions to a class of mixed type Monge–Ampère equations in the plane. More precisely, the equation changes type to finite order across two smooth curves intersecting transversely at a point. Existence of C^∞ global solutions to a corresponding class of linear mixed type equations is also established. These results are motivated by and may be applied to the problem of prescribed Gaussian curvature for graphs, the isometric embedding problem for 2-dimensional Riemannian manifolds into Euclidean 3-space, and also transonic fluid flow.

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1 Introduction

In this paper, we will study a class of Monge–Ampère equations of mixed-type. One source of interest in these equations arises from the equation of prescribed Gaussian curvature. Let u be a C^2 function defined in a domain $\Omega \subset \mathbb{R}^2$ and suppose that the graph of u has Gaussian curvature $K(x)$ at the point $(x, u(x))$, $x \in \Omega$. It follows that u satisfies the equation

$$\det D^2 u = K(x)(1 + |Du|^2)^2.$$

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This equation is elliptic if K is positive and hyperbolic if K is negative, and hence is of mixed type when K changes sign. Another source of Monge–Ampère equations comes from the isometric embedding problem for 2-dimensional Riemannian manifolds into $\mathbb{R}^3$. See chapter three in [8] for details. In [21], Lin proved the existence of local isometric embeddings of surfaces into $\mathbb{R}^3$ if the Gaussian curvature changes sign cleanly. In other words, the Gaussian curvature changes sign to first order across a curve. In this case the Darboux equation, a basic equation associated with the isometric embedding problem is elliptic on one side of the curve and hyperbolic on the other. Such a result was generalized by the first named author in [5] and [6]. Recently, we [9] discussed a case in which the Gaussian curvature changes sign in a more complicated way and proved the existence of sufficiently smooth isometric embeddings. For further results on this and related problems see [4–17], [20], and [21].

Mixed type equations also arise naturally in many other areas. Recently, there have been several survey articles on this subject. In [22], Morawetz gives a detailed account of the historical background and known results on mixed type equations and transonic flows. In [23], Otway presents a detailed review on mixed type equations and Riemannian–Lorentzian metrics. The most intensively studied equation of mixed type is the Tricomi equation [24]

$$u_{yy} + yu_{xx} = f.$$

The plane is divided into two parts by the x-axis. The Tricomi equation is elliptic in the upper half plane and hyperbolic in the lower half plane. Many results have been obtained in various settings for this equation. Nonetheless, beyond the equations of the Tricomi family, the theory of mixed type equations is fairly underdeveloped. However this lack of development is not due to a lack of applications or well-motivated problems. Mixed type equations which change type in a way more complicated than that of the Tricomi case also arise naturally in many circumstances. For instance, as far back as in 1929, Bateman [1] presented several models for the 2-dimensional motion of compressible fluids. One of these models is given by a class of elliptic-hyperbolic equations in the unit disk which change type in the following way. The unit disk is divided into four regions by two straight lines through the origin. These equations are elliptic in a pair of opposing regions and hyperbolic in another pair of opposing regions. (See Fig. 1 on p. 612 in [1].)

In this paper, we study smooth solutions to a class of mixed type Monge–Ampère equations in the plane which change type in a way similar to that in [1]. The model equation has the following form

$$u_{xx}u_{yy} - u_{xy}^2 = (x^2 - y^2)\psi(x, y, u, u_x, u_y), \tag{1.1}$$

where ψ is a positive smooth function in $B_1 \times \mathbb{R} \times \mathbb{R}^2$. Here B_1 is the unit disk in $\mathbb{R}^2$. We are interested in the question of whether or not (1.1) admits a *smooth* solution u, defined in some neighborhood of the origin. We note that (1.1) is a Monge–Ampère type equation of mixed type. The unit ball $B_1 \subset \mathbb{R}^2$ is divided into four components by $\{|x| = |y|\}$. The equation (1.1) is elliptic in $\{|x| > |y|\}$ and hyperbolic in $\{|x| < |y|\}$.

The following result is a special case of a more general result that we will prove in Sect. 6.

Theorem 1.1 *Let ψ be a positive smooth function in $B_1 \times \mathbb{R} \times \mathbb{R}^2$. Then there exists a smooth solution u of* (1.1) *in B_r for some $r \in (0, 1)$.*

We should point out that $x^2 - y^2$ can be replaced by any function with a similar behavior, such as $y^2 - x^2$. This is due to the invariance of the Monge–Ampère operator by orthogonal transformations.

In order to prove Theorem 1.1, it is essential to analyze the corresponding linear equation. It turns out that it suffices to consider

$$u_{yy} + (x^2 - y^2)u_{xx} = f. \tag{1.2}$$

Again, the plane is divided into four components by $\{|x| = |y|\}$. The equation (1.2) is elliptic in $\{|x| > |y|\}$ and hyperbolic in $\{|x| < |y|\}$. The lines of degeneracy $\{|x| = |y|\}$ are non-characteristic. Moreover, the boundaries $\partial\{y > |x|\}$ and $\partial\{y < -|x|\}$ are space-like for the corresponding hyperbolic regions $\{y > |x|\}$ and $\{y < -|x|\}$, respectively. Hence equation (1.2) is considerably more complicated than the Tricomi equation, however we are still able to establish the following theorem, which is a special case of a more general result proven in Sect. 5.

Theorem 1.2 *Let f be a smooth function in $\bar{B}_1 \subset \mathbb{R}^2$. Then there exists a smooth solution u of (1.2) in B_1. Moreover, for any positive integer s, u satisfies*

$$\|u\|_{H^s(B_1)} \leq C_s \|f\|_{H^{s+5}(B_1)}, \tag{1.3}$$

where C_s is a positive constant depending only on s.

We point out that (1.2) is a small perturbation of the linearization for (1.1), at a suitably chosen approximate solution. It should be emphasized that the form of the degenerate coefficient $x^2 - y^2$ plays an important role in the solvability of (1.2). If $x^2 - y^2$ is replaced by other quadratic functions, then it may not be possible to solve the new equation. For instance, the approach and methods used in this paper do not yield solutions of

$$u_{yy} + (y^2 - x^2)u_{xx} = f. \tag{1.4}$$

This equation is different from (1.2), in that (1.4) is elliptic in $\{|x| < |y|\}$ and hyperbolic in $\{|x| > |y|\}$. We note that the y-direction, which may be considered as the time direction, does not always point into the hyperbolic regions. In this sense, the linear Eq. (1.2) is more rigid than the nonlinear Eq. (1.1).

The proof of Theorem 1.2 consists of two steps. In the first step, we construct a smooth solution in the elliptic regions $\{|y| < x\}$ and $\{|y| < -x\}$. This is achieved by solving the homogeneous Dirichlet problem. Such a solution then naturally yields Cauchy data for the hyperbolic regions along the lines of degeneracy. In the second step, we construct a smooth solution in the hyperbolic regions $\{y > |x|\}$ and $\{y < -|x|\}$, by solving the Cauchy problem. The solution constructed in Theorem 1.2 vanishes along the degenerate set $\{|x| = |y|\} \cap B_1$. It is clear from the proof in this paper that one can prescribe the solution arbitrarily (as a smooth function) on $\{|x| = |y|\} \cap B_1$. A similar idea was used by Han [7] in the discussion of higher dimensional Tricomi equations and related Monge–Ampère equations.

The difficulty in solving both the Dirichlet problem in the elliptic regions and the Cauchy problem in the hyperbolic regions arises from two distinct aspects of this problem. First, the equation is degenerate on the boundary. Second, there is an angular point (i.e., the origin) on the boundary of each domain.

Boundary value problems for (strictly) elliptic differential equations in domains with angular points have been studied extensively. The regularity results are in fact not encouraging. Well known examples of harmonic functions in sector domains demonstrate that these solutions are not necessarily smooth. Furthermore, in general, solutions of degenerate elliptic differential equations exhibit worse regularity than those of (strictly) elliptic differential equations. Hence, it seems unrealistic to expect, at first glance, that solutions of the degenerate elliptic equation studied here should be smooth in domains with angular points. However, it is precisely due to the degeneracy at the angular points that we are able to prove that the solutions have this high degree of regularity up to the boundary. The *degeneracy* plays an important *positive* role. In fact, we are not aware of any other cases where degeneracy actually improves the regularity.

In contrast to the extensive studies of elliptic equations in nonsmooth domains, little is known about the Cauchy problem for hyperbolic equations when the initial curve is nonsmooth. Our first task here is to prove that the Cauchy problem is well posed for (strictly) hyperbolic equations in domains whose initial curves contain angular points. Compatibility conditions are needed at the angular points in order to ensure the regularity of solutions. (See Lemma 3.3 for details.) As in the elliptic case, the *degeneracy* along the initial curve surprisingly plays a *positive* role in passing the existence and regularity result from strict hyperbolicity to degenerate hyperbolicity. In fact, it demonstrates that any such initial curve is space-like for the hyperbolic regions. This plays an important role in the proof of the well-posedness of degenerate hyperbolic equations in domains whose initial curves have angular points.

This paper is organized as follows. In Sect. 2, we will construct smooth solutions for the Dirichlet problem in the elliptic regions and derive necessary estimates. Smooth solutions to the Cauchy problem for uniformly hyperbolic equations in domains with angular points on the boundary will be established in Sect. 3. Estimates independent of the hyperbolicity constant will then be derived in Sect. 4. In Sect. 5, we will state and prove a general theorem of which Theorem 1.2 is a special case. Finally in Sect. 6, we will discuss a class of Monge–Ampère type equations and study the appropriate iterations to prove a result which generalizes Theorem 1.1.

2 Elliptic regions

In this section, we will study a class of degenerate elliptic differential equations in planar domains with angular singularities. We will construct smooth solutions if the degeneracy occurs at angular points.

For any $\kappa > 0$, let $\mathcal{C}_\kappa$ be a cone in $\mathbb{R}^2$ with vertex at the origin given by

$$\mathcal{C}_\kappa = \{(x, y); 0 < |y| < \kappa x\}.$$

Let Ω_κ be a bounded domain in $\mathbb{R}^2$ such that

$$\Omega_\kappa \cap B_1 = \mathcal{C}_\kappa \cap B_1,$$

and

$$\partial\Omega_\kappa \setminus \{0\} \text{ is smooth.}$$

Consider the equation

$$u_{yy} + K u_{xx} + b_1 u_x + b_2 u_y + cu = f \quad \text{in } \Omega_\kappa, \tag{2.1}$$

where K, b_i and c are smooth functions in $\bar{\Omega}_\kappa$. In the following, we assume

$$K > 0 \text{ in } \Omega_\kappa \quad \text{and} \quad K = 0 \text{ on } \partial\Omega_\kappa \cap B_1. \tag{2.2}$$

There are two major difficulties in studying (2.1). First, (2.1) is degenerate on a portion of the boundary $\partial\Omega_\kappa \cap B_1$. Second, there is an angular singularity on the boundary. Usually, solutions of degenerate elliptic differential equations exhibit a worse regularity than those of (strictly) elliptic differential equations. It is well known that solutions of (strictly) elliptic differential equations in domains with angular singularities are in general not smooth. The regularity depends on the angle in an essential way; the smaller the angle, the better the regularity of solutions. However, it is entirely different for equations which are degenerate at angular points. In our case, we are able to construct smooth solutions of (2.1). Moreover, we can prove that any solutions of (2.1) are in fact smooth if its Dirichlet value on the boundary is smooth and satisfies a compatibility condition up to infinite order at the angular point. The *degeneracy* plays an important *positive* role in the proof of the smoothness of solutions at the angular point.

We will prove the following result.

Theorem 2.1 *Let K, b_i, c and f be smooth functions in $\bar{\Omega}_\kappa$ satisfying (2.2), $c \le 0$ in Ω_κ and*

$$|b_1| \le C_b\left(\sqrt{K} + |\partial_x K|\right) \quad \text{in } \Omega_\kappa. \tag{2.3}$$

Then (2.1) admits a smooth solution in $\bar{\Omega}_\kappa$ with $u = 0$ on $\partial\Omega_\kappa$. Moreover, for any integer $m \ge 1$, u satisfies

$$\|u\|_{H^m(\Omega_\kappa)} \le C_m \|f\|_{H^{m+1}(\Omega_\kappa)}, \tag{2.4}$$

where C_m is a positive constant depending only on C_b and the C^m-norms of K, b_i and c.

To prove Theorem 2.1, we regularize (2.1) by replacing K by $K + \delta$ for any $\delta > 0$. Then the new equation is uniformly elliptic and hence admits a unique solution $u_\delta \in H_0^1(\Omega_\kappa)$. In order to pass limit as $\delta \to 0$, we need to derive estimates of u_δ independent of δ. The condition (2.3) is introduced to overcome the degeneracy of K along $\partial\Omega_\kappa \cap B_1$.

In the following, we consider

$$\mathcal{L}u \equiv u_{yy} + au_{xx} + b_1u_x + b_2u_y + cu = f \quad \text{in } \Omega_\kappa, \tag{2.5}$$

where a, b_i and c are smooth functions in $\bar{\Omega}_\kappa$. We assume

$$a_0 \le a \le 1 \quad \text{in } \Omega_\kappa, \tag{2.6}$$

for a positive constant $a_0 \in (0, 1)$.

It is obvious that (2.5) is uniformly elliptic. Hence (2.5) admits a solution $u \in H_0^1(\Omega_\kappa)$ and classical results for uniformly elliptic differential equations on smooth domains apply in any subdomains of $\bar{\Omega}_\kappa$ away from the origin. Specifically, for any $r \in (0, 1)$ and any $k \ge 2$, there holds

$$\|u\|_{H^k(\Omega_\kappa\setminus B_r)} \le C_{k,r}\|f\|_{H^{k-2}(\Omega_\kappa)},$$

where $C_{k,r}$ is a positive constant depending on k, r, a_0 and C^{k-2}-norms of a, b_i and c. In general, $C_{k,r} \to \infty$ as $r \to 0$ or $a_0 \to 0$. Therefore, we need to derive an estimate which is independent of the lower bound of a. Moreover, the regularity of u close to the origin needs special attentions.

We first consider boundary points away from the origin. We set for any $\varepsilon > 0$

$$D_\varepsilon = \{(x, y); |x| < 1,\ 0 < y < \varepsilon\} \subset \mathbb{R}^2. \tag{2.7}$$

We denote by $\partial_h^+ D_\varepsilon$, $\partial_h^- D_\varepsilon$ and $\partial_v D_\varepsilon$ the horizontal top, horizontal bottom and vertical boundaries respectively. By an appropriate transform, a neighborhood of any point on $\partial\Omega_\kappa \setminus \{0\}$ is changed to D_ε for an $\varepsilon > 0$. We consider (2.5) in D_ε and assume

$$|b_1| \le C_b(\sqrt{a} + |\partial_x a|) \quad \text{in } D_\varepsilon, \tag{2.8}$$

for some positive constant C_b.

Lemma 2.2 and Corollary 2.3 below provide energy estimates of solutions in narrow domains.

Lemma 2.2 *Suppose a, b_1, b_2 and c are smooth functions in D_ε satisfying (2.6) and (2.8) and u is a smooth solution of (2.5) with $u = 0$ on $\partial_h^- D_\varepsilon$. If*

$$\varepsilon(|c^+|_{L^\infty(D_\varepsilon)} + |a_{xx}|_{L^\infty(D_\varepsilon)} + |b_{1,x}|_{L^\infty(D_\varepsilon)} + |b_{2,y}|_{L^\infty(D_\varepsilon)} + 1)^{\frac{1}{2}} < 1,$$

then for any cutoff function $\varphi = \varphi(x)$ on $(-1, 1)$

$$\begin{aligned}&\|\varphi u\|_{L^2(D_\varepsilon)} + \|\varphi u_y\|_{L^2(D_\varepsilon)} + \|\varphi\sqrt{a}u_x\|_{L^2(D_\varepsilon)} \\ &\quad \le C_0\left(\|u\|_{L^2(\partial_h^+ D_\varepsilon)} + \|u_y\|_{L^2(\partial_h^+ D_\varepsilon)} + \|\sqrt{\varphi a}u\|_{L^2(D_\varepsilon)} + \|\varphi f\|_{L^2(D_\varepsilon)}\right),\end{aligned} \tag{2.9}$$

where C_0 is a positive constant depending only φ, a, C_b and the supnorm of b_2.

Proof For convenience, we set

$$M = |c^+|_{L^\infty(D_\varepsilon)} + |a_{xx}|_{L^\infty(D_\varepsilon)} + |b_{1,x}|_{L^\infty(D_\varepsilon)} + |b_{2,y}|_{L^\infty(D_\varepsilon)} + 1. \tag{2.10}$$

Multiplying (2.5) by $\varphi^2 u$ and integrating over D_ε, we obtain

$$\begin{aligned}\int_{D_\varepsilon}(\varphi^2 u_y^2 + \varphi^2 a u_x^2) &= \int_{D_\varepsilon}\left(\varphi^2 a u u_x - \frac{1}{2}(\varphi^2 a)_x u^2 + \frac{1}{2}\varphi^2 b_1 u^2\right)_x \\ &\quad + \int_{D_\varepsilon}\left(\varphi^2 u u_y + \frac{1}{2}\varphi^2 b_2 u^2\right)_y + \int_{D_\varepsilon}\varphi^2\left(c + \frac{1}{2}a_{xx} - \frac{1}{2}b_{1,x} - \frac{1}{2}b_{2,y}\right)u^2 \\ &\quad + \int_{D_\varepsilon}\left((\varphi\varphi_{xx} + \varphi_x^2)a + \varphi\varphi_x(2a_x - b_1)\right)u^2 - \int_{D_\varepsilon}\varphi^2 u f.\end{aligned} \tag{2.11}$$

We first note that there is no boundary integral over $\partial_v D_\varepsilon$ since $\varphi = 0$ there and there is no boundary integral on $\partial_h^- D_\varepsilon$ since $u = 0$ there. Next, we note that $\varphi_x^2 \le C_\varphi \varphi$ on $(-1, 1)$ for some positive constant C_φ. Since $a \ge 0$ in D_1, we also have

$$|\partial_x a| \le C_a\sqrt{a} \quad \text{in } \mathrm{supp}\varphi \times (-1, 1), \tag{2.12}$$

for some positive constant C_a depending only on suppφ and the C^2-norm of a. Then by (2.8), (2.12) and the Cauchy inequality, we have

$$(\varphi\varphi_{xx} + \varphi_x^2)a + \varphi\varphi_x(2a_x - b_1) \le \varphi^2 + C_0\varphi a,$$

where C_0 is a positive constant depending only on φ, C_a and C_b. With (2.10), we get

$$\begin{aligned}\int_{D_\varepsilon}(\varphi^2 u_y^2 + \varphi^2 a u_x^2) &\le C_0\int_{y=\varepsilon}\varphi^2(u^2 + u_y^2) + C_0\int_{D_\varepsilon}\varphi a u^2 \\ &\quad + M\int_{D_\varepsilon}\varphi^2 u^2 + \int_{D_\varepsilon}\varphi^2 f^2,\end{aligned}$$

where C_0 is a positive constant depending only φ, C_a, C_b and the supnorm of b_2. A simple integration over y yields

$$u^2(x, y) \le y\int_0^\varepsilon u_y^2(x, t)dt,$$

and then

$$\int_{D_\varepsilon} \varphi^2 u^2 \le \frac{1}{2}\varepsilon^2 \int_{D_\varepsilon} \varphi^2 u_y^2.$$

By a simple substitution, we get

$$\int_{D_\varepsilon} (\varphi^2 u_y^2 + \varphi^2 a u_x^2) \le C_0 \int_{y=\varepsilon} \varphi^2(u^2 + u_y^2) + C_0 \int_{D_\varepsilon} \varphi a u^2 + \frac{1}{2}\varepsilon^2 M \int_{D_\varepsilon} \varphi^2 u_y^2 + \int_{D_\varepsilon} \varphi^2 f^2.$$

With $\varepsilon\sqrt{M} \le 1$, we then obtain

$$\int_{D_\varepsilon} (\varphi^2 u_y^2 + \varphi^2 a u_x^2) \le C_0 \left\{ \int_{y=\varepsilon} \varphi^2(u^2 + u_y^2) + \int_{D_\varepsilon} \varphi a u^2 + \int_{D_\varepsilon} \varphi^2 f^2 \right\},$$

and hence

$$\int_{D_\varepsilon} (\varphi^2 u^2 + \varphi^2 u_y^2 + \varphi^2 a u_x^2) \le C_0 \left\{ \int_{y=\varepsilon} \varphi^2(u^2 + u_y^2) + \int_{D_\varepsilon} \varphi a u^2 + \int_{D_\varepsilon} \varphi^2 f^2 \right\}.$$

This implies (2.9) easily. □

Corollary 2.3 *Suppose a, b_1, b_2 and c are smooth functions in D_ε satisfying (2.6) and (2.8) and u is a smooth solution of (2.5) with $u = 0$ on $\partial_h^- D_\varepsilon$. If for an integer $s \ge 1$,*

$$s\varepsilon(|c^+|_{L^\infty(D_\varepsilon)} + |a_{xx}|_{L^\infty(D_\varepsilon)} + |b_{1,x}|_{L^\infty(D_\varepsilon)} + |b_{2,y}|_{L^\infty(D_\varepsilon)} + 1)^{\frac{1}{2}} < 1,$$

then for any cutoff function $\varphi = \varphi(x)$ on $(-1, 1)$

$$\|\varphi u\|_{H^s(D_\varepsilon)} \le C_s \left(\sum_{k=0}^{s+1} \|D^k u\|_{L^2(\partial_h^+ D_\varepsilon)} + \|u\|_{L^2(D_\varepsilon)} + \|f\|_{H^s(D_\varepsilon)} \right), \tag{2.13}$$

where C_s is a positive constant depending on φ, C_b and the C^s-norms of a, b_1, b_2 and c.

We emphasize that C_s is independent of $\inf a$.

Proof We first claim for any integer $s \ge 0$

$$\begin{aligned} &\|\varphi\partial_x^s u\|_{L^2(D_\varepsilon)} + \|\varphi\partial_y\partial_x^s u\|_{L^2(D_\varepsilon)} + \|\varphi\sqrt{a}\partial_x^{s+1} u\|_{L^2(D_\varepsilon)} \\ &\quad\le C\Bigg(\|\partial_x^s u\|_{L^2(\partial_h^+ D_\varepsilon)} + \|\partial_y\partial_x^s u\|_{L^2(\partial_h^+ D_\varepsilon)} + \|\sqrt{\varphi a}\partial_x^s u\|_{L^2(D_\varepsilon)} \\ &\qquad + \sum_{k=0}^{s-1} \|\varphi\partial_x^k u\|_{L^2(D_\varepsilon)} + \sum_{k=0}^{s-1} \|\varphi\partial_y\partial_x^k u\|_{L^2(D_\varepsilon)} + \|\varphi\partial_x^s f\|_{L^2(D_\varepsilon)} \Bigg), \end{aligned} \tag{2.14}$$

where C is a positive constant depending on φ, C_b and the C^s-norms of a, b_1, b_2 and c.

We first assume (2.14) for any $s \geq 0$ and prove (2.13). By $(2.14)_{s-1}$ and $(2.14)_s$, with different cutoff functions, we obtain

$$\begin{aligned}
&\|\varphi\partial_x^s u\|_{L^2(D_\varepsilon)} + \|\varphi\partial_y\partial_x^{s-1}u\|_{L^2(D_\varepsilon)} \\
&\leq C\Bigg(\sum_{k=s-1}^{s}\|\partial_x^k u\|_{L^2(\partial_h^+ D_\varepsilon)} + \sum_{k=s-1}^{s}\|\partial_y\partial_x^k u\|_{L^2(\partial_h^+ D_\varepsilon)} + \|\sqrt[4]{\varphi}\sqrt{a}\partial_x^{s-1}u\|_{L^2(D_\varepsilon)} \\
&\quad + \sum_{k=0}^{s-1}\|\sqrt{\varphi}\partial_x^k u\|_{L^2(D_\varepsilon)} + \sum_{k=0}^{s-2}\|\sqrt{\varphi}\partial_y\partial_x^k u\|_{L^2(D_\varepsilon)} + \sum_{k=s-1}^{s}\|\sqrt{\varphi}\partial_x^k f\|_{L^2(D_\varepsilon)}\Bigg).
\end{aligned}$$

Note by (2.5)

$$\partial_{yy}u = -a\partial_{xx}u - b_1\partial_x u - b_2\partial_y u - cu + f \quad \text{in } D_\varepsilon.$$

It is obvious that derivatives of u of order s can be obtained easily in terms of $\partial_x^s u$ and lower order derivatives of u. Hence we obtain

$$\begin{aligned}
\sum_{i+j=s}\|\varphi\partial_x^i\partial_y^j u\|_{L^2(D_\varepsilon)} &\leq C\Big(\sum_{k=s-1}^{s}\|\partial_x^k u\|_{L^2(\partial_h^+ D_\varepsilon)} + \sum_{k=s-1}^{s}\|\partial_y\partial_x^k u\|_{L^2(\partial_h^+ D_\varepsilon)} \\
&\quad + \sum_{i+j\leq s-1}\|\sqrt{\varphi}\partial_x^i\partial_y^j u\|_{L^2(D_\varepsilon)} + \sum_{i+j\leq s}\|\sqrt{\varphi}\partial_x^i\partial_y^j f\|_{L^2(D_\varepsilon)}\Big).
\end{aligned}$$

This implies (2.13) by a simple induction.

Next, we prove (2.14). Applying ∂_x^s to (2.5), we get

$$\partial_y^2\partial_x^s u + a\partial_x^2\partial_x^s u + \tilde{b}_1\partial_x\partial_x^s u + b_2\partial_y\partial_x^s u + \tilde{c}\partial_x^s u = f_s, \tag{2.15}$$

where

$$\begin{aligned}
\tilde{b}_1 &= b_1 + sa_x, \\
\tilde{c} &= c + s(b_1)_x + \frac{1}{2}s(s-1)a_{xx},
\end{aligned}$$

and

$$f_s = \partial_x^s f + \sum_{i=0}^{s-1}\big(c_{s,i-2}\partial_x^{s-i+2}a + c_{s,i-1}\partial_x^{s-i+1}b_1 + c_{s,i}\partial_x^{s-i}c\big)\partial_x^i u + \sum_{i=0}^{s-1}c_{s,i}\partial_x^{s-i}b_2\partial_y\partial_x^i u,$$

where $c_{s,i}$ is a positive constant for $i = 0, 1, \ldots, s-1$ with $c_{s,-2} = c_{s,-1} = 0$. Note that (2.15) has the same structure as (2.5). So we can proceed as in the proof of Lemma 2.2 to get an estimate of $\partial_x^s u$. We only need to note that the corresponding coefficient for $\varphi^2(\partial_x^s u)^2$, as compared with that for $\varphi^2 u^2$ in (2.11), is given by

$$\tilde{c} + \frac{1}{2}a_{xx} - \frac{1}{2}\tilde{b}_{1,x} - \frac{1}{2}b_{2,y} = c + \frac{1}{2}(s-1)^2 a_{xx} + (s - \frac{1}{2})b_{1,x} - \frac{1}{2}b_{2,y}.$$

By Lemma 2.2, we have for $\varepsilon \leq (s\sqrt{M})^{-1}$

$$\begin{aligned}
&\|\varphi\partial_x^s u\|_{L^2(D_\varepsilon)} + \|\varphi\partial_y\partial_x^s u\|_{L^2(D_\varepsilon)} + \|\varphi\sqrt{a}\partial_x^{s+1}u\|_{L^2(D_\varepsilon)} \\
&\leq C\big(\|\partial_x^s u\|_{L^2(\partial_h^+ D_\varepsilon)} + \|\partial_y\partial_x^s u\|_{L^2(\partial_h^+ D_\varepsilon)} + \|\sqrt{\varphi a}\partial_x^s u\|_{L^2(D_\varepsilon)}\| + \|\varphi f_s\|_{L^2(D_\varepsilon)}\big).
\end{aligned}$$

With the explicit expression of f_s, we get (2.14) easily. □

Next, we study solutions in a neighborhood of the origin. We first recall some results for (strictly) elliptic differential equations in domains with an angular singularity on the boundary. Main references are [18], Chapter 4 and Chapter 5 in [3] or Chapter 6 in [19].

For any nonnegative integer m, define the space $V^m(\mathcal{C}_\kappa)$ as the closure of $C_0^\infty(\bar{\mathcal{C}}_\kappa \setminus \{0\})$ with respect to the norm

$$\|u\|_{V^m(\mathcal{C}_\kappa)} = \Big(\sum_{|\alpha| \le m} \int_{\mathcal{C}_\kappa} r^{2(|\alpha|-m)} |D^\alpha u|^2 \Big)^{1/2}.$$

To illustrate how the regularity depends on the angle of the cone, we consider

$$\begin{aligned} \Delta u &= f \quad \text{in } \mathcal{C}_\kappa, \\ u &= 0 \quad \text{on } \partial\mathcal{C}_\kappa. \end{aligned} \tag{2.16}$$

Let $\kappa = \tan(\alpha/2)$ for an $\alpha \in (0, \pi)$. Obviously, $u = r^{\frac{\pi}{\alpha}} \cos(\pi\theta/\alpha)$ is a solution of the homogeneous (2.16). It is easy to check that such a u is in $V^m(\mathcal{C}_\kappa \cap B_1)$ provided

$$(m-1)\alpha < \pi.$$

In general, the regularity $u \in V^m(\mathcal{C}_\kappa)$ cannot be improved if $(m-1)\alpha/\pi$ is not an integer. Hence solutions of (2.16) exhibit a better regularity in smaller cones. This turns out to be a general result.

We consider a slightly more general case. For a constant $a > 0$, we consider

$$\begin{aligned} u_{yy} + a u_{xx} &= f \quad \text{in } \mathcal{C}_\kappa, \\ u &= 0 \quad \text{on } \partial\mathcal{C}_\kappa. \end{aligned} \tag{2.17}$$

By introducing

$$x = \sqrt{a}s, \quad y = t,$$

we have

$$\begin{aligned} u_{tt} + u_{ss} &= f \quad \text{in } \mathcal{C}_{\sqrt{a}\kappa}, \\ u &= 0 \quad \text{on } \partial\mathcal{C}_{\sqrt{a}\kappa}. \end{aligned}$$

Lemma 2.4 *Let $\kappa, a > 0$ be constants and $u \in H_0^1(\mathcal{C}_\kappa)$ be the unique solution of (2.17) for an $f \in L^2(\mathcal{C}_\kappa)$. Then for any integer $m \ge 2$ satisfying*

$$2(m-1)\arctan(\sqrt{a}\kappa) < \pi, \tag{2.18}$$

if $f \in V^{m-2}(\mathcal{C}_\kappa)$, then u is in $V^m(\mathcal{C}_\kappa)$ and satisfies

$$\|u\|_{V^m(\mathcal{C}_\kappa)} \le C\|f\|_{V^{m-2}(\mathcal{C}_\kappa)},$$

where C is a positive constant depending only on m, a and κ.

Note that (2.18) always holds for $m = 2$.

Remark 2.5 If (2.18) is violated, then u is not necessarily in $V^m(\mathcal{C}_\kappa)$. To illustrate this, we consider (2.16), or (2.17) with $a = 1$. We write $\kappa = \tan(\alpha/2)$ for an $\alpha \in (0, \pi)$ and let $m > 2$ be an integer such that $(m-1)\alpha/\pi$ is not an integer. If $f \in V^{m-2}(\mathcal{C}_\kappa)$, then any solution u of (2.16) admits a decomposition

$$u = \sum_j c_j r^{\frac{j\pi}{\alpha}} \cos\frac{j\pi\theta}{\alpha} + w,$$

where $w \in V^m(\mathcal{C}_\kappa)$ and the summation is extended over all integer j in the interval $(\alpha/\pi, (m-1)\alpha/\pi)$.

For solutions of (2.5), the regularity is governed by the corresponding result for the constant coefficient operator $\partial_{yy} + a(0)\partial_{xx}$.

Lemma 2.6 *Let κ be a constant, a, b_1, b_2 and c be smooth functions in Ω_κ satisfying (2.6) and $u \in H_0^1(\Omega_\kappa)$ be a solution of (2.5) for an $f \in L^2(\Omega_\kappa)$. Then for any integer $m \geq 2$ satisfying*

$$2(m-1)\arctan(\sqrt{a(0)}\kappa) < \pi, \tag{2.19}$$

if $\xi f \in V^{m-2}(\mathcal{C}_\kappa)$, then ηu is in $V^m(\mathcal{C}_\kappa)$ and satisfies

$$\|\eta u\|_{V^m(\mathcal{C}_\kappa)} \leq C(\|\xi u\|_{L^2(\Omega_\kappa)} + \|\xi f\|_{V^{m-2}(\mathcal{C}_\kappa)}),$$

where ξ and η are two arbitrary cutoff functions in B_1 with $\xi = 1$ on the support of η and C is a positive constant depending only on m, $a(0)$, κ, ξ, η and C^{m-2}-norms of a, b_i and c.

Later on, we will only use the regularity assertion, instead of estimates, in Lemma 2.6. Now we begin to derive estimates of u close to the origin independent of $\inf a$. The main result for this part is the following lemma.

Lemma 2.7 *Let m be an integer, $f \in H^m(\mathcal{C}_\kappa \cap B_1)$ and u be an H^1-solution of (2.23) in $\mathcal{C}_1 \cap B_1$ satisfying $u = 0$ on $\theta = \pm\alpha$. Then there exist constants δ_m and κ_m such that, if $\kappa \leq \kappa_m$ and $\sqrt{a(0)}\kappa \leq \delta_m$, then $u \in H^m(\mathcal{C}_1 \cap B_1)$ and*

$$\|u\|_{H^m(\mathcal{C}_\kappa \cap \{x<\frac{1}{2}\})} \leq C_m\left(\sum_{i=0}^{m+1} \|D^i u\|_{L^2(\mathcal{C}_\kappa \cap \{x=\frac{1}{2}\})} + \|f\|_{H^m(\mathcal{C}_\kappa \cap B_1)}\right), \tag{2.20}$$

where δ_m is a positive constant depending only on m, κ_m is a positive constant depending only on the C^2-norms of a, b_i, c and C_m is a positive constant depending only on the C^m-norms of a, b_i and c.

We emphasize that C_m is independent of $\inf a$. The proof of Lemma 2.7 is complicated. We first establish some auxiliary lemmas.

Lemma 2.8 *Let m be a nonnegative integer, a, b_1, b_2 and c be smooth functions in Ω_κ satisfying (2.6) and u be an H^2-solution of (2.5) with $u = 0$ on $\partial\Omega_\kappa \cap B_1$. Then there exists a positive constant η_m depending only on m such that, if*

$$\kappa^2 a(0) \leq \eta_m, \tag{2.21}$$

then there exists a polynomial $\mathcal{P}_m(u)$ of degree m such that $\mathcal{P}_m(u) = 0$ on $\partial\Omega_\kappa \cap B_1$, any coefficient c_k in the homogeneous part of degree k, for any $k \leq m$, in $\mathcal{P}_m(u)$ satisfies

$$|c_k| \leq C_k \sum_{|\alpha| \leq k-2} |D^\alpha f(0)|,$$

and

$$\mathcal{L}(u - \mathcal{P}_m(u)) = (m-2)\text{-th remainder of } f - \tilde{\mathcal{L}}(\mathcal{P}_m(u)),$$

where C_k is a positive constant depending only on δ_m, κ, and the C^{k-1}-norms of a, b_i and c, and $\tilde{\mathcal{L}} = (a - a(0))\partial_{xx} + b_1\partial_x + b_2\partial_y + c$.

It is easy to see from the proof below that $\mathcal{P}_m(u)$ is the m-th Taylor polynomial of u at 0 if u is C^m in a neighborhood of the origin.

Proof We first consider the transform $(x, y) \mapsto (x/\kappa, y)$. Then (2.5) has the form

$$u_{yy} + \kappa^2 a u_{xx} + \kappa b_1 u_x + b_2 u_y + cu = f \quad \text{in } \tilde{\Omega}_1,$$

for a domain $\tilde{\Omega}_1 \subset \mathbb{R}^2$ with $\tilde{\Omega}_1 \cap B_1 = \mathcal{C}_1 \cap B_1$. In the following, we simply assume $\kappa = 1$.

Let $\mathcal{L}_0 = \partial_{yy} + a(0)\partial_{xx}$. We first note $\partial_x u(0) = \partial_y u(0) = 0$ since $u = 0$ on $y = \pm\kappa x$. Set

$$u = \mathcal{P}_m(u) + \mathcal{R}_m(u) = Q_2 + \cdots + Q_m + \mathcal{R}_m(u),$$

where Q_k is a homogeneous polynomial of degree k for $k = 2, \ldots, m$, with $Q_2 = \cdots = Q_m = 0$ on $\partial\Omega_1 \cap B_1$. Hence for $k = 2, \ldots, m$, Q_k has the form

$$Q_k = (x^2 - y^2) \sum_{i=0}^{k-2} c_{k-2,i} x^{k-2-i} y^i.$$

Then

$$\mathcal{L}\big(\mathcal{R}_m(u)\big) + \mathcal{L}_0 Q_2 + \cdots + \mathcal{L}_0 Q_m = \tilde{f},$$

where

$$\tilde{f} = f - (\mathcal{L} - \mathcal{L}_0)\big(\mathcal{P}_m(u)\big) = f - \sum_{k=2}^{m} (\mathcal{L} - \mathcal{L}_0) Q_k.$$

Note that $\mathcal{L}_0 Q_k$ is a homogeneous polynomial of degree $k-2$. We set

$$\mathcal{L}_0 Q_k = \text{the } (k-2)\text{-th homogeneous part of } \tilde{f}, \quad \text{for each } k = 2, \ldots, m. \tag{2.22}$$

Then

$$\mathcal{L}\big(\mathcal{R}_m(u)\big) = \ (m-2)\text{-th remainder of } f - (\mathcal{L} - \mathcal{L}_0)(\mathcal{P}_m(u)).$$

We claim that we can solve successively $Q_2, Q_3, \ldots, Q_m$. In fact, a simple calculation shows

$$\begin{aligned} \mathcal{L}_0 Q_k = & \sum_{i=0}^{k-2} \big((k-i)(k-i-1)a(0) - (i+2)(i+1)\big) c_{k-2,i} \\ & - \sum_{i=2}^{k-2} (k-i)(k-i-1)a(0) c_{k-2,i-2} + \sum_{i=0}^{k-4} (i+2)(i+1) c_{k-2,i+2}. \end{aligned}$$

If we write (2.22) as a linear system for $c_{k-2,0}, c_{k-2,1}, \ldots, c_{k-2,k-2}$, the $(k-1) \times (k-1)$ coefficient matrix is obviously invertible if $a(0) = 0$ and hence invertible if $a(0)$ is small. It is easy to see that $c_{k-2,0}, c_{k-2,1}, \ldots, c_{k-2,k-2}$ solving (2.22) is a linear combination of $D^\alpha f(0)$, $|\alpha| \le k$. □

To discuss the regularity of solutions close to the origin, we need to consider (2.5) in polar coordinates. We note

$$x = r\cos\theta, \quad y = r\sin\theta.$$

It is easy to see that any $D^\alpha u$, for some $|\alpha| = m$, is a linear combination of

$$\frac{1}{r^{m-i}}\partial_r^i\partial_\theta^j u, \quad 1 \le i+j \le m,$$

with coefficients given by smooth functions of θ.

In polar coordinates, (2.5) has the form

$$\tilde{a}_{11}r^2u_{rr} + 2\tilde{a}_{12}ru_{r\theta} + u_{\theta\theta} + \tilde{b}_1ru_r + \tilde{b}_2u_\theta + \tilde{c}u = \tilde{f}, \tag{2.23}$$

where

$$\begin{aligned}
\tilde{a}_{11} &= \frac{\sin^2\theta + a\cos^2\theta}{\cos^2\theta + a\sin^2\theta},\\
\tilde{a}_{12} &= \frac{(1-a)\sin\theta\cos\theta}{\cos^2\theta + a\sin^2\theta},\\
\tilde{b}_1 &= \frac{\cos^2\theta + a\sin^2\theta + b_1r\cos\theta + b_2r\sin\theta}{\cos^2\theta + a\sin^2\theta},\\
\tilde{b}_2 &= \frac{-2(1-a)\cos\theta\sin\theta - b_1r\sin\theta + b_2r\cos\theta}{\cos^2\theta + a\sin^2\theta},\\
\tilde{c} &= \frac{r^2c}{\cos^2\theta + a\sin^2\theta},
\end{aligned}$$

and

$$\tilde{f} = \frac{r^2 f}{\cos^2\theta + a\sin^2\theta}.$$

Lemma 2.9 *Let u be a C^m-solution of (2.23). Then for any integer k, l with $1 \le k+l \le m$*

$$\frac{1}{r^{m-k}}\partial_r^k\partial_\theta^l u = \sum_{i=0}^{m}\frac{1}{r^{m-i}}\left(c_i\partial_r^i u + d_i\partial_r^i\partial_\theta u\right) + \sum_{i=2}^{m}\sum_{j=0}^{m-i}\frac{1}{r^{m-i}}e_{ij}\partial_r^{i-2}\partial_\theta^j\tilde{f},$$

where $d_m = 0$ and all coefficients c_i, d_i and e_{ij} are functions depending on derivatives of $\tilde{a}_{11}, \tilde{a}_{12}, \tilde{b}_1, \tilde{b}_2$ and $\tilde{c}$ (with respect to r and θ) up to the order $m-2$.

The proof is by a simple induction based on (2.23) and hence omitted. Therefore, in order to estimate $D^m u$, we only need to estimate

$$\frac{1}{r^{m-i}}\partial_r^i\partial_\theta^j u, \quad 0 \le j \le 1,\ 0 \le i+j \le m.$$

In the following, we assume $\kappa \le 1$ and consider (2.5) in

$$R = \{(x, y) \in \Omega_\kappa; x < 1/2\},$$

or the equivalent (2.23) in

$$R = \{(r, \theta); 0 < r < r(\theta), -\alpha < \theta < \alpha\},$$

where $\alpha \in (0, \pi/2)$ with $\tan\alpha = \kappa$ and $r = r(\theta)$ corresponds to $x = 1/2$, hence $r(\theta) = 1/(2\cos\theta)$.

Lemma 2.10 *Let μ be a positive constant and u be a C^2-solution of (2.5) in R satisfying $u = 0$ on $\theta = \pm\alpha$ and*

$$\int_R \left(\frac{u^2}{r^\mu} + \frac{u_r^2}{r^{\mu-2}}\right) < \infty.$$

Then there exists a sufficiently small κ_0 such that, if

$$\kappa(|a|_{C^2} + |b_i|_{C^1} + |c| + 1)^{\frac{1}{2}} < \kappa_0,$$

then

$$\int_R \left(\frac{u^2}{r^\mu} + \frac{1}{r^{\mu-2}}(u_r \sin\theta + \frac{1}{r}u_\theta \cos\theta)^2\right) \le C_0 \int_{\partial_v^+ R} (u^2 + u_r^2) + \int_R \frac{f^2}{r^{\mu-4}}, \quad (2.24)$$

where C_0 is a positive constant depending only on the C^1-norm of a and the L^∞-norm of b_i.

The proof is similar to that of Lemma 2.2.

Proof We multiply (2.23) by $-u/r^\mu$ and get by a straightforward calculation

$$\begin{aligned}
&-\frac{1}{2r}\left(\frac{2\tilde{a}_{11}uu_r}{r^{\mu-3}} - \big(2\tilde{a}_{12,\theta} + r\tilde{a}_{11,r} - (\mu-3)\tilde{a}_{11} - \tilde{b}_1\big)\frac{u^2}{r^{\mu-2}}\right)_r \\
&-\left(\frac{uu_\theta}{r^\mu} + \frac{2\tilde{a}_{12}uu_r}{r^{\mu-1}} + \frac{\tilde{b}_2u^2}{2r^\mu}\right)_\theta + \frac{1}{r^{\mu-2}}\left(\frac{u_\theta^2}{r^2} + \frac{2\tilde{a}_{12}u_\theta u_r}{r} + \tilde{a}_{11}u_r^2\right) = \Lambda\frac{u^2}{r^\mu} - \frac{u\tilde{f}}{r^\mu},
\end{aligned} \quad (2.25)$$

where

$$\begin{aligned}
\Lambda =&\frac{1}{2}\big(2r\tilde{a}_{12,\theta r} - 2(\mu-2)\tilde{a}_{12,\theta} + r^2\tilde{a}_{11,rr} - 2(\mu-3)r\tilde{a}_{11,r} \\
&+ (\mu-3)(\mu-2)\tilde{a}_{11} - r\tilde{b}_{1,r} + (\mu-2)\tilde{b}_1 - \tilde{b}_{2,\theta} + 2c\big).
\end{aligned} \quad (2.26)$$

A simple calculation shows

$$\begin{aligned}
&\frac{u_\theta^2}{r^2} + \frac{2\tilde{a}_{12}u_\theta u_r}{r} + \tilde{a}_{11}u_r^2 \\
&\quad= \frac{1}{\cos^2\theta + a\sin^2\theta}\left(\left(u_r\sin\theta + \frac{1}{r}u_\theta\cos\theta\right)^2 + a\left(u_r\cos\theta - \frac{1}{r}u_\theta\sin\theta\right)^2\right) \\
&\quad\ge \left(u_r\sin\theta + \frac{1}{r}u_\theta\cos\theta\right)^2 + a\left(u_r\cos\theta - \frac{1}{r}u_\theta\sin\theta\right)^2 = u_y^2 + au_x^2.
\end{aligned}$$

Now we integrate (2.25) with respect to $rdrd\theta$ in

$$R_{\bar{r}} = \{(r,\theta); \bar{r} < r < r(\theta), -\alpha < \theta < \alpha\},$$

for any $\bar{r} < 1/2$. Since $u = 0$ on $\theta = \pm\alpha$, there is no boundary integral on $\theta = \pm\alpha$. By the Cauchy inequality, we have

$$\begin{aligned}
\int_{R_{\bar{r}}} \frac{1}{r^{\mu-2}}(u_y^2 + au_x^2) \le C_0\int_{-\alpha}^{\alpha}\left(\frac{u^2}{r^\mu} + \frac{u_r^2}{r^{\mu-2}}\right)\Big|_{r=\bar{r}} d\theta + C_0\int_{r=r(\theta)} (u^2 + u_r^2) \\
+ \int_{R_{\bar{r}}} (\Lambda+1)\frac{u^2}{r^\mu} + \int_{R_{\bar{r}}} \frac{\tilde{f}^2}{r^\mu},
\end{aligned}$$

where C_0 depends on the L^∞-norms of $\tilde{a}_{12,\theta}$, $r\tilde{a}_{11,r}$, $\tilde{a}_{11}$ and $\tilde{b}_1$. Next, we write

$$\int_{R\cap\{r<\frac{1}{2}\}} \left(\frac{u^2}{r^\mu}+\frac{u_r^2}{r^{\mu-2}}\right)=\int_0^{\frac{1}{2}}\frac{1}{r}\int_{-\alpha}^{\alpha}\left(\frac{u^2}{r^{\mu-2}}+\frac{u_r^2}{r^{\mu-4}}\right)d\theta dr.$$

Then there exists a sequence $r_i\to 0$ such that

$$\int_{-\alpha}^{\alpha}\left(\frac{u^2}{r^{\mu-2}}+\frac{u_r^2}{r^{\mu-4}}\right)\bigg|_{r=r_i} d\theta\to 0 \quad \text{as } r_i\to 0.$$

By taking $\bar{r}=r_i\to 0$, we have

$$\int_R \frac{1}{r^{\mu-2}}(u_y^2+au_x^2)\le C_0\int_{r=r(\theta)}(u^2+u_r^2)+\int_R(\Lambda+1)\frac{u^2}{r^\mu}+\int_R\frac{\tilde{f}^2}{r^\mu}. \tag{2.27}$$

For any $(r,\theta)\in R$, the corresponding (x,y) satisfies $|y|<\kappa x$ and $x<1/2$. Since $u(x,-\kappa x)=0$, we have

$$u(x,y)=\int_{-\kappa x}^{y}u_y(x,t)dt,$$

and

$$u^2(x,y)\le 2\kappa x\int_{-\kappa x}^{\kappa x}u_y^2(x,t)dt.$$

Note that $x^2\le r^2=x^2+y^2\le(\kappa^2+1)x^2$. This implies

$$\frac{u^2(x,y)}{(x^2+y^2)^{\frac{\mu}{2}}}\le 2(\kappa^2+1)^{\frac{\mu-2}{2}}\frac{\kappa x}{x^2+y^2}\int_{-\kappa x}^{\kappa x}\frac{u_y^2(x,t)}{(x^2+t^2)^{\frac{\mu-2}{2}}}dt.$$

An integration in $R=\{(x,y)\in\Omega_\kappa;\, x<1/2\}$ yields

$$\int_R\frac{u^2}{r^\mu}\le 4\kappa^2(\kappa^2+1)^{\frac{\mu-2}{2}}\int_R\frac{u_y^2}{r^{\mu-2}}.$$

If κ is small so that

$$4\kappa^2(\kappa^2+1)^{\frac{\mu-2}{2}}(|\Lambda|_{L^\infty}+1)\le\frac{1}{2},$$

we then have by (2.27)

$$\int_R\left(\frac{u^2}{r^\mu}+\frac{u_y^2}{r^{\mu-2}}+\frac{au_x^2}{r^{\mu-2}}\right)\le C_0\int_{r=r(\theta)}(u^2+u_r^2)+\int_R\frac{\tilde{f}^2}{r^\mu}.$$

This implies (2.24). □

Lemma 2.11 *Let m be an integer and u be a C^{m+2}-solution of (2.23) in R satisfying $u = 0$ on $\theta = \pm\alpha$ and*

$$\int_R \sum_{i=0}^{m+1} \frac{(\partial_r^i u)^2}{r^{2(m-i)}} < \infty.$$

If

$$m\kappa(|a|_{C^2} + |b_i|_{C^1} + |c|_{L^\infty} + 1)^{\frac{1}{2}} < \kappa_0, \tag{2.28}$$

then

$$\int_R \left(\sum_{i=0}^{m} \frac{(\partial_r^i u)^2}{r^{2(m-i)}} + \sum_{i=0}^{m-1} \frac{(\partial_\theta \partial_r^i u)^2}{r^{2(m-i)}} \right) \le C_m \int_{\partial_v^+ R} \sum_{i=0}^{m+1} (\partial_r^i u)^2 + \int_R \sum_{i=0}^{m} \frac{(\partial_r^i f)^2}{r^{2(m-i-2)^+}}, \tag{2.29}$$

where κ_0 is as in Lemma 2.10 and C_m is a positive constant depending only on the C^m-norms of a, b_i and c.

Proof For any $s = 0, 1, \dots, m$, we apply ∂_r^s to (2.23) to get

$$\begin{aligned} &\tilde{a}_{11} r^2 (\partial_r^s u)_{rr} + 2\tilde{a}_{12} r (\partial_r^s u)_{r\theta} + (\partial_r^s u)_{\theta\theta} + \tilde{b}_1^{(s)} r (\partial_r^s u)_r + \tilde{b}_2^{(s)} (\partial_r^s u)_\theta \\ &+ \tilde{c}^{(s)} \partial_r^s u = \tilde{f}^{(s)} - \tilde{d}^{(s)} \partial_\theta \partial_r^{s-1} u, \end{aligned} \tag{2.30}$$

where

$$\begin{aligned} \tilde{b}_1^{(s)} &= \tilde{b}_1 + s(r\tilde{a}_{11,r} + 2\tilde{a}_{11}), \\ \tilde{b}_2^{(s)} &= \tilde{b}_2 + 2s(r\tilde{a}_{12})_r, \\ \tilde{c}^{(s)} &= c + s\partial_r(r\tilde{b}_1) + \frac{1}{2} s(s-1)(r^2\tilde{a}_{11})_{rr}, \\ \tilde{d}^{(s)} &= s\tilde{b}_{2,r} + s(s-1)(r\tilde{a}_{12})_{rr}, \end{aligned}$$

and

$$\begin{aligned} \tilde{f}^{(s)} = \partial_r^s(r^2 f) &- \sum_{i=0}^{s-2} \big(2c_{s,i-1} \partial_r^{s-i+1}(r\tilde{a}_{12}) + c_{s,i} \partial_r^{s-i} \tilde{b}_2\big) \partial_\theta \partial_r^i u \\ &- \sum_{i=0}^{s-1} \big(c_{s,i-2} \partial_r^{s-i+2}(r^2\tilde{a}_{11}) + c_{s,i-1} \partial_r^{s-i+1}(r\tilde{b}_1) + c_{s,i} \partial_r^{s-i} c\big) \partial_r^i u, \end{aligned}$$

where $c_{s,i}$ is a constant depending only on s and i with $c_{s,-2} = c_{s,-1} = 0$. Since (2.30) has a similar structure as (2.23), we may apply Lemma 2.10 to (2.30). If

$$\int_R \left(\frac{(\partial_r^s u)^2}{r^\mu} + \frac{(\partial_r^{s+1} u)^2}{r^{\mu-2}} \right) < \infty,$$

and

$$4\kappa^2(\kappa^2 + 1)^{\frac{\mu-2}{2}} (|\Lambda_s|_{L^\infty} + |\tilde{d}^{(s)}|_{L^\infty} + 1) \le \kappa_0,$$

we obtain

$$\int_R \left(\frac{(\partial_r^s u)^2}{r^\mu} + \frac{1}{r^{\mu-2}}\left(\partial_r^{s+1} u \sin\theta + \frac{1}{r}\partial_\theta \partial_r^s u \cos\theta\right)^2\right)$$
$$\leq C_0 \int_{\partial_v^+ R} ((\partial_r^s u)^2 + (\partial_r^{s+1} u)^2) + s^2 \int_R \frac{(\partial_\theta \partial_r^{s-1} u)^2}{r^\mu} + \int_R \frac{(\tilde{f}^{(s)})^2}{r^\mu},$$

where Λ_s is as Λ in (2.26) with $\tilde{a}_{ij}, \tilde{b}_i, \tilde{c}$ replaced by $\tilde{a}_{ij}^{(s)}, \tilde{b}_i^{(s)}, \tilde{c}^{(s)}$. For each $s = 0, 1, \ldots, m$, we take $\mu = 2(m-s)$ and then obtain

$$\int_R \left(\frac{(\partial_r^s u)^2}{r^{2(m-s)}} + \frac{1}{r^{2(m-s-1)}}\left(\partial_r^{s+1} u \sin\theta + \frac{1}{r}\partial_\theta \partial_r^s u \cos\theta\right)^2\right)$$
$$\leq C_0 \int_{\partial_v^+ R} ((\partial_r^s u)^2 + (\partial_r^{s+1} u)^2) + s^2 \int_R \frac{(\partial_\theta \partial_r^{s-1} u)^2}{r^{2(m-s)}} + \int_R \frac{(\tilde{f}^{(s)})^2}{r^{2(m-s)}}.$$

Note that

$$(\tilde{f}^{(s)})^2 \leq C_s \left(\sum_{i=s-2}^{s} r^{2(i-s+2)}(\partial_r^i f)^2 + \sum_{i=0}^{s-1}(\partial_r^i u)^2 + \sum_{i=0}^{s-2}(\partial_\theta \partial_r^i u)^2\right),$$

where C_s depends on the C^s-norms of $\tilde{a}_{ij}, \tilde{b}_i$ and $\tilde{c}$. Hence we have

$$\int_R \left(\frac{(\partial_r^s u)^2}{r^{2(m-s)}} + \frac{1}{r^{2(m-s-1)}}(\partial_r^{s+1} u \sin\theta + \frac{1}{r}\partial_\theta \partial_r^s u \cos\theta)^2\right)$$
$$\leq C_0 \int_{\partial_v^+ R} ((\partial_r^s u)^2 + (\partial_r^{s+1} u)^2) + s^2 \int_R \frac{(\partial_\theta \partial_r^{s-1} u)^2}{r^{2(m-s+1)}}$$
$$+ \int_R \left(\sum_{i=s-2}^{s} \frac{(\partial_r^i f)^2}{r^{2(m-i-2)}} + \sum_{i=0}^{s-1} \frac{(\partial_r^i u)^2}{r^{2(m-i)}} + \sum_{i=0}^{s-2} \frac{(\partial_\theta \partial_r^i u)^2}{r^{2(m-i)}}\right). \tag{2.31}$$

Now we claim for any $k = 0, 1, \ldots, m$

$$\int_R \left(\frac{(\partial_r^k u)^2}{r^{2(m-k)}} + \frac{(\partial_\theta \partial_r^{k-1} u)^2}{r^{2(m-k+1)}}\right) \leq C_s \left(\int_{\partial_v^+ R} \sum_{i=0}^{k+1}(\partial_r^i u)^2 + \int_R \sum_{i=0}^{k} \frac{(\partial_r^i f)^2}{r^{2(m-i-2)^+}}\right). \tag{2.32}$$

Note that $(2.32)_0$ is simply a part of $(2.31)_0$. Now we assume that (2.32) holds for $k = 0, 1, \ldots, s \leq m-1$ and prove (2.32) for $k = s+1$. By $(2.31)_s$ and $(2.32)_0, \cdots, (2.32)_s$, we have

$$\int_R \frac{1}{r^{2(m-s-1)}}\left(\partial_r^{s+1} u \sin\theta + \frac{1}{r}\partial_\theta \partial_r^s u \cos\theta\right)^2 \leq C_s \left(\int_{\partial_v^+ R} \sum_{i=0}^{k+1}(\partial_r^i u)^2 + \int_R \sum_{i=0}^{k} \frac{(\partial_r^i f)^2}{r^{2(m-i-2)^+}}\right),$$

or

$$\int_R \frac{1}{r^{2(m-s)}}(\partial_\theta \partial_r^s u)^2 \le \kappa^2 \int_R \frac{1}{r^{2(m-s-1)}}(\partial_r^{s+1} u)^2 + C_s \left(\int_{\partial_v^+ R} \sum_{i=0}^{k+1} (\partial_r^i u)^2 + \int_R \sum_{i=0}^{k} \frac{(\partial_r^i f)^2}{r^{2(m-i-2)^+}} \right), \tag{2.33}$$

where we used $|\tan\theta| \le \tan\alpha = \kappa$ for any $|\theta| < \alpha$. Next, by $(2.31)_{s+1}$ and $(2.32)_0, \cdots, (2.32)_s$, we have

$$\int_R \frac{1}{r^{2(m-s-1)}}(\partial_r^{s+1} u)^2 \le (s+1)^2 \int_R \frac{1}{r^{2(m-s)}}(\partial_\theta \partial_r^s u)^2 + C_{s+1} \left(\int_{\partial_v^+ R} \sum_{i=0}^{s+2} (\partial_r^i u)^2 + \int_R \sum_{i=0}^{s+1} \frac{(\partial_r^i f)^2}{r^{2(m-i-2)^+}} \right). \tag{2.34}$$

If $\kappa(s+1) < 1/2$, (2.33) and (2.34) imply

$$\int_R \frac{1}{r^{2(m-s)}}(\partial_\theta \partial_r^s u)^2 \le C_{s+1} \left(\int_{\partial_v^+ R} \sum_{i=0}^{s+2} (\partial_r^i u)^2 + \int_R \sum_{i=0}^{s+1} \frac{(\partial_r^i f)^2}{r^{2(m-i-2)^+}} \right).$$

This, together with (2.34), yields (2.32) for $k = s+1$. □

Now we prove Lemma 2.7.

Proof of Lemma 2.7 We will only estimate the L^2-norms of $D^\alpha u$ for $|\alpha| = m$. We will first subtract a polynomial of an appropriate degree from u. By Lemma 2.8, if $\kappa^2 a(0)$ is small, we may find a polynomial P of degree $m-1$ such that $P = 0$ on $\partial\mathcal{C}_\kappa \cap B_1$, any coefficient c_k of degree k, for $k \le m-1$, in P satisfies

$$|c_k| \le C_k \sum_{|\alpha| \le k-2} |D^\alpha f(0)|,$$

and

$$\mathcal{L}\big(u - P\big) = f_m \equiv \ (m-3)\text{-th remainder of } f - (\mathcal{L} - (\partial_{yy} + a(0)\partial_{xx}))P,$$

where C_k is a positive constant depending only on δ_m, κ, and the C^{m-1}-norms of a, b_i and c. Then the Sobolev embedding theorem yields

$$|c_k| \le C_{m-1} \|f\|_{H^{m-1}(\mathcal{C}_\kappa \cap B_1)}. \tag{2.35}$$

Note that $f_m \in V^{m-2}(\mathcal{C}_\kappa \cap B_1)$ and $\xi(u-P) \in H_0^1(B_1)$ for any cutoff function ξ in B_1. By Lemma 2.6, if $\kappa^2 a(0)$ is small, then $u - P \in V^m(\mathcal{C}_\kappa \cap B_1)$. By Lemma 2.11, if (2.28) holds, then

$$\int_R \left(\sum_{i=0}^{m} \frac{(\partial_r^i (u-P))^2}{r^{2(m-i)}} + \sum_{i=0}^{m-1} \frac{(\partial_\theta \partial_r^i (u-P))^2}{r^{2(m-i)}} \right)$$
$$\leq C_m \int_{\partial_v^+ R} \sum_{i=0}^{m+1} (\partial_r^i (u-P))^2 + \int_R \sum_{i=0}^{m} \frac{(\partial_r^i f_m)^2}{r^{2(m-i-2)^+}}.$$

Now we apply Lemma 2.9 to $\mathcal{L}(u-P) = f_m$ to get

$$\int_R \sum_{|\alpha|=m} |D^\alpha (u-P)|^2 \leq C_m \Bigg(\int_{\partial_v^+ R} \sum_{i=0}^{m+1} \left(\partial_r^i (u-P)\right)^2 + \int_R \sum_{i=0}^{m} \sum_{j=0}^{m-i-2} \frac{\left(\partial_r^i \partial_\theta^j f_m\right)^2}{r^{2(m-i-2)^+}} \Bigg).$$

Then we obtain by (2.35)

$$\sum_{|\alpha|=m} \|D^\alpha u\|_{L^2(\mathcal{C}_\kappa \cap \{x<\frac{1}{2}\})} \leq C_m \Big(\sum_{i=0}^{m+1} \|D^i u\|_{L^2(\mathcal{C}_\kappa \cap \{x<\frac{1}{2}\})} + \|f\|_{H^m(\mathcal{C}_\kappa \cap B_1)} \Big).$$

This ends the proof. □

Now, we are ready to prove Theorem 2.1.

Proof of Theorem 2.1 We first note that $c \leq 0$ in Ω_κ. For any $\delta > 0$, we consider

$$\mathcal{L}_\delta u \equiv u_{yy} + (K+\delta)u_{xx} + b_1 u_x + b_2 u_y + cu = f \quad \text{in } \Omega_\kappa. \tag{2.36}$$

This is a uniformly elliptic differential equation in $\bar{\Omega}_\kappa$. Hence there exists a solution $u_\delta \in H_0^1(\Omega_\kappa)$. By the classical theory of uniform elliptic differential equations, we know $u \in C^\infty(\bar{\Omega}_\kappa \setminus \{0\}) \cap C(\bar{\Omega}_\kappa)$. In the following, we derive estimates on u_δ independent of δ. For brevity, we simply write $u = u_\delta$.

We first estimate u itself. We claim

$$|u|_{L^\infty(\Omega_\kappa)} \leq C|f|_{L^\infty(\Omega_\kappa)}. \tag{2.37}$$

To see this, we set

$$w(y) = e^{\alpha d} - e^{\alpha y},$$

where d is chosen so that $d > y$ for any $(x, y) \in \bar{\Omega}_\kappa$ and $\alpha > 0$ is chosen so that $\mathcal{L}_\delta w \leq -1$. Then (2.37) follows from a simple comparison of $\pm u$ with $|f|_{L^\infty(\Omega)} w$.

Next, we discuss derivatives of u. We note that (2.36) is elliptic in any subset Ω' of $\bar{\Omega}_\kappa$ away from the two rays $\theta = \pm \arctan \kappa$. Then by the standard H^m-estimates for solutions of elliptic differential equations (e.g., Theorem 8.10 in [2]), we have for any $m \geq 2$

$$\|u\|_{H^m(\Omega')} \leq C_m \left(\|u\|_{L^2(\Omega_\kappa)} + \|f\|_{H^{m-2}(\Omega_\kappa)} \right), \tag{2.38}$$

where C_m is a positive constant depending on the distance between $\partial\Omega'$ and the two rays $\theta = \pm \arctan \kappa$, the ellipticity constant in Ω' and the C^{m-2}-norms of K, b_i and c.

Next, we claim for any $p \in \partial\Omega_\kappa \cap \partial\mathcal{C}_\kappa$ and any $m \geq 1$, there exists a neighborhood U of p such that

$$\|u\|_{H^m(U\cap\Omega_\kappa)} \leq C_m \left(\|u\|_{L^2(\Omega_\kappa)} + \|f\|_{H^m(\Omega_\kappa)} \right), \tag{2.39}$$

where C_m is a positive constant depending on the distance between U and the origin, and the C^m-norms of K, b_i and c. To see this, we introduce a transform which takes p to the origin, the ray $\theta = \arctan\kappa$ or $\theta = -\arctan\kappa$ to the x-axis, and a neighborhood of p in Ω_κ to $D_\varepsilon = (-1, 1) \times (0, \varepsilon)$. By Corollary 2.3, for any cutoff function $\varphi = \varphi(x)$ in $(-1, 1)$ and any $m \geq 0$, there holds

$$\|\varphi u\|_{H^m(D_\varepsilon)} \leq C_m \left(\sum_{k=0}^{m+1} \|D^k u\|_{L^2(\partial_h^+ D_\varepsilon)} + \|u\|_{L^2(D_\varepsilon)} + \|f\|_{H^m(D_\varepsilon)} \right),$$

as long as ε is small. In fact, we may apply Corollary 2.3 in $D_t = (-1, 1) \times (0, t)$ for any $t \in (\varepsilon/2, \varepsilon)$ and then integrate with respect to t in $(\varepsilon/2, \varepsilon)$. Then we get

$$\|\varphi u\|_{H^m(D_{\varepsilon/2})} \leq C_m \left(\sum_{k=0}^{m+1} \|D^k u\|_{L^2(D_\varepsilon \setminus D_{\varepsilon/2})} + \|u\|_{L^2(D_\varepsilon)} + \|f\|_{H^m(D_\varepsilon)} \right).$$

The first term in the right-hand side can be estimated by (2.38). Hence, we get (2.39) easily for an appropriate U. We should note that U depends on m. It is obvious that U does not contain the origin.

With (2.38) and (2.39) and a simple covering, we obtain for any $r > 0$ and

$$\|u\|_{H^m(\Omega_\kappa \setminus B_r)} \leq C_m(\|u\|_{L^2(\Omega_\kappa)} + \|f\|_{H^m(\Omega_\kappa)}), \tag{2.40}$$

where C_m depends only on r and the C^m-norms of K, b_i and c. We emphasize that C_m does not depend on δ.

Next, we discuss the regularity of u in $\Omega_\kappa \cap B_r$. We claim for any integer m there exists an $\varepsilon = \varepsilon(m)$ such that if $\delta < \varepsilon^4$ there holds

$$\|u\|_{H^m(\Omega_\kappa \cap \{x<\varepsilon^5/2\})} \leq C_m \left(\sum_{i=0}^{m+1} \|D^i u\|_{L^2(\Omega_\kappa \cap \{x=\varepsilon^5/2\})} + \|f\|_{H^m(\Omega_\kappa \cap \{x<\varepsilon^5/2\})} \right), \tag{2.41}$$

where C_m is a positive constant depending only on m and the C^m-norms of K, b_i and c. To prove this, we set

$$x = \varepsilon^5 s, \quad y = \varepsilon^4 t.$$

Then

$$|y| < \kappa x \;\Leftrightarrow\; |t| < \varepsilon\kappa s.$$

Let $v(s, t) = u(\varepsilon^5 s, \varepsilon^4 t)$. Then v satisfies

$$v_{tt} + a_\delta v_{ss} + \varepsilon^3 b_1 v_s + \varepsilon^4 b_2 v_t + \varepsilon^8 c v = \varepsilon^8 f \quad \text{in } \Omega_{\varepsilon\kappa} \cap B_1,$$

where $a_\delta = (K + \delta)\varepsilon^{-2}$ and K, b_i and c are evaluated at $(\varepsilon^5 s, \varepsilon^4 t)$. Note $K(0, 0) = 0$ and hence for $(s, t) \in \Omega_{\varepsilon\kappa} \cap B_1$

$$K(\varepsilon^5 s, \varepsilon^4 t) \leq |DK|_{L^\infty} \sqrt{\varepsilon^{10} s^2 + \varepsilon^8 t^2} \leq \varepsilon^4 |DK|_{L^\infty}.$$

This implies

$$(\varepsilon\kappa)^2 a_\delta(0, 0) = \kappa^2 \delta,$$

and

$$a_\delta \leq \varepsilon^2 |DK|_{L^\infty} + \frac{\delta}{\varepsilon^2} \quad \text{in } \Omega_{\varepsilon\kappa} \cap B_1.$$

Now we take ε small so that $\kappa\varepsilon^2 \leq \delta_m$ and $\kappa\varepsilon \leq \kappa_m$, where δ_m and κ_m are as in Lemma 2.7. Then if $\delta < \varepsilon^4$, Lemma 2.7 implies $v \in H^k(\Omega_{\varepsilon\kappa})$ and

$$\|v\|_{H^m(\Omega_{\varepsilon\kappa}\cap\{s<1/2\})} \leq C_m\left(\sum_{i=0}^{m+1}\|D^i v\|_{L^2(\Omega_{\varepsilon\kappa}\cap\{s=1/2\})} + \|f\|_{H^m(\Omega_{\varepsilon\kappa}\cap\{s<1/2\})}\right), \quad (2.42)$$

where C_m is a positive constant depending only on m and the C^m-norms of K, b_i and c. Obviously, (2.42) implies (2.41). With a similar trick, we then get

$$\|u\|_{H^m(\Omega_\kappa\cap\{x<\varepsilon^5/2\})} \leq C_m\left(\sum_{i=0}^{m+1}\|D^i u\|_{L^2(\Omega_\kappa\cap\{\varepsilon^5/2<x<\varepsilon^5\})} + \|f\|_{H^m(\Omega_\kappa\cap\{x<\varepsilon^5\})}\right). \quad (2.43)$$

With (2.40) and (2.43), we conclude the following result: For any integer m there exists an $\varepsilon = \varepsilon(m)$ such that the solution u_δ of (2.36) with $u_\delta = 0$ on $\partial\Omega_\kappa$ for $\delta < \varepsilon^4$ satisfies

$$\|u_\delta\|_{H^m(\Omega_\kappa)} \leq C_m(\|u\|_{L^2(\Omega_\kappa)} + \|f\|_{H^{m+1}(\Omega_\kappa)}),$$

where C_m is a positive constant depending only on m and the C^m-norms of K, b_i and c. With (2.37) and the Sobolev embedding theorem, we obtain for any $m \geq 1$

$$\|u_\delta\|_{H^m(\Omega_\kappa)} \leq C_m\|f\|_{H^{m+1}(\Omega_\kappa)}.$$

It is easy to get a sequence of $\delta \to 0$ and a $u \in \cap_{m=1}^\infty H^m(\Omega_\kappa) \cap H_0^1(\Omega_\kappa)$ such that

$$u_\delta \to u \quad \text{in } H^m(\Omega_\kappa) \text{ for any } m \text{ as } \delta \to 0.$$

Therefore, u is a solution of (2.1) and satisfies $u = 0$ on $\partial\Omega_\kappa$ and (2.4). □

Remark 2.12 It is clear that Theorem 2.1 still holds if Ω_κ is replaced by $\Omega_{\kappa_1,\kappa_2}$ with the property that $\partial\Omega_{\kappa_1,\kappa_2} \setminus \{0\}$ is smooth and that in a small neighborhood of the origin $\partial\Omega_{\kappa_1,\kappa_2}$ is given by smooth functions $y = \kappa_1(x)$ and $y = \kappa_2(x)$ over a small interval $[0, d]$ with $\kappa_1(0) = \kappa_2(0) = 0$ and $\kappa_1'(0) > 0 > \kappa_2'(0)$. To see this, we simply note that there exists a smooth transform in $\bar{\Omega}_{\kappa_1,\kappa_2}$ such that $F(\Omega_{\kappa_1,\kappa_2} \cap U) = \mathcal{C}_\kappa \cap V$ for a positive constant κ and neighborhoods U and V of the origin.

Remark 2.13 We also note that $c \leq 0$ can be replaced by $c \leq \varepsilon$ for $\varepsilon > 0$ sufficiently small. This is standard for elliptic differential equations.

3 The cauchy problem in non-smooth hyperbolic regions

In this section, we will discuss Cauchy problems for hyperbolic equations in $\mathbb{R}^2$ when the initial curve has an angular point. We will discuss uniformly hyperbolic equations here and treat degenerate hyperbolic equations in the next section.

It is well known that the Cauchy problem for linear hyperbolic differential equation is well-posed in a domain whose boundary is a smooth non-characteristic curve. A standard example of such a domain is the upper half plane. However, we cannot apply directly results for smooth domains to non-smooth domains. In this section, we will prove by hand the existence of solutions of Cauchy problems for hyperbolic equations if the initial curve is not smooth and has an angular point. The method is based on energy estimates and is particularly designed for non-flat domains. The regularity of these solutions depends essentially on a class of *compatibility conditions* of Cauchy data and nonhomogeneous terms at angular points.

Throughout this section, we fix a function $y=\kappa(x)$ on $\mathbb{R}$ with $\kappa(0)=0$ satisfying

$$y=\kappa(x) \text{ is Lipschitz in } \mathbb{R} \text{and smooth for any } x\neq 0,$$

and

$$y=\kappa(x) \text{ is strictly decreasing for } x<0 \text{ and strictly increasing for } x>0.$$

Hence for any $\tau>0$, $\kappa(x)=\tau$ has two roots, one positive and one negative. An important example of such a function is given by $y=\kappa|x|$ for a positive constant κ.

For a fixed positive constant y_0, we set

$$\Omega_{\kappa,y_0}=\{(x,y);\kappa(x)<y<y_0\}.$$

For brevity, we simply write Ω instead of Ω_{κ,y_0}. We denote by $\partial_b\Omega$ and $\partial_t\Omega$ the bottom and top boundaries of Ω, i.e.,

$$\partial_b\Omega=\{(x,y);\ y=\kappa(x)<y_0\},\quad \partial_t\Omega=\{(x,y);\ \kappa(x)<y_0, y=y_0\}.$$

In the following, we consider

$$Lu\equiv u_{yy}-(au_x)_x+b_1u_x+b_2u_y+cu=f \quad \text{in } \Omega, \tag{3.1}$$

where a, b_1, b_2 and c are smooth functions in Ω satisfying

$$a\geq a_0 \quad \text{in } \Omega, \tag{3.2}$$

for a positive constant a_0. Obviously, $y=\kappa(x)$ is space-like if

$$a\kappa_x^2\leq \eta_0 \quad \text{on } \partial_b\Omega, \tag{3.3}$$

for a constant $\eta_0\in(0,1)$. Our goal is to prove that the Cauchy problem of (3.1) in Ω is well-posed for Cauchy data prescribed on $\partial_b\Omega$. We point out that $\partial_b\Omega$, as an initial curve, is not smooth and has an angular point.

For any nonnegative integers $m\geq l$, we define $H^{(m,l)}(\Omega)$ $(H_{0b}^{(m,l)}(\Omega))$ to be the closure of all $C^\infty(\Omega)$ functions (which vanish to all orders at $\partial_b\Omega$), in the norm

$$\| u \|_{(m,l)}^2=\int_\Omega\sum_{j=0}^{l}\sum_{i=0}^{m-j}(\partial_x^i\partial_y^j u)^2.$$

Obviously, the usual Sobolev space $H^m(\Omega)$ is a subset of $H^{(m,l)}(\Omega)$. The $L^2(\Omega)$ inner product will as usual be denoted by $(\cdot,\cdot)$. A simple calculation shows that the formal adjoint L^* of L is given by

$$L^*u=u_{yy}-(au_x)_x-(b_1u)_x-(b_2u)_y+cu.$$

It is convenient to first establish an existence result for (3.1) with homogeneous Cauchy data and with f vanishing to high order on $\partial_b\Omega$.

Lemma 3.1 *Let m be a positive integer, $a\in C^{m+1}(\bar{\Omega})$ and $b_1,b_2,c\in C^m(\bar{\Omega})$ satisfying (3.2) and (3.3). Then for any $f\in H_{0b}^{(m,0)}(\Omega)$, there exists a $u\in H_{0b}^{(m+1,1)}(\Omega)$ satisfying*

$$(u,L^*v)=(f,v) \quad \textit{for any } v\in C^\infty(\Omega) \textit{ with } v=v_y=0 \textit{ on } \partial_t\Omega. \tag{3.4}$$

We note that (3.3) holds automatically for arbitrary nonnegative a if $\partial_b\Omega$ is a horizontal line, i.e., $\kappa\equiv 0$. It is clear from the proof that η_0 in (3.3) is allowed to be 1 in Lemma 3.1.

Proof Let $\widehat{C}^{\infty}(\Omega)$ consist of all $C^{\infty}(\Omega)$ functions v with $v = v_y = 0$ on $\partial_t\Omega$. We consider a fixed $v \in \widehat{C}^{\infty}(\Omega)$. For a large constant λ to be determined, we consider

$$\begin{aligned} \sum_{s=0}^{m}(-1)^s\lambda^{-s}\partial_x^{2s}\varphi &= v \quad \text{in } \Omega, \\ \varphi = \cdots = \partial_x^{m-1}\varphi = 0 &\quad \text{on } \partial_b\Omega. \end{aligned}$$

This is a boundary value problem related to an ODE for each $y \in (0, y_0)$, and therefore the theory of such equations guarantees the existence of a unique solution $\varphi \in C^{2m}(\Omega)$. Set

$$w(x, y) = \int_{\kappa|x|}^{y} e^{\lambda t}\varphi(x, t)dt.$$

Then w satisfies

$$\begin{aligned} \sum_{s=0}^{m}(-1)^s\lambda^{-s}\partial_x^{2s}(e^{-\lambda y}w_y) = v \quad \text{in } \Omega, \\ w = w_y = \partial_x w_y = \cdots = \partial_x^{m-1}w_y = 0 \quad \text{on } \partial_b\Omega. \end{aligned} \tag{3.5}$$

We note that w satisfies extra boundary conditions

$$\partial_x^i\partial_y^j w|_{\partial_b\Omega} = 0, \quad \text{for } i + j \leq m. \tag{3.6}$$

To see this, we simply differentiate $w = 0$ along $\partial_b\Omega$ to get

$$n_2 w_x - n_1 w_y = 0 \quad \text{on } \partial_b\Omega, \tag{3.7}$$

where (n_1, n_2) is the outward unit normal vector of $\partial_b\Omega$. With $w_y = 0$ on $\partial_b\Omega$, we get easily $w_x = 0$ on $\partial_b\Omega$. A simple induction argument then yields (3.6). We note that

$$(n_1, n_2) = \frac{1}{\sqrt{1+\kappa_x^2}}(\kappa_x, -1).$$

By taking λ sufficiently large, we claim

$$\left(Lw, \sum_{s=0}^{m}(-1)^s\lambda^{-s}\partial_x^{2s}(e^{-\lambda y}w_y)\right) \geq C\|w\|_{(m+1,1)}^2, \tag{3.8}$$

where C is a positive constant depending on m, a_0, the C^{m+1}-norm of a and the C^m-norms of b_1, b_2 and c. To prove this, we integrate by parts each term in the left hand side of (3.8) repeatedly with the help of (3.6). First for $1 \leq s \leq m$, we have

$$\left(w_{yy} - (aw_x)_x, (-1)^s e^{-\lambda y}\partial_x^{2s} w_y\right)$$
$$= \int_\Omega e^{-\lambda y}\partial_x^s\partial_y^2 w\partial_x^s\partial_y w + \int_\Omega e^{-\lambda y}\partial_x^{s+1} w_y\partial_x^s(aw_x) - \delta_{sm}\int_{\partial\Omega} e^{-\lambda y}\partial_x^{s-1}\partial_{yy} w\partial_x^s\partial_y wn_1$$
$$= \int_\Omega e^{-\lambda y}\left(\frac{1}{2}\lambda(\partial_x^s\partial_y w)^2 + \frac{1}{2}(\lambda a - a_y)(\partial_x^{s+1}w)^2 - \partial_x^s\partial_y w\partial_x\left(\sum_{i=1}^s C_s^i\partial_x^i a\partial_x^{s+1-i}w\right)\right)$$
$$+ \int_{\partial\Omega} e^{-\lambda y}\left(\frac{1}{2}a(\partial_x^{s+1}w)^2 n_2 + \frac{1}{2}(\partial_x^s\partial_y w)^2 n_2 - \delta_{sm}\partial_x^{s-1}\partial_{yy}w\partial_x^s\partial_y wn_1\right),$$

and

$$\left(b_1 w_x + b_2 w_2 + cw, (-1)^s e^{-\lambda y}\partial_x^{2s} w_y\right)$$
$$= \int_\Omega e^{-\lambda y}\partial_x^s(b_1 w_x + b_2 w_y + cw)\partial_x^s\partial_y w.$$

For $s = 0$, we simply have

$$(Lw, e^{-\lambda y}w) = \int_\Omega e^{-\lambda y}\left(\frac{1}{2}w_y^2 + \frac{1}{2}(\lambda a - a_y)w_x^2 + (b_1 w_x + b_2 w_y + cw)w_y\right).$$

By taking λ large enough and using a Poincaré type inequality to estimate $\|w\|_{L^2(\Omega)}$, we obtain

$$\left(Lw, \sum_{s=0}^m (-1)^s\lambda^{-s}e^{-\lambda y}\partial_x^{2s}w_y\right) \ge C\|w\|^2_{(m+1,1)}$$
$$+\lambda^{-m}\int_{\partial\Omega} e^{-\lambda y}\left(\frac{1}{2}a(\partial_x^{m+1}w)^2 n_2 + \frac{1}{2}(\partial_x^m\partial_y w)^2 n_2 - \partial_x^{m-1}\partial_{yy}w\partial_x^m\partial_y wn_1\right),$$

where λ and C are positive constants depending on m, a_0, the C^{m+1}-norm of a and the C^m-norms of b_1, b_2 and c. Note that the boundary integral is nonnegative on $\partial_t\Omega$. We now study the boundary integral on $\partial_b\Omega$. We first note $\partial_x^{m-1}\partial_y w = \partial_x^m w = 0$ on $\partial_b\Omega$ by (3.6). Then by an argument as similar as in proving (3.7), we have

$$n_1\partial_x^{m-1}\partial_{yy}w - n_2\partial_x^m\partial_y w = 0, \quad n_1\partial_x^m\partial_y w - n_2\partial_x^{m+1}w = 0 \quad \text{on } \partial_b\Omega.$$

It follows that the boundary integral on $\partial_b\Omega$ is given by

$$\int_{\partial\Omega} e^{-\lambda y}\left(\frac{1}{2}a(\partial_x^{m+1}w)^2 n_2 + \frac{1}{2}(\partial_x^m\partial_y w)^2 n_2 - \partial_x^{m-1}\partial_{yy}w\partial_x^m\partial_y wn_1\right)$$
$$= \frac{1}{2}\int_{\partial\Omega} e^{-\lambda y}\left(a\frac{n_1^2}{n_2^2} - 1\right)(\partial_x^m\partial_y w)^2 n_2 = \frac{1}{2}\int_{\partial\Omega} e^{-\lambda y}\left(a\kappa_x^2 - 1\right)(\partial_x^m\partial_y w)^2 n_2.$$

This is nonnegative by (3.3) and $n_2 < 0$ on $\partial_b\Omega$. Then (3.8) holds.

Next we claim

$$\|v\|_{(-m,0)} \le C\|w\|_{(m+1,1)}. \tag{3.9}$$

Here $\|\cdot\|_{(-m,0)}$ is the norm on the dual space $H_{0b}^{(-m,0)}(\Omega)$ of $H_{0b}^{(m,0)}(\Omega)$. This dual space may be obtained as the completion of $L^2(\Omega)$ in the norm $\|\cdot\|_{(-m,0)}$. To get (3.9), we simply note

$$\|v\|_{(-m,0)} = \sup_{z\in H_{0b}^{(m,0)}(\Omega)} \frac{|(v,z)|}{\|z\|_{(m,0)}}$$
$$= \sup_{z\in H_{0b}^{(m,0)}(\Omega)} \frac{\left|\left(\sum_{s=0}^m (-1)^s\lambda^{-s}\partial_x^{2s}(e^{-\lambda y}w_y), z\right)\right|}{\|z\|_{(m,0)}} \le C\|w\|_{(m+1,1)}.$$

Now, a simple integration by parts yields

$$(w, L^*v) = (Lw, v) \quad \text{for any } v \in \widehat{C}^\infty(\Omega).$$

By (3.8), we obtain

$$\|w\|_{(m+1,1)}\|L^*v\|_{(-m-1,-1)} \ge (w, L^*v) = (Lw, v)$$
$$= \left(Lw, \sum_{s=0}^m (-1)^s\lambda^{-s}\partial_x^{2s}(e^{-\lambda y}w_y)\right) \ge C\|w\|_{(m+1,1)}^2,$$

and hence with (3.9)

$$\|v\|_{(-m,0)} \le C\|L^*v\|_{(-m-1,-1)} \quad \text{for any } v \in \widehat{C}^\infty(\Omega). \tag{3.10}$$

Consider the linear functional $F : L^*\widehat{C}^\infty(\Omega) \to \mathbb{R}$ given by

$$F(L^*v) = (f, v).$$

By (3.10), we have

$$|F(L^*v)| \le \|f\|_{(m,0)}\|v\|_{(-m,0)} \le C\|f\|_{(m,0)}\|L^*v\|_{(-m-1,-1)}.$$

Hence F is a bounded linear functional on the subspace $L^*\widehat{C}^\infty(\Omega)$ of $H_{0b}^{(-m-1,-1)}(\Omega)$. Thus we can apply the Hahn-Banach Theorem to obtain a bounded extension of F defined on $H_{0b}^{(-m-1,-1)}(\Omega)$ such that $\|F\| \le C\|f\|_{(m,0)}$. It follows that there exists a $u \in H_{0b}^{(m+1,1)}(\Omega)$ such that

$$F(z) = (u, z) \quad \text{for any } z \in H_{0b}^{(-m-1,-1)}(\Omega).$$

Now restrict z back to $L^*\widehat{C}^\infty(\Omega)$ to obtain (3.4). □

Next, we discuss the regularity of solutions in Lemma 3.1 in usual Sobolev spaces. The Sobolev space of square integrable derivatives up to and including order m will be denoted by $H^m(\Omega)$ with norm $\|\cdot\|_m$, and the completion of $C^\infty(\Omega)$ functions which vanish to all order at $\partial_b\Omega$ in the norm $\|\cdot\|_m$ will be denoted by $H_{0b}^m(\Omega)$.

Corollary 3.2 *Under the hypotheses of Lemma 3.1, if $f \in H_{0b}^m(\Omega)$, there exists a unique solution $u \in H_{0b}^{m+1}(\Omega)$ of (3.1).*

Proof Obviously, $H_{0b}^m(\Omega) \subset H_{0b}^{(m,0)}(\Omega)$. Let $u \in H_{0b}^{(m+1,1)}(\Omega)$ be the function given in Lemma 3.1 so that (3.4) holds.

We first consider $m = 1$. By $u \in H_{0b}^{(2,1)}(\Omega)$, we have $u, u_x \in H^1(\Omega)$ with $u = u_x = 0$ on $\partial_b\Omega$ in the $L^2(\partial_b\Omega)$ sense. We integrate by parts to obtain

$$-(u_y + b_2u, v_y) = \left(f + (au_x)_x - b_1u_x + (b_{2,y} - c)u, v\right) \quad \text{for any } v \in \widehat{C}^\infty(\Omega). \tag{3.11}$$

A standard argument using difference quotients in the y-direction implies $(u_y + b_2u)_y \in L^2_{\text{loc}}(\Omega)$ and

$$(u_y + b_2u)_y = f + (au_x)_x - b_1u_x + (b_{2,y} - c)u.$$

Then $u_{yy} \in L^2_{\text{loc}}(\Omega)$ and

$$u_{yy} - (au_x)_x + b_1u_x + b_2u_y + cu = f \quad \text{in } \Omega.$$

This implies easily that $u_{yy} \in L^2(\Omega)$ and hence $u \in H^2(\Omega)$. An integration by parts of (3.11) then yields $u_y = 0$ on $\partial_b\Omega$ in the $L^2(\partial_b\Omega)$ sense. Last, by $u = |\nabla u| = 0$ on $\partial_b\Omega = 0$, (3.8) with $m = 1$ yields

$$\left(Lu, \sum_{s=0}^{1}(-1)^s\lambda^{-s}\partial_x^{2s}(e^{-\lambda y}u_y)\right) \geq C\|u\|^2_{(2,1)},$$

from which the uniqueness follows.

Now we assume $m \geq 2$. We already proved that $u \in H^2(\Omega)$ and that (3.4) holds. We need to prove

$$\partial_x^i\partial_y^j u \in L^2(\Omega) \quad \text{for any } i + j \leq m + 1,$$

and

$$\partial_x^i\partial_y^j u|_{\partial_b\Omega} = 0 \quad \text{for any } i + j \leq m.$$

This follows easily from (3.1), $u \in H_{0b}^{(m+1,1)}(\Omega)$ and $f \in H_{0b}^m(\Omega)$. □

Corollary 3.2 yields the existence of a regular solution of (3.1) for homogeneous Cauchy data and f vanishing to high order on $\partial_b\Omega$. However, our main concern is to solve (3.1) for general f and Cauchy data

$$u = \varphi, \quad u_y = \psi \quad \text{on } \partial_b\Omega. \tag{3.12}$$

Since $\partial_b\Omega$ has an angular point at the origin, there is a natural compatibility condition which we will derive next. As $\partial_b\Omega$ is the graph given by $y = \kappa(x)$ over $\mathbb{R}$, we may assume φ and ψ are functions of $x \in \mathbb{R}$.

Lemma 3.3 *Let $m \geq 2$ be an integer and $\varphi \in C(\partial_b\Omega) \cap C^m(\partial_b\Omega \setminus \{0\})$, $\psi \in C(\partial_b\Omega) \cap C^{m-1}(\partial_b\Omega\setminus\{0\})$ and $f \in C^{m-2}(\bar{\Omega})$. Suppose (3.3) is satisfied. Then there exists a $u \in C^m(\bar{\Omega})$ such that*

$$u = \varphi, \; u_y = \psi, \; \text{and } \partial^\alpha(Lu - f) = 0 \quad \text{on } \partial_b\Omega, \tag{3.13}$$

for any $|\alpha| \leq m - 2$ if and only if there hold compatibility conditions $\mathcal{C}_i(\varphi, \psi, f)$ for $i = 1, \ldots, m$.

The compatibility condition $\mathcal{C}_i(\varphi, \psi, f)$ is imposed on (one-sided) derivatives of φ, ψ, f and κ up to order i at the origin. The formulation of such a condition will be given in the proof below, from which it is clear that $\mathcal{C}_m(\varphi, \psi, f)$ makes sense for $\varphi \in C(\partial_b\Omega) \cap C^m(\partial_b\Omega \setminus \{0\})$, $\psi \in C(\partial_b\Omega) \cap C^{m-1}(\partial_b\Omega \setminus \{0\})$ and $f \in C^{m-2}(\bar{\Omega})$.

Proof First, we assume there exists a function $u \in C^1(\bar{\Omega})$ satisfying (3.12). Then a simple differentiation yields

$$u_x + \kappa_x u_y = \varphi_x \quad \text{on } \partial_b\Omega \setminus \{0\},$$

or

$$u_x = \varphi_x - \kappa_x \psi \quad \text{on } \partial_b\Omega \setminus \{0\}.$$

Letting $x \to 0+$ and $x \to 0-$, we have a compatibility condition

$$\varphi_x(0+) - \kappa_x(0+)\psi(0) = \varphi_x(0-) - \kappa_x(0-)\psi(0),$$

or

$$\psi(0)\big(\kappa_x(0+) - \kappa_x(0-)\big) = \varphi_x(0+) - \varphi_x(0-). \tag{3.14}$$

If (3.14) holds, then

$$u_x(0) = -\frac{\kappa_x(0+)\varphi_x(0-) + \kappa_x(0-)\varphi_x(0+)}{\kappa_x(0+) - \kappa_x(0-)}.$$

It is easy to check that for any $\varphi \in C(\partial_b\Omega) \cap C^1(\partial_b\Omega \setminus \{0\})$ and $\psi \in C(\partial_b\Omega)$ satisfying (3.14), there exists a $u \in C^1(\bar{\Omega})$ satisfying (3.12). We denote by $\mathcal{C}_1(\varphi, \psi, f)$ the compatibility condition (3.14), which in fact is independent of f.

The discussion for higher order derivatives is more complicated. For an integer $m \geq 2$, we assume we already derived $\mathcal{C}_i(\varphi, \psi, f)$ for $i = 1, \ldots, m-1$. Now let $u \in C^m(\bar{\Omega})$ satisfy (3.13). For any multi-index $\alpha \in \mathbb{Z}_+^2$ with $|\alpha| = m - 2$, a simple calculation yields

$$\partial_x^{\alpha_1}\partial_y^{\alpha_2} u(p) = \begin{cases} a^{\frac{\alpha_2}{2}}(p)\partial_x^m u(p) + \cdots & \text{if } \alpha_2 \text{ is even,} \\ a^{\frac{\alpha_2-1}{2}}(p)\partial_x^{m-1}\partial_y u(p) + \cdots & \text{if } \alpha_2 \text{ is odd,} \end{cases}$$

where $\cdots$ denotes a linear combination of derivatives of u at p with order $\leq m-1$ and derivatives of f at p with order $\leq m-2$. Now we apply ∂_x^m to $u = \varphi$ and ∂_x^{m-1} to $u_y = \psi$ and evaluate at $p \in \partial_b\Omega \setminus \{0\}$. Then we get on $\partial_b\Omega \setminus \{0\}$

$$\sum_{i=0}^{m} C_m^i \kappa_x^i \partial_x^{m-i}\partial_y^i u|_p = \varphi^{(m)} + \cdots, \ \textstyle\sum_{i=0}^{m-1} C_{m-1}^i \kappa_x^i \partial_x^{m-1-i}\partial_y^{i+1} u|_p = \psi^{(m-1)} + \cdots,$$

where $\cdots$ denotes derivatives of u at p with order $\leq m-1$. By a simple substitution of $\partial^\alpha u(p)$ with $\alpha_2 \geq 2$, we obtain at $p \in \partial_b\Omega \setminus \{0\}$

$$\begin{aligned} &\left(\sum_{0\leq 2i\leq m} C_m^{2i}\kappa_x^{2i}a^i\right)\partial_x^m u + \left(\sum_{0\leq 2i+1\leq m} C_m^{2i+1}\kappa_x^{2i+1}a^i\right)\partial_x^{m-1}\partial_y u \\ &= \varphi^{(m)} + \cdots, \ \left(\sum_{0\leq 2i+1\leq m-1} C_{m-1}^{2i+1}\kappa_x^{2i+1}a^{i+1}\right)\partial_x^m u \\ &+ \left(\sum_{0\leq 2i\leq m-1} C_{m-1}^{2i}\kappa_x^{2i}a^i\right)\partial_x^{m-1}\partial_y u = \psi^{(m-1)} + \cdots, \end{aligned}$$

where $\cdots$ denotes a linear combination of derivatives of u at p with order $\leq m-1$ and derivatives of f at p with order $\leq m-2$. This is a 2×2 linear system for $\partial_x^m u(p)$ and $\partial_x^{m-1}\partial_y u(p)$. A straightforward calculation shows that the determinate of the coefficient matrix is given by

$$(1 - a\kappa_x^2)^{m-1}|_p,$$

which is nonzero by (3.3). This implies that $\partial_x^m u$ and $\partial_x^{m-1}\partial_y u$, and hence all other m-th order derivatives of u, at $p \in \partial_b\Omega \setminus \{0\}$, can be expressed as a linear combination of $\varphi^{(m)}(p)$, $\psi^{(m-1)}(p)$, derivatives of u at p with order $\leq m-1$ and derivatives of f at p with order $\leq m-2$. Now we consider $p=0$. In this case, there are four linear equations for $\partial_x^m u(0)$ and $\partial_x^{m-1}\partial_y u(0)$ arising from $x \to 0+$ and $x \to 0-$. This implies that there are two compatibility conditions similar to (3.14) involving $\varphi^{(m)}(0+)$, $\varphi^{(m)}(0-)$, $\psi^{(m-1)}(0+)$, $\psi^{(m-1)}(0-)$, $\kappa^{(i)}(0+)$ and $\kappa^{(i)}(0-)$, $i=1,\dots,m$. We denote by $\mathcal{C}_m(\varphi,\psi,f)$ this compatibility condition. If $\mathcal{C}_m(\varphi,\psi,f)$ is satisfied, then $\partial_x^m u(0)$ and $\partial_x^{m-1}\partial_y u(0)$, and hence all other m-th order derivatives of u at 0, can be expressed as a linear combination of $\varphi^{(m)}(0+)$, $\varphi^{(m)}(0-)$, $\psi^{(m-1)}(0+)$, $\psi^{(m-1)}(0-)$, derivatives of u at 0 with order $\leq m-1$ and derivatives of f at 0 with order $\leq m-2$. □

Now we are ready to solve the Cauchy problem (3.1) and (3.12).

Theorem 3.4 *Let $m \geq 2$ be an integer and $\varphi \in H^{m+1}(\partial_b\Omega)$, $\psi \in H^m(\partial_b\Omega)$ and $f \in H^m(\Omega)$. Suppose (3.2), (3.3) and $\mathcal{C}_i(\varphi,\psi,f)$, $i=1,\dots,m$, are satisfied. Then the Cauchy problem (3.1) and (3.12) admits a unique solution $u \in H^m(\Omega)$. Moreover,*

$$\|u\|_{m,\Omega} \leq C(\|\varphi\|_{m+1,\partial_b\Omega} + \|\psi\|_{m,\partial_b\Omega} + \|f\|_{m,\Omega}), \tag{3.15}$$

where C is a positive constant depending only on m, a_0, η_0, the C^{m+1}-norm of a and the C^m-norms of b_1, b_2 and c.

Here and thereafter, we denote by $\|\cdot\|_{m,\Omega}$ and $\|\cdot\|_{m,\partial_b\Omega}$ the H^m-norms in Ω and $\partial_b\Omega$ respectively.

Proof By the Sobolev embedding, we have $\varphi \in C(\partial_b\Omega) \cap C^m(\partial_b\Omega \setminus \{0\})$, $\psi \in C(\partial_b\Omega) \cap C^{m-1}(\partial_b\Omega \setminus \{0\})$ and $f \in C^{m-2}(\Omega)$. Hence the compatibility condition $\mathcal{C}_i(\varphi,\psi,f)$ makes sense for $i=1,\dots,m$. By Lemma 3.3, there exists a $v \in C^m(\bar{\Omega})$ such that $v=\varphi$, $v_y=\psi$ and $\partial^\alpha(f-Lv)=0$ on $\partial_b\Omega$ for any $|\alpha| \leq m-2$. This implies $f-Lv \in H_{0b}^{m-1}(\Omega)$. By Corollary 3.2, there exists a $w \in H_{0b}^m(\Omega)$ such that $Lw = f - Lv$. Then $u = v+w$ is a required solution.

We note that (3.15) is the classical energy estimates. The proof is identical to that for Cauchy problems with the initial curve as the x-axis. For example, the H^1-estimate is based on integrating the product of (3.1) and u_y. We omit details. □

In this paper, we only need the existence part in Theorem 3.4. The estimate (3.15) depends on the lower bound a_0 of a and is not sufficient for our application. In the next section, we will derive an estimate independent of a_0 under extra assumptions on a.

4 A priori estimates in the hyperbolic regions

In this section, we will derive estimates of the solutions established in the previous section which are independent of the hyperbolicity constant. Such estimates will enable us to establish the existence of solutions to the Cauchy problem for degenerate hyperbolic equations when the initial curve has angular points.

Let $y=\kappa(x)$ and $\Omega \subset \mathbb{R}^2$ be as defined in the beginning of Sect. 3. We consider an equation of the following form

$$Lu \equiv u_{yy} - aKu_{xx} + b_1u_x + b_2u_y + cu = f \quad \text{in } \Omega \tag{4.1}$$

with the Cauchy data

$$u=\varphi,\ u_y=\psi \quad \text{on } \partial_b\Omega, \tag{4.2}$$

where a, b_1, b_2, c and K are smooth functions satisfying

$$\lambda \le a \le \Lambda \quad \text{in } \Omega, \tag{4.3}$$

$$0 < K \le 1 \quad \text{in } \Omega, \tag{4.4}$$

and

$$|b_1| \le C_b\big(\sqrt{K}+|K_x|\big) \quad \text{in } \Omega, \tag{4.5}$$

for positive constants $\lambda \le \Lambda$ and C_b. We always assume that $\partial_b\Omega$ is space-like, i.e.,

$$aK\kappa_x^2 \le \eta_0, \tag{4.6}$$

for a constant $\eta_0 \in (0, 1)$. In the following, we also assume

$$K_x^2 \le C_K^2 K_y \quad \text{in } \Omega, \tag{4.7}$$

and

$$\big(y-\kappa(x)\big)^d \le C_K K(x, y) \quad \text{for any } (x, y) \in \Omega, \tag{4.8}$$

where C_K is a positive constant and d is a positive integer. Note that (4.7) implies in particular $K_y \ge 0$.

Here, K is allowed to be zero along $\partial_b\Omega$. If this happens, (4.1) is degenerate there and (4.6) holds automatically. Conditions (4.5), (4.7) and (4.8) are introduced to overcome the degeneracy. The condition (4.8) of the finite degree degeneracy is essential in our arguments. It is not clear whether results in this section still hold without this assumption.

An example of Ω and K is given by

$$\Omega = \{(x, y);\ |x| < y < 1\},$$

and

$$K(x, y) = y^2 - x^2.$$

Obviously, (4.7) and (4.8) are satisfied for $\kappa(x) = |x|$ and $d = 2$.

Our intention is to derive energy estimates. We first derive an estimate on H^1-norms.

Lemma 4.1 *Let a, b_1, b_2, c and K be C^d-functions in $\bar{\Omega}$ satisfying (4.3)–(4.8) and u be an H^{d+3}-solution of (4.1)–(4.2) for $\varphi \in H^{d+2}(\partial_b\Omega)$, $\psi \in H^{d+1}(\partial_b\Omega)$ and $f \in H^{d+1}(\Omega)$. Then*

$$\|u\|_{1,\Omega} \le C\big(\|\varphi\|_{d+2,\partial_b\Omega} + \|\psi\|_{d+1,\partial_b\Omega} + \|f\|_{d+1,\Omega}\big), \tag{4.9}$$

where C is a positive constant depending on λ, Λ, C_b, C_K, η_0 and the C^d-norms of a, b_1, b_2, c and K.

We note that (4.9) exhibits a loss of derivatives and such a loss depends on the degree to which coefficients degenerate along the boundary.

Proof Multiplying $2e^{-\mu y}u_y/K$ to (4.1), we get

$$\partial_y\left(e^{-\mu y}\left(\frac{u_y^2}{K}+au_x^2\right)\right)-2\partial_x(e^{-\mu y}au_xu_y)+e^{-\mu y}\left(\mu+\frac{K_y}{K}\right)\frac{u_y^2}{K}+e^{-\mu y}\left(\mu-\frac{a_y}{a}\right)au_x^2$$
$$=-2e^{-\mu y}a_xu_xu_y-2e^{-\mu y}b_2\frac{u_y^2}{K}-2e^{-\mu y}b_1\frac{u_xu_y}{K}-2e^{-\mu y}c\frac{uu_y}{K}+2e^{-\mu y}\frac{u_yf}{K}.$$

By combining with

$$\partial_y\left(e^{-\mu y}\frac{u^2}{K}\right)+e^{-\mu y}\left(\mu+\frac{K_y}{K}\right)\frac{u^2}{K}=2e^{-\mu y}\frac{uu_y}{K},$$

we obtain

$$\begin{aligned}
&\partial_y\left(e^{-\mu y}\left(\frac{u^2}{K}+\frac{u_y^2}{K}+au_x^2\right)\right)-2\partial_x(e^{-\mu y}au_xu_y)\\
&\quad+e^{-\mu y}\left(\mu+\frac{K_y}{K}\right)\left(\frac{u^2}{K}+\frac{u_y^2}{K}\right)+e^{-\mu y}\left(\mu-\frac{a_y}{a}\right)au_x^2\\
&=-2e^{-\mu y}a_xu_xu_y-2e^{-\mu y}\frac{b_1}{K}u_xu_y\\
&\quad-2e^{-\mu y}b_2\frac{u_y^2}{K}+2e^{-\mu y}(1-c)\frac{uu_y}{K}+2e^{-\mu y}\frac{u_yf}{K}.
\end{aligned}\tag{4.10}$$

We point out again that $\partial_yK\geq 0$ by (4.7). We first consider the second term in the right hand side. By (4.5) and (4.7), we have

$$\frac{|b_1|}{\sqrt{K}}\leq C_b\left(C_K\sqrt{\frac{K_y}{K}}+1\right).$$

By the Cauchy inequality, we get

$$\begin{aligned}
\left|2e^{-\mu y}\frac{b_1}{K}u_xu_y\right|&=\left|2e^{-\mu y}\frac{b_1}{\sqrt{K}}\cdot\frac{u_y}{\sqrt{K}}u_x\right|\leq e^{-\mu y}\left(\varepsilon\frac{b_1^2}{K}\cdot\frac{u_y^2}{K}+\frac{1}{\varepsilon}u_x^2\right)\\
&\leq e^{-\mu y}\left(2\varepsilon C_b^2(C_K^2\frac{K_y}{K}+1)\cdot\frac{u_y^2}{K}+\frac{1}{\varepsilon}u_x^2\right),
\end{aligned}$$

for any $\varepsilon>0$. By choosing $\varepsilon>0$ small enough and applying the Cauchy inequality to other terms in the right hand side of (4.10), we obtain

$$\begin{aligned}
&\partial_y\left(e^{-\mu y}(\frac{u^2}{K}+\frac{u_y^2}{K}+au_x^2)\right)-2\partial_x(e^{-\mu y}au_xu_y)\\
&\quad+(\mu-\mu_0)e^{-\mu y}\left(\frac{u^2}{K}+\frac{u_y^2}{K}+au_x^2\right)\leq e^{-\mu y}\frac{f^2}{K},
\end{aligned}$$

where μ_0 is a positive constant depending only on $\inf a$, $|a|_{C^1}$, $|b_2|_{L^\infty}$, $|c|_{L^\infty}$, C_b and C_K. By a simple integration, we have

$$(\mu-\mu_0)\int_\Omega e^{-\mu y}\Big(\frac{u^2}{K}+\frac{u_y^2}{K}+au_x^2\Big)\le\int_\Omega e^{-\mu y}\frac{f^2}{K}$$
$$+\int_{\partial_b\Omega}\frac{e^{-\mu y}}{\sqrt{1+\kappa_x^2}}\Big(\frac{u^2}{K}+\frac{u_y^2}{K}+au_x^2+2a\kappa_x u_x u_y\Big),$$

where the integral over $\partial_t\Omega$, having the correct sign, is already dropped. By the Cauchy inequality and (4.6), we get

$$2a|\kappa_x u_x u_y|\le\frac{u_y^2}{K}+aK\kappa_x^2\cdot au_x^2\le\frac{u_y^2}{K}+au_x^2.$$

Therefore, by (4.3) and taking μ large enough, we obtain

$$\int_\Omega\left(\frac{u^2}{K}+\frac{u_t^2}{K}+u_x^2\right)\le C\int_{\partial_b\Omega}\frac{1}{\sqrt{1+\kappa_x^2}}\left(\frac{u^2}{K}+\frac{u_y^2}{K}+u_x^2\right)+\int_\Omega\frac{f^2}{K}. \tag{4.11}$$

We should note that the boundary integral in the right hand side of (4.11) makes sense only when $u=u_y=0$ on $\partial_b\Omega$ if $K=0$ on $\partial_b\Omega$.

To eliminate $1/K$ from the right-hand side of (4.11), we introduce an auxiliary function. It is easy to see that there exists a $v\in H^{d+2}(\Omega)$ such that

$$D^\alpha v=D^\alpha u\quad\text{on }\partial_b\Omega\text{ for any }|\alpha|\le d+1,$$

and

$$\|v\|_{d+2,\Omega}\le C\sum_{|\alpha|\le d+2}\|D^\alpha u\|_{0,\partial_b\Omega}. \tag{4.12}$$

Obviously, v satisfies

$$v=\varphi,\ v_y=\psi\quad\text{on }\partial_b\Omega,$$

and

$$\partial_y^i(f-Lv)=0\quad\text{on }\partial_b\Omega,\ \text{for any }i=0,1,\ldots,d-1. \tag{4.13}$$

Then we have

$$L(u-v)=f-Lv\quad\text{in }\Omega,$$
$$u-v=0,\ (u-v)_y=0\quad\text{on }\partial_b\Omega.$$

By applying (4.11) to $u-v$, we obtain

$$\int_\Omega\left(\frac{(u-v)^2}{K}+\frac{(u_y-v_y)^2}{K}+(u_x-v_x)^2\right)\le C\int_\Omega\frac{(f-Lv)^2}{K}. \tag{4.14}$$

With (4.4), we have

$$\int_\Omega\left(u^2+u_y^2+u_x^2\right)\le C\int_\Omega\left(v^2+v_y^2+v_x^2\right)+C\int_\Omega\frac{(f-Lv)^2}{K}. \tag{4.15}$$

Next, we eliminate the factor $1/K$ in the last integral in (4.15). With (4.13) and (4.8), a simple calculation yields

$$\big((f-Lv)(x,y)\big)^2 \le \big(y-\kappa(x)\big)^d \int_{\kappa(x)}^{y} \big(\partial_y^d (f-Lv)(x,t)\big)^2 dt \quad \text{for any}(x,y)\in\Omega,$$

and

$$\int_\Omega \frac{(f-Lv)^2}{K} \le C\int_\Omega \big(\partial_y^d (f-Lv)(x,s)\big)^2 \le C\big(\|f\|_{d,\Omega}+\|v\|_{d+2,\Omega}\big)^2.$$

Hence, we obtain

$$\|u\|_{1,\Omega} \le C\|v\|_{d+2,\Omega} + C\|f\|_{d,\Omega}. \tag{4.16}$$

With the help of (4.12), (4.1) and the trace theorem, we get

$$\begin{aligned}\|v\|_{d+2,\Omega} &\le C\sum_{|\alpha|\le d+2}\|D^\alpha u\|_{0,\partial_b\Omega}\\ &\le C\big(\|\varphi\|_{d+2,\partial_b\Omega}+\|\psi\|_{d+1,\partial_b\Omega}+\|f\|_{d,\partial_b\Omega}\big)\\ &\le C\big(\|\varphi\|_{d+2,\partial_b\Omega}+\|\psi\|_{d+1,\partial_b\Omega}+\|f\|_{d+1,\Omega}\big),\end{aligned}$$

where C is a positive constant depending only on the C^d-norms of a, b_1, b_2, c and K. This implies (4.9) easily. □

Remark 4.2 It is clear that we have

$$\begin{aligned}&\int_\Omega \Big(\frac{(u-v)^2}{K}+\frac{(u_y-v_y)^2}{K}+(u_x-v_x)^2\Big)\\ &\quad\le C\big(\|\varphi\|_{d+2,\partial_b\Omega}+\|\psi\|_{d+1,\partial_b\Omega}+\|f\|_{d+1,\Omega}\big)^2.\end{aligned} \tag{4.17}$$

This will be used in the proof of Lemma 4.3 below.

Next, we derive estimates of derivatives of u.

Lemma 4.3 *For an integer $m \ge 1$, let a, b_1, b_2, c and K be C^{m+d-1}-functions in $\bar\Omega$ satisfying (4.3)–(4.8) and u be an H^{m+d+2}-solution of (4.1)–(4.2) for $\varphi \in H^{m+d+1}(\partial_b\Omega)$, $\psi \in H^{m+d}(\partial_b\Omega)$ and $f \in H^{m+d}(\Omega)$. Then*

$$\|u\|_{m,\Omega} \le C\big(\|\varphi\|_{m+d+1,\partial_b\Omega}+\|\psi\|_{m+d,\partial_b\Omega}+\|f\|_{m+d,\Omega}\big), \tag{4.18}$$

where C is a positive constant depending on λ, Λ, C_b, C_K, η_0 and the C^{m+d-1}-norms of a, b_1, b_2, c and K.

Proof We prove by induction. We note that Lemma 4.1 corresponds the case $m = 1$. Let s be a positive integer $\le m-1$. Apply ∂_x^s to (4.1) to get

$$L_s(\partial_x^s u) = f_s, \tag{4.19}$$

where

$$L_s = \partial_{yy} - aK\partial_{xx} + \big(b_1 - s(aK)_x\big)\partial_x + b_2\partial_y + \Big(c + sb_{1,x} - \frac{s(s-1)}{2}(aK)_{xx}\Big),$$

and

$$f_s = \partial_x^s f + \sum_{i=2}^{s-1} c'_{s,i}\partial_x^{s+2-i}(aK)\partial_x^i u + \sum_{i=1}^{s-1} c''_{s,i}\partial_x^{s+1-i} b_1 \partial_x^i u + \sum_{i=0}^{s-1} c'''_{s,i}\partial_x^{s-i} c\partial_x^i u + \sum_{i=0}^{s-1} c''''_{s,i}\partial_x^{s+2-i} b_2 \partial_x^i \partial_y u,$$

for some constants $c'_{s,i}$, $c''_{s,i}$, $c'''_{s,i}$ and $c''''_{s,i}$. We will write

$$f_s = \partial_x^s f + \sum_{i=0}^{s-1} \Gamma'_{si}\partial_x^i u + \sum_{i=0}^{s-1} \Gamma''_{si}\partial_x^i \partial_y u.$$

We should note that L_s has the same structure as L. As in the proof of Lemma 4.1, we construct a function $v_s \in H^{d+2}(\Omega)$ such that

$$D^\alpha v_s = D^\alpha(\partial_x^s u) \quad \text{on } \partial_b\Omega \text{ for any } |\alpha| \le d+1,$$

and

$$\|v_s\|_{d+2,\Omega} \le C \sum_{|\alpha|\le d+2} \|D^\alpha(\partial_x^s u)\|_{0,\partial_b\Omega}.$$

Similar to (4.14), we have

$$\int_\Omega \left(\frac{(\partial_x^s u - v_s)^2}{K} + \frac{(\partial_x^s \partial_y u - \partial_y v_s)^2}{K} + (\partial_x^{s+1} u - \partial_x v_s)^2 \right) \le C \int_\Omega \frac{(f_s - L_s v_s)^2}{K}, \tag{4.20}$$

where C is positive constant depending only on $\inf a$, $|a|_{C^2}$, $|K|_{C^2}$, $|b_1|_{C^1}$, $|b_2|_{C^1}$, $|c|_{L^\infty}$, C_b and C_K. We write

$$f_s - L_s v_s = \tilde{f}_s + \sum_{i=0}^{s-1} \Gamma'_{si}(\partial_x^i u - v_i) + \sum_{i=0}^{s-1} \Gamma''_{si}\partial_y(\partial_x^i u - v_i),$$

where $v_0, \ldots, v_{s-1}$ are constructed for $u, \ldots, \partial_x^{s-1} u$ as v_s for $\partial_x^s u$, and

$$\tilde{f}_s = (\partial_x^s f - L_s v_s) + \sum_{i=0}^{s-1} \Gamma'_{si} v_i + \sum_{i=0}^{s-1} \Gamma''_{si}\partial_y v_i.$$

This implies

$$\int_\Omega \frac{(f_s - L_s v_s)^2}{K} \le C \left(\int_\Omega \frac{\tilde{f}_s^2}{K} + \sum_{i=0}^{s-1} \int_\Omega \frac{(\partial_x^i u - v_i)^2}{K} + \sum_{i=0}^{s-1} \int_\Omega \frac{(\partial_x^i \partial_y u - \partial_y v_i)^2}{K} \right),$$

where C is a positive constant depending only on the C^s-norms of aK, b_1, b_2 and c. Note

$$\partial_y^i \tilde{f}_s = 0 \quad \text{on } \partial_b\Omega \text{ for} i = 0, \ldots, d-1.$$

Therefore, we get

$$\int_\Omega \frac{\tilde f_s^2}{K} \le \int_\Omega \left(\partial_y^d \tilde f\right)^2$$
$$\le \int_\Omega |\partial_y^d(\partial_x^s f - L_s v_s)|^2 + \sum_{i=0}^{s-1}\int_\Omega |\partial_y^d(\Gamma'_{si}v_i)|^2 + \sum_{i=0}^{s-1}\int_\Omega |\partial_y^d(\Gamma''_{si}\partial_y v_i)|^2$$
$$\le C\big(\|f\|^2_{s+d,\Omega} + \|v_s\|^2_{d+2,\Omega} + \sum_{i=0}^{s-1}\|v_i\|^2_{d+1,\Omega}\big),$$

where C is a positive constant depending only on the C^{s+d}-norms of aK, b_1, b_2 and c. For each $i = 0, \ldots, s$, we have

$$\|v_i\|_{d+2,\Omega} \le C \sum_{|\alpha|\le d+2} \|D^\alpha \partial_x^i u\|_{0,\partial_b\Omega}$$
$$\le C\big(\|\varphi\|_{i+d+2,\partial_b\Omega} + \|\psi\|_{i+d+1,\partial_b\Omega} + \|f\|_{i+d,\partial_b\Omega}\big)$$
$$\le C\big(\|\varphi\|_{i+d+2,\partial_b\Omega} + \|\psi\|_{i+d+1,\partial_b\Omega} + \|f\|_{i+d+1,\Omega}\big),$$

where C depends on the C^{i+d}-norms of a, b_1, b_2, c and K. In summary, we obtain

$$\int_\Omega \left(\frac{\left(\partial_x^s u - v_s\right)^2}{K} + \frac{\left(\partial_y\partial_x^s u - \partial_y v_s\right)^2}{K} + (\partial_x^{s+1}u - \partial_x v_s)^2\right)$$
$$\le C\left(\sum_{i=0}^{s-1}\int_\Omega \left(\frac{(\partial_x^i u - v_i)^2}{K} + \frac{\left(\partial_y\partial_x^i u - \partial_y v_i\right)^2}{K}\right)\right.$$
$$\left. + C\left(\|\varphi\|^2_{s+d+2,\partial_b\Omega} + \|\psi\|^2_{s+d+1,\partial_b\Omega} + \|f\|^2_{s+d+1,\Omega}\right)\right), \tag{4.21}$$

where C depends on the C^{s+d}-norms of a, b_1, b_2, c and K. By a simple induction starting from (4.17), we obtain

$$\int_\Omega \left((\partial_x^s u)^2 + (\partial_y\partial_x^s u)^2 + (\partial_x^{s+1}u)^2\right)$$
$$\le C\big(\|\varphi\|^2_{s+d+2,\partial_b\Omega} + \|\psi\|^2_{s+d+1,\partial_b\Omega} + \|f\|^2_{s+d+1,\Omega}\big).$$

All other derivatives of u of order $s+1$ can be obtained from (4.1). □

Now we prove the main result in this section.

Theorem 4.4 *For an integer $m \ge 2$, let a, b_1, b_2, c and K be C^{m+d-1}-functions in $\bar\Omega$ satisfying (4.3)–(4.8) and $\varphi \in H^{m+d+1}(\partial_b\Omega)$, $\psi \in H^{m+d}(\partial_b\Omega)$ and $f \in H^{m+d}(\Omega)$. If $\mathcal{C}_i(\varphi, \psi, f)$ holds for $i = 1, \ldots, m+d-2$, then (4.1)–(4.2) admits a unique $H^{m+d+2}(\Omega)$-solution u and such a u satisfies*

$$\|u\|_{m,\Omega} \le C\big(\|\varphi\|_{m+d+1,\partial_b\Omega} + \|\psi\|_{m+d,\partial_b\Omega} + \|f\|_{m+d,\Omega}\big), \tag{4.22}$$

where C is a positive constant depending on λ, Λ, C_b, C_K, η_0 and the C^{m+d-1}-norms of a, b_1, b_2, c and K. Moreover, if a, b_1, b_2, c, K and f are $H^s(\bar{\Omega})$ and φ and ψ are $H^s(\partial_b\Omega)$ for any $s \geq 1$ and $\mathcal{C}_i(\varphi, \psi, f)$ holds for any $i \geq 1$, then u is smooth and satisfies (4.22) for any $m \geq 1$.

Proof For a positive sequence $\varepsilon \to 0$, we consider an equation of the following form

$$L_\varepsilon u \equiv u_{yy} - a(K+\varepsilon)u_{xx} + b_1 u_x + b_2 u_y + cu = f_\varepsilon \quad \text{in } \Omega \tag{4.23}$$

with the Cauchy data

$$u = \varphi_\varepsilon, \; u_y = \psi_\varepsilon \quad \text{on} \partial_b\Omega, \tag{4.24}$$

where φ_ε, ψ_ε and f_ε are chosen so that

$$\varphi_\varepsilon \to \varphi \text{ in } H^{m+d+1}(\partial_b\Omega), \quad \psi_\varepsilon \to \psi \text{ in } H^{m+d}(\partial_b\Omega), \quad f_\varepsilon \to f \text{ in } H^{m+d}(\Omega),$$

and

$$C_i(\varphi_\varepsilon, \psi_\varepsilon, f_\varepsilon) \text{ holds for} L_\varepsilon \quad \text{for any } i = 1, \ldots, m+d-2.$$

We note that L_ε in (4.23) is strictly hyperbolic in $\bar{\Omega}$. By Theorem 3.4, (4.23)–(4.24) admits a solution $u_\varepsilon \in H^{m+d}(\Omega)$. By Lemma 4.3, u_ε satisfies

$$\|u_\varepsilon\|_{m,\Omega} \leq C\big(\|\varphi_\varepsilon\|_{m+d+1,\partial_b\Omega} + \|\psi_\varepsilon\|_{m+d,\partial_b\Omega} + \|f_\varepsilon\|_{m+d,\Omega}\big),$$

where C is a positive constant depending on λ, Λ, C_b, C_K, η_0 and the C^{m+d-1}-norms of a, b_1, b_2, c and K. We finish the proof by letting $\varepsilon \to 0$. □

5 Proof of Theorem 1.2

In this section, we will prove a result of which Theorem 1.2 is a special case.

We consider an equation of the following form

$$Lu \equiv u_{yy} + aKu_{xx} + b_1 u_x + b_2 u_y + cu = f \quad \text{in } B_2 \subset \mathbb{R}^2, \tag{5.1}$$

where a, b_1, b_2, c and K are smooth in B_2. We always assume

$$a \geq \lambda \quad \text{in } B_2, \tag{5.2}$$

for a positive constant λ. Concerning K, we assume

$$\begin{aligned} &\{K=0\} \text{ consists of two curves given by smooth functions } y = \gamma_i(x), \\ &\text{where } y = \gamma_1(x) \text{ is decreasing and } y = \gamma_2(x) \text{ is increasing and} \\ &\gamma_1(0) = 0, \gamma_2(0) = 0, \gamma_1'(0) \neq \gamma_2'(0). \end{aligned} \tag{5.3}$$

By setting

$$\kappa_1(x) = \max\{\gamma_1(x), \gamma_2(x)\}, \quad \kappa_2(x) = \min\{\gamma_1(x), \gamma_2(x)\},$$

we note that $\kappa_1(x)$ and $\kappa_2(x)$ are smooth at any $x \neq 0$, $\kappa_i(0) = 0$ and $\kappa_1(x) > 0$ and $\kappa_2(x) < 0$ for any $x \neq 0$. Obviously, $y = \kappa_1(x)$ and $y = \kappa_2(x)$ divide B_2 into four regions. We denote by Ω_+ and Ω_- the union of the two regions containing the x-coordinate axis and the y-coordinate axis, respectively. We further assume that

$$K > 0 \text{ in } \Omega_+ \quad \text{and} \quad K < 0 \text{ in } \Omega_-. \tag{5.4}$$

Moreover, we assume that

$$K_x^2 \le C_K^2 |K_y| \quad \text{in } \Omega_-, \tag{5.5}$$

and

$$\big|y - \kappa(x)\big|^d \le C_K |K(x, y)| \quad \text{for any } (x, y) \in \Omega_-, \tag{5.6}$$

where C_K is a positive constant and d is a positive integer. Concerning coefficients b_1 and c, we assume

$$|b_1| \le C_b\big(\sqrt{K} + |K_x|\big) \quad \text{in } \Omega, \tag{5.7}$$

for a positive constant C_b and

$$c \le 0 \quad \text{in } \Omega_+. \tag{5.8}$$

We note that (5.5) and (5.6) are assumed only in Ω_- and (5.8) only in Ω_+.

Now we explain briefly the roles of these assumptions. The curves $y = \gamma_1(x)$ and $y = \gamma_2(x)$ divide B_2 into four regions, in two of which (5.1) is elliptic and in another two (5.1) is hyperbolic by (5.4). For any one of the regions, the origin is an angular point. For any hyperbolic region, the part of the boundary containing the origin is space-like. The assumption (5.7) is the so-called *Levy condition*. It is needed in both elliptic regions and hyperbolic regions. The condition (5.8) is used to ensure the existence of solutions of the Dirichlet problem in elliptic regions. The assumptions (5.5) and (5.6) are needed to overcome the degeneracy in the hyperbolic regions.

For Eq. (1.2) in Theorem 1.2, we have $K(x, y) = x^2 - y^2$, $\kappa_1(x) = |x|$, $\kappa_2(x) = -|x|$ and $d = 2$.

We now present a result more general than Theorem 1.2 and only formulate it for the infinite differentiability.

Theorem 5.1 *Let a, b_1, b_2, c and K be smooth functions in $B_2 \subset \mathbb{R}^2$ satisfying (5.2)–(5.8). Then for any smooth function f in B_2, there exists a smooth solution u of (5.1) in B_1. Moreover, for any nonnegative integer s, u satisfies*

$$\|u\|_{H^s(B_1)} \le c_s \|f\|_{H^{s+d+3}(B_2)}, \tag{5.9}$$

where c_s is a positive constant depending only on s, λ, C_K, C_b, the C^1-norm of γ_i, $i = 1, 2$, and the C^{s+d+2}-norms of a, b_1, b_2, c and K.

Proof Throughout the proof, we denote by C_s a positive constant depending only on s, λ, C_K, C_b, the C^1-norm of γ_i, $i = 1, 2$, and the C^s-norms of a, b_1, b_2, c and K.

We first smoothen the corner of $\partial\Omega_+$ at ∂B_2 and consider (5.1) in Ω_+. By Theorem 2.1, there exists a smooth solution u of (5.1) in Ω_+ with $u = 0$ on $\partial\Omega_+$. Moreover, for any integer $s \ge 1$, u satisfies

$$\|u\|_{H^s(\Omega_+)} \le C_s \|f\|_{H^{s+1}(\Omega_+)}. \tag{5.10}$$

By the trace theorem, we obtain

$$\sum_{|\alpha| \le s} \|D^\alpha u\|_{L^2(\partial\Omega_+)} \le C_{s+1} \|f\|_{H^{s+2}(\Omega_+)}. \tag{5.11}$$

Next, we assume $y = 1$ intersects $y = \kappa_1(x)$ for a positive x and a negative x in B_2. If not, we may extend K appropriately outside B_2 to achieve this. Now we set

$$\Omega_{-1} = \Omega_- \cap \{0 < y < 1\},$$

and

$$\varphi = 0, \quad \psi = u_y \quad \text{on } \partial_b \Omega_{-1},$$

where $\partial_b \Omega_{-1}$ is the lower portion of $\partial \Omega_{-1}$. We consider (5.1) in Ω_{-1} with the Cauchy data

$$u = \varphi, \ u_y = \psi \quad \text{on } \partial_b \Omega_{-1}. \tag{5.12}$$

Since φ and ψ are boundary values of a smooth solution u in $\bar{\Omega}_+$, it is easy to check that compatibility conditions $\mathcal{C}_i(\varphi, \psi, f)$ are satisfied for any $i \geq 1$ by Lemma 3.3. By Theorem 4.4, there exists a smooth solution u of (5.1) in Ω_{-1} satisfying (5.12). Moreover, for any integer $s \geq 1$, u satisfies

$$\|u\|_{H^s(\Omega_{-1})} \leq C_{s+d}\big(\|\varphi\|_{H^{s+d+1}(\partial_b \Omega_{-1})} + \|\psi\|_{H^{s+d}(\partial_b \Omega_{-1})} + \|f\|_{H^{s+d}(\Omega_{-1})}\big).$$

With (5.11), we have easily

$$\|u\|_{H^s(\Omega_{-1})} \leq C_{m+d+2}\big(\|f\|_{H^{s+d+3}(\Omega_+)} + \|f\|_{H^{s+d}(\Omega_{-1})}\big).$$

A similar argument can be applied to

$$\Omega_{-2} = \Omega_- \cap \{-1 < y < 0\}.$$

Therefore we obtain a function u which is a smooth solution of (5.1) in $\Omega_+ \cap B_1$ and $\Omega_- \cap B_1$. It is easy to see that u is smooth across $\partial \Omega_+ \cap B_1$ and especially at the origin. The estimate (5.9) also follows easily. □

Remark 5.2 We also note that $c \leq 0$ in (5.8) can be replaced by $c \leq \varepsilon$ for $\varepsilon > 0$ sufficiently small. Refer to Remark 2.13.

The estimate (5.9) is not sufficient for the iteration process when solving the nonlinear equations. For this, we need a stronger estimate.

Theorem 5.3 *Let a, b_1, b_2, c and K be smooth functions in $B_2 \subset \mathbb{R}^2$ satisfying (5.2)–(5.8). Then for any smooth function f in B_2, there exists a smooth solution u of (5.1) in B_1. Moreover, for any nonnegative integer s, u satisfies*

$$\|u\|_{H^s(B_1)} \leq c_s \left(||f||_{H^{s+d+3}(B_1)} + \Lambda_s ||f||_{H^{d+3}(B_1)}\right), \tag{5.13}$$

where c_s is a constant depending only on s, λ, C_K, C_b, the C^1-norm of γ_i, $i = 1, 2$, and where

$$\Lambda_s = \|a\|_{H^{s+d+4}(B_1)} + \sum_{i=1}^{2} \|b_i\|_{H^{s+d+4}(B_1)} + \|c\|_{H^{s+d+4}(B_1)} + \|K\|_{H^{s+d+4}(B_1)} + 1.$$

We note that all estimates in Sects. 2–4 are standard energy estimates. Hence, we obtain (5.13) with the help of interpolation inequalities. We skip the details.

6 Proof of Theorem 1.1

In this section, we will prove a result of which Theorem 1.1 is a special case.

Consider an equation of the following form

$$\det(D^2 u) = K(x, y)\psi(x, y, u, Du) \quad \text{in } B_1 \subset \mathbb{R}^2, \tag{6.1}$$

where K is smooth in B_1 and ψ is smooth in $B_1 \times \mathbb{R} \times \mathbb{R}^2$. We always assume

$$\psi \geq \lambda \quad \text{in } B_1 \times \mathbb{R} \times \mathbb{R}^2, \tag{6.2}$$

for a positive constant λ. Concerning K, we assume that K satisfies (5.3)–(5.6). In other words, we assume

$$\begin{aligned} &\{K = 0\} \text{ consists of two curves given by smooth functions } y = \gamma_i(x),\\ &\text{where } y = \gamma_1(x) \text{ is decreasing and } y = \gamma_2(x) \text{ is increasing and}\\ &\gamma_1(0) = 0, \gamma_2(0) = 0, \gamma_1'(0) \neq \gamma_2'(0). \end{aligned} \tag{6.3}$$

By setting

$$\kappa_1(x) = \max\{\gamma_1(x), \gamma_2(x)\}, \quad \kappa_2(x) = \min\{\gamma_1(x), \gamma_2(x)\},$$

we note that $\kappa_1(x)$ and $\kappa_2(x)$ are smooth at any $x \neq 0$, $\kappa_i(0) = 0$ and $\kappa_1(x) > 0$ and $\kappa_2(x) < 0$ for any $x \neq 0$. Obviously, $y = \kappa_1(x)$ and $y = \kappa_2(x)$ divide B_1 into four regions. We denote by Ω_+ and Ω_- the union of the two regions containing the x-coordinate axis and the y-coordinate axis, respectively. We further assume

$$K > 0 \text{ in } \Omega_+ \quad \text{and} \quad K < 0 \text{ in } \Omega_-. \tag{6.4}$$

Moreover, we assume

$$K_x^2 \leq C_K^2 |K_y| \quad \text{in } \Omega_-, \tag{6.5}$$

and

$$\left|y - \kappa(x)\right|^d \leq C_K |K(x, y)| \quad \text{for any } (x, y) \in \Omega_-, \tag{6.6}$$

where C_K is a positive constant and d is a positive integer.

We now point out the difference between the assumptions on K for (5.1) and (6.1). For linear equations having the specific form of (5.1), the conditions on K are assumed with respect to this *particular* coordinate system. However, the Monge–Ampère operator is invariant by orthogonal transformations. Hence, conditions on K for (6.1) in this section are assumed in *some* coordinate system.

We now present a result more general than Theorem 1.1 and only formulate it for the case of infinite differentiability.

Theorem 6.1 *Let ψ be a smooth function satisfying* (6.2) *and let K be a smooth function in B_1 satisfying* (6.3)–(6.6). *Then there exists a smooth solution u of* (6.1) *in B_r for some $r \in (0, 1)$.*

The proof of Theorem 6.1 is based on Nash–Moser iterations. An important step in such an iteration process consists of appropriate estimates for solutions of the linearized equations. In the case of the degenerate Monge–Ampère Eq. (6.1), the linearized equations are hard to classify. A crucial observation by Han [7] is that the linearization of Monge–Ampère equations can be decomposed into two parts, one of which has type determined solely by K and another which may be considered as quadratic error with respect to the iteration process.

In the following, we denote points in $\mathbb{R}^2$ by (x_1, x_2) instead of (x, y) and write $x = (x_1, x_2) \in \mathbb{R}^2$. Set

$$\tilde{\mathcal{F}}(u) = \det(D^2u) - K\psi(x, u, Du). \tag{6.7}$$

To proceed, we temporarily replace $x \in \mathbb{R}^2$ by $\tilde{x} \in \mathbb{R}^2$, replace ψ by $\tilde{\psi}$, and write $\widetilde{\partial}_i$ instead of $\partial_{\tilde{x}_i}$. Then (6.7) has the form

$$\tilde{\mathcal{F}}(u) = \det(\widetilde{D}^2 u) - K\tilde{\psi}(\tilde{x}, u, \widetilde{D}u).$$

All functions are evaluated at $\widetilde{x}$. For $\varepsilon > 0$ set

$$\tilde{x} = \varepsilon^2 x,$$

and

$$u(\tilde{x}) = \frac{1}{2}\tilde{x}_1^2 + \varepsilon^5 w\left(\frac{\tilde{x}}{\varepsilon^2}\right).$$

Now we evaluate $\tilde{\mathcal{F}}(u)$ in terms of w. Set

$$\mathcal{F}(w; \varepsilon) = \mathcal{F}(w) = \frac{1}{\varepsilon}\tilde{\mathcal{F}}(u),$$

or

$$\mathcal{F}(w) = \frac{1}{\varepsilon}\left\{ \det\left((1-\delta_{i2})\delta_{1j} + \varepsilon\partial_{ij} w\right) - K\psi\right\}, \tag{6.8}$$

where

$$\psi(\varepsilon, x, w, Dw) = \tilde{\psi}\left(\varepsilon^2 x, \frac{1}{2}\varepsilon^4 x_1^2 + \varepsilon^5 w(x), \varepsilon^2(1-\delta_{i2})x_i + \varepsilon^3\partial_i w(x)\right). \tag{6.9}$$

Note that the arguments of $\tilde{\psi}$ are $\tilde{x}$, u and $\tilde{D}u$ in terms of w in the x-coordinates. All known functions are evaluated at $\tilde{x} = \varepsilon^2 x$. By taking ε small enough, we may assume $\mathcal{F}(w)$ is well defined in $B_1 \subset \mathbb{R}^2$. Letting $w = 0$ in (6.8), we have

$$\mathcal{F}(0) = -\frac{1}{\varepsilon}K\psi.$$

By $K = K(\varepsilon^2 x)$ and $K(0) = 0$, there holds

$$\mathcal{F}(0) = \varepsilon F_0(\varepsilon, x),$$

for some smooth function F_0 in ε and x. We also have

$$\psi(\varepsilon, x, w, Dw) \geq \lambda,$$

for any $x \in B_1$, any ε small and any $w \in C^\infty(B_1)$.

Now we discuss the linearized operator $\mathcal{F}'(w)$ of $\mathcal{F}$ at w. For convenience, we set

$$(\Phi_{ij}) = \left((1-\delta_{i2})\delta_{1j} + \varepsilon\partial_{ij} w\right).$$

A straightforward calculation yields

$$\mathcal{F}'(w)\rho = \Phi^{ij}\partial_{ij}\rho + a_i\partial_i\rho + a\rho, \tag{6.10}$$

where (Φ^{ij}) is the matrix of cofactors of (Φ_{ij}), i.e.,

$$\Phi^{11} = \varepsilon\partial_{22}w, \quad \Phi^{12} = -\varepsilon\partial_{12}w, \quad \Phi^{22} = 1 + \varepsilon\partial_{11}w, \tag{6.11}$$

and

$$a_i = a_i(\varepsilon, x, w, Dw) = -\varepsilon^2 K\partial_{\tilde{\partial}_i u}\tilde{\psi}, \quad a = a(\varepsilon, x, w, Dw) = -\varepsilon^4 K\partial_u\tilde{\psi}. \tag{6.12}$$

As in (6.9), $\partial_{\tilde{\partial}_i u}\tilde{\psi}$ and $\partial_u\tilde{\psi}$ are evaluated at

$$\Big(\varepsilon^2 x, \frac{1}{2}\varepsilon^4 x_1^2 + \varepsilon^5 w(x), \varepsilon^2(1-\delta_{i2})x_i + \varepsilon^3\partial_i w(x)\Big).$$

Obviously, a_i and a are smooth in ε, x, w and Dw.

By (6.8), we have

$$\det(\Phi^{ij}) = \varepsilon\mathcal{F}(w) + K\psi. \tag{6.13}$$

It is not clear how K determines the type of the linear operator $\mathcal{F}'(w)$ in (6.10). Next, we shall introduce a new coordinate system and rewrite (6.10).

Lemma 6.2 *For any $\varepsilon \in (0, \varepsilon_0]$ and any smooth function w with $|w|_{C^2} \le 1$, there exists a transformation $T : B_1 \to T(B_1)$, smooth in ε, x, D^2w and D^3w, of the form*

$$x \mapsto y = (y_1(x), y_2(x)) \tag{6.14}$$

such that in the new coordinates y the operator $\mathcal{F}'(w)$ is given by

$$\begin{aligned}\mathcal{F}'(w)\rho &= a_{22}\partial_{y_2y_2}\rho + \big(K\psi + \varepsilon\mathcal{F}(w)\big)a_{11}\partial_{y_1y_1}\rho \\ &\quad + \big(b_{10}K + b_{11}\partial_{y_1}K + \varepsilon\tilde{b}_{10}\mathcal{F}(w) + \tilde{b}_{11}\partial_{y_1}(\mathcal{F}(w)\big)\partial_{y_1}\rho + b_2\partial_{y_2}\rho + cK\rho,\end{aligned} \tag{6.15}$$

where a_{11}, a_{22}, b_{10}, b_{11}, $\tilde{b}_{10}$, $\tilde{b}_{11}$, b_2 and c are smooth functions in ε, y, w, Dw, D^2w, D^3w and D^4w, with

$$a_{ii} = 1 + O(\varepsilon) \quad \text{for } i = 1, 2.$$

Moreover, for $i = 1, 2$, $y_i = y_i(x)$ in (6.14) satisfies

$$|y_i - x_i| \le c\varepsilon,$$

and for any $s \ge 0$

$$\|y_i\|_{H^s} \le c(1 + \|w\|_{H^{s+2}}),$$

for some positive constant c.

This is Lemma 2.2 in [7] for $n = 2$ (p. 430). The proof for $n = 2$ is easy. We outline the proof for completeness.

Proof By (6.11), we have

$$\Phi^{ij} = \delta_{i2}\delta_{j2} + O(\varepsilon) \quad \text{for any } 1 \le i, j \le 2. \tag{6.16}$$

First, we set

$$y_2 = x_2. \tag{6.17}$$

Next, we consider the following equation for y_1

$$\begin{aligned}\Phi^{12}\partial_1 y_1 + \Phi^{22}\partial_2 y_2 &= 0,\\ y_1(x_1, 0) &= x_1.\end{aligned} \tag{6.18}$$

The coefficient of $\partial_2 y_1$ is given by Φ^{22}, which is not zero for small ε. Hence for small ε, (6.18) always has a unique solution y_1 in B_1, smooth in ε, x and D^2w. Moreover,

$$y_1(x) = x_1 + O(\varepsilon). \tag{6.19}$$

Obviously, $y = y(x)$ forms a new coordinate system. This defines the transformation T in (6.14).

In the new coordinates y, the operator $\mathcal{F}'(w)$ has the following form

$$\mathcal{F}'(w)\rho = b_{ij}\partial_{y_i y_j}\rho + b_i\partial_{y_i}\rho + a\rho, \tag{6.20}$$

where

$$b_{ij} = \sum_{k,l=1}^{2} \Phi^{kl}\partial_k y_i \partial_l y_j,$$

and

$$b_i = \sum_{k,l=1}^{2} \Phi^{kl}\partial_{kl} y_i + \sum_{k=1}^{2} a_k \partial_k y_i.$$

We now claim that

$$b_{11} = \frac{1}{\Phi^{22}}\det(\Phi^{ij})(\partial_1 y_1)^2, \quad b_{12} = 0, \quad b_{22} = \Phi^{22}, \tag{6.21}$$

and

$$b_1 = \partial_1\left(\frac{\det(\Phi^{ij})}{\Phi^{22}}\partial_1 y_1\right) + \sum_{k=1}^{2} a_k\partial_k y_1, \quad b_2 = a_2. \tag{6.22}$$

To prove the claim, we note that the expressions for b_{22} and b_2 follow from (6.17) and those for b_{12} and b_{11} follow from (6.18). To calculate b_1, we have by (6.16)

$$\sum_{k=1}^{2} \partial_k \Phi^{kl} = 0.$$

Then the first term in b_1 in (6.22) can be written as

$$\sum_{k,l=1}^{2} \Phi^{kl}\partial_{kl} y_1 = \sum_{k,l=1}^{2} \partial_k(\Phi^{kl}\partial_l y_1).$$

Then the expression for b_1 follows again from (6.18).

By substituting (6.21) and (6.22) in (6.20), we have

$$\begin{aligned}\mathcal{F}'(w)\rho &= \Phi^{22}\partial_{y_2 y_2}\rho + \frac{1}{\Phi^{22}}\det(\Phi^{ij})(\partial_1 y_1)^2\partial_{y_1 y_1}\rho \\ &\quad + \left(\partial_1\big(\frac{\det(\Phi^{ij})}{\Phi^{22}}\partial_1 y_1\big) + \sum_{k=1}^{2} a_k\partial_k y_1\right)\partial_{y_1}\rho + a_2\partial_{y_2}\rho + a\rho.\end{aligned}$$

Recalling (6.12), (6.13), (6.16) and (6.19), we conclude the proof. □

Next, we write $\mathcal{F}'(w)$ in (6.15) as

$$\mathcal{F}'(w)\rho = \mathcal{L}(w)\rho + \varepsilon\mathcal{F}(w)\sum_{i,j=1}^{2} \tilde{a}_{ij}\partial_{ij}\rho + \varepsilon\sum_{i,j=1}^{2}\big(\tilde{b}_{j0}\mathcal{F}(w) + \tilde{b}_{ij}\partial_i(\mathcal{F}(w))\big)\partial_j\rho, \tag{6.23}$$

where $\tilde{a}_{ij}$ and $\tilde{b}_{ij}$ are functions smooth in ε, x, D^2w, D^3w and D^4w, and $\mathcal{L}(w)$ has the following form in $T(B_1)$

$$\mathcal{L}(w)\rho = a_{22}\partial_{y_2y_2}\rho + a_{11}K\partial_{y_1y_1}\rho + (b_1K + \tilde{b}_1\partial_{y_1}K)\partial_{y_1}\rho + b_2\partial_{y_2}\rho + cK\rho, \quad (6.24)$$

where $a_{11}, a_{22}, b_1, \tilde{b}_1, b_2$ and c are functions smooth in ε, y, w, Dw, D^2w, D^3w and D^4w. We point out that, in the new coordinate system (y_1, y_2) in (6.14), the operator $\mathcal{L}(w)$ in (6.24) has a special structure. Both a_{11} and a_{22} are positive and there is a factor of K in the coefficient of $\partial_{y_1y_1}\rho$. Hence, the operator $\mathcal{L}(w)$ is elliptic if $K > 0$ and hyperbolic if $K < 0$. We emphasize that the type of $\mathcal{L}(w)$ is determined *solely* by K and is independent of w, the function at which the linearized operator is evaluated. This is crucial for the iterations. Next, we note that the correction terms that were added in (6.23) are *quadratic* in $\mathcal{F}(w)$ and ρ, and their derivatives. Hence they can be relegated to the quadratic error in the iteration process, that is, they may be ignored when solving the linearized equation.

By writing

$$a_{22}^{-1}\mathcal{L}(w)\rho = \partial_{y_2y_2}\rho + \frac{a_{11}}{a_{22}}K\partial_{y_1y_1}\rho + \frac{1}{a_{22}}(b_1K + \tilde{b}_1\partial_{y_1}K)\partial_{y_1}\rho + \frac{b_2}{a_{22}}\partial_{y_2}\rho + \frac{c}{a_{22}}K\rho,$$

we note that the coefficients in the right hand side satisfy (5.2)–(5.7). (The notation here is different from that used in the previous section.) The coefficient of ρ may not be nonpositive. However, it is small as $K = K(\varepsilon^2x)$ and $K(0) = 0$. By Theorem 5.3 and Remark 5.2, for any smooth function f in B_1 and any $\varepsilon > 0$ sufficiently small, there exists a smooth function ρ in B_1 such that

$$\mathcal{L}(w)(\rho \circ T^{-1}) = f \circ T^{-1} \quad \text{in } T(B_1),$$

where T is the transformation given by (6.14). (In the following, we abuse notation and simply write $\mathcal{L}(w)\rho = f$ in B_1.) Moreover, if $\|w\|_{H^4(B_1)} \leq 1$, then for any nonnegative integer s

$$\|\rho\|_{H^s(B_1)} \leq c_s \left(||f||_{H^{s+d+3}(B_1)} + (\|w\|_{H^{s+d+8}(B_1)} + 1)||f||_{H^{d+3}(B_1)}\right), \qquad (6.25)$$

where c_s is a constant depending only on s, λ, C_K, the C^1-norm of γ_i, $i = 1, 2$ and the H^{s+d+8}-norm of K. Here, we use the fact that a_{11}, a_{22}, b_1, b_2 and c are functions smooth in ε, y, w, Dw, D^2w, D^3w and D^4w.

The proof of Theorem 6.1 is based on Nash–Moser iterations. A general result for the existence of local smooth solutions is formulated in [8]. (Refer to Theorem 7.4.1 on p. 130 [8].) However, the linearized equations of (6.1) do not satisfy the condition listed there. Specifically, solutions of the linearized equations of (6.1) do not satisfy the estimate (7.4.5) in [8]. As we have discussed, the linearization of Monge–Ampère equations can be decomposed into two parts, one of which can be used to form a linear equation whose solutions satisfy the estimate (7.4.5) in [8] and another which may be considered as quadratic error. Therefore the iteration process in the proof of Theorem 7.4.1 can be performed. We now outline the proof.

Proof of Theorem 6.1 Now we can use iterations to solve $\mathcal{F}(\cdot, \varepsilon) = 0$ for ε sufficiently small as in the proof of Theorem 7.4.1 on p. 130 [8]. The estimate (6.25) plays the same role as (7.4.5) there. We begin the iteration by setting $w_0 = 0$. Then w_ℓ is constructed by induction on ℓ as follows. Suppose $w_0, w_1, \ldots, w_\ell$ have been chosen. Let ρ_ℓ be a solution of

$$\mathcal{L}(w_\ell)\rho_\ell = -\mathcal{F}(w_\ell). \qquad (6.26)$$

Here, ρ_ℓ is chosen to satisfy

$$\|\rho_\ell\|_{H^s(B_1)} \leq c_s \left(||\mathcal{F}(w_\ell)||_{H^{s+d+3}(B_1)} + (\|w_\ell\|_{H^{s+d+8}(B_1)} + 1)||\mathcal{F}(w_\ell)||_{H^{d+3}(B_1)}\right),$$

for any $s \geq 0$. Now we define

$$w_{\ell+1} = w_\ell + S_\ell \rho_\ell, \tag{6.27}$$

where $\{S_\ell\}$ is an appropriately chosen family of smoothing operators. We point out that (6.26) replaces (7.4.8) in [8]. We may proceed as in the proof of Theorem 7.4.1 [8] with minor modifications. By Taylor expansion and (6.27), we have

$$\begin{aligned}
&\mathcal{F}(w_{\ell+1}) - \mathcal{F}(w_\ell)\\
&\quad = \mathcal{F}'(w_\ell)(w_{\ell+1} - w_\ell) + Q(w_\ell; w_{\ell+1} - w_\ell)\\
&\quad = \mathcal{F}'(w_\ell)(S_\ell\rho_\ell) + Q(w_\ell; S_\ell\rho_\ell)\\
&\quad = \mathcal{L}(w_\ell)(S_\ell\rho_\ell) + \big(\mathcal{F}'(w_\ell) - \mathcal{L}(w_\ell)\big)(S_\ell\rho_\ell) + Q(w_\ell; S_\ell\rho_\ell)\\
&\quad = \mathcal{L}(w_\ell)\rho_\ell + \mathcal{L}(w_\ell)(S_\ell - 1)\rho_\ell + \big(\mathcal{F}'(w_\ell) - \mathcal{L}(w_\ell)\big)(S_\ell\rho_\ell) + Q(w_\ell; S_\ell\rho_\ell),
\end{aligned}$$

where $Q(w_\ell; S_\ell\rho_\ell)$ is the quadratic error. Then (7.4.23) on p. 133 of [8] may be modified accordingly. We point out that, by (6.23) and (6.24), the difference of $\mathcal{F}'(w_\ell)(S_\ell\rho_\ell)$ and $\mathcal{L}(w_\ell)(S_\ell\rho_\ell)$ consists of *quadratic* expressions in $\mathcal{F}(w_\ell)$ and $S_\ell\rho_\ell$, and their derivatives, which may be estimated in a way similar to $Q(w_\ell; S_\ell\rho_\ell)$. The rest of the proof is the same as that of Theorem 7.4.1 [8], and is therefore not included here. □

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