COMMUNICATIONS IN
ANALYSIS AND GEOMETRY
Volume 18, Number 4, 649–704, 2010

On the local isometric embedding in $\mathbb{R}^3$ of surfaces with Gaussian curvature of mixed sign

Qing Han and Marcus Khuri

We study the old problem of isometrically embedding a two-dimensional Riemannian manifold into Euclidean three-space. It is shown that if the Gaussian curvature vanishes to finite order and its zero set consists of two Lipschitz curves intersecting transversely at a point, then local sufficiently smooth isometric embeddings exist.

1. Introduction

Does every smooth two-dimensional Riemannian manifold admit a smooth local isometric embedding into $\mathbb{R}^3$, or heuristically, can every abstract surface be visualized at least locally? This natural question was first posed in 1873 by Schlaefli [16], and remarkably has remained to a large extent unanswered. It is the purpose of this paper to provide a general sufficient condition under which local embeddings exist.

The local isometric embedding problem for surfaces is equivalent to finding local solutions of a particular Monge–Ampère equation, usually referred to as the Darboux equation. The primary difficulty in analyzing this equation arises from the fact that it changes from elliptic to hyperbolic type, whenever the Gaussian curvature of the given metric passes from positive to negative curvature. Consequently, the hypotheses of any result must take into account the manner in which the Gaussian curvature, K, vanishes. The classical results deal with the cases in which the curvature does not vanish, or the metric is analytic. It was not until 1985/1986 that the first degenerate cases (when K vanishes) were treated, by Lin. He showed the existence of sufficiently smooth embeddings if the metric is sufficiently smooth and $K \geq 0$ [11], or $K(0) = 0$, $|\nabla K(0)| \neq 0$ [12]. Smooth embeddings of smooth surfaces were obtained by Han *et al.* [5] when $K \leq 0$ and ∇K possesses a certain nondegeneracy, and by Han [3] when K vanishes across a single smooth curve (see also [1, 2, 8] for related results). Lastly if $K = |\nabla K(0)| = 0$, $|\nabla^2 K(0)| \neq 0$ then Khuri [9] has proven the existence of sufficiently smooth embeddings for sufficiently smooth surfaces. For more details on this problem and other related topics the reader is referred to [4]. Here we will show

Theorem 1.1. *Let $g \in C^{m_*}$, $m_* \geq 36(N+10)$, be a Riemannian metric defined on a neighborhood of the origin in the plane, with Gaussian curvature K vanishing there to finite order N. If the zero set $K^{-1}(0)$ consists of two C^{m_*} curves intersecting transversely at the origin, then g admits a C^m, $m \leq \frac{1}{12}m_* - N - 24$, isometric embedding into $\mathbb{R}^3$ on some neighborhood of the origin.*

Remark 1.1. Our methods actually treat a slightly more general situation in that the zero set $K^{-1}(0)$ may consist of more than two curves intersecting transversely at the origin. However, in this setting there should not be more than two regions on which $K \geq 0$.

The embeddings produced by this theorem are referred to as sufficiently smooth, since higher regularity of the metric implies higher regularity for the embedding. However, this theorem does not guarantee that C^∞ metrics give rise to C^∞ embeddings, as the methods used here require the domain of existence to shrink whenever higher regularity is demanded of the solution. On the other hand, it is likely that techniques similar to those found in [3, 6] may lead to a C^∞ version of Theorem 1.1. We also point out that counterexamples to the existence of local isometric embeddings have been found for metrics of low regularity, by Pogorelov [15] when $g \in C^{2,1}$, and by Nadirashvili and Yuan [13] who recently generalized this result. Moreover counterexamples to the local solvability of smooth Monge–Ampère equations have been found in [10]. Yet it is still very much an open question, whether or not there are any smooth (or sufficiently smooth) counterexamples to the local isometric embedding problem.

As mentioned above this problem is equivalent to finding local solutions of a particular Monge–Ampère equation. To see this we use a standard method originally introduced by Weingarten [20]. That is, we search for a function $z(u,v)$ defined in a neighborhood of the origin such that the new metric $g - dz^2$ is flat. Note that $g - dz^2$ will be Riemannian as long as $|\nabla_g z| < 1$. Since flat metrics are locally isometric to Euclidean space, there exist two C^m functions $x(u,v)$, $y(u,v)$ (if $g \in C^m$ and $z \in C^{m+1}$, see [7]) such that $g - dz^2 = dx^2 + dy^2$. The map $(u,v) \mapsto (x(u,v), y(u,v), z(u,v))$ then provides the desired embedding. Furthermore, a straightforward calculation shows that $g - dz^2$ is flat if and only if z satisfies

$$\det \operatorname{Hess}_g z = K(\det g)(1 - |\nabla_g z|^2). \tag{1.1}$$

This Monge–Ampère equation is the so-called Darboux equation, and the question of its local solvability is equivalent to the local isometric embedding problem.

It is trivial to construct approximate local solutions of (1.1), and thus it is natural to use an implicit function theorem to prove existence. Under the hypotheses of Theorem 1.1 the linearization will be of mixed type, and thus we will necessarily lose derivatives when it is inverted. This suggests that we use a version of the Nash–Moser implicit function theorem, which essentially reduces the problem to a study of the linearized equation. We will show that the linearization has a particularly nice canonical form, when an appropriate coordinate system is chosen and certain perturbation terms that behave like quadratic error in the Nash–Moser iteration are removed. This was first observed by Han in [3]. More precisely, the canonical form is given by

$$Lu = (aKu_x)_x + bu_{yy} + cKu_x + du_y, \tag{1.2}$$

where $a, b > 0$. The significance of this particular structure is that it explicitly illustrates how the Gaussian curvature affects the type of the linearization. Moreover it is also important that the first-order coefficient cK vanishes whenever the principal symbol changes type, as this leads to the so-called Levi conditions [14] in the hyperbolic regions, which facilitate the making of estimates. Under the assumptions of Theorem 1.1 on the Gauss curvature, there are four separate regions of elliptic or hyperbolic type for (1.2), each having a Lipschitz smooth boundary. We will develop the appropriate existence and regularity theory for (1.2) in each of these regions, and show that combined with a Nash–Moser iteration this leads to a corresponding solution of the nonlinear equation (1.1) in each region. These separate solutions will then be patched together to form a solution on a full neighborhood of the origin.

This paper is organized as follows. In Section 2 we obtain the canonical form (1.2), and in Sections 3 and 4 the linear existence theory is established in the elliptic and hyperbolic regions, respectively. Lastly in Section 5 we use a version of the Nash–Moser iteration to solve (1.1) in the elliptic and hyperbolic regions separately, and also show how the solutions obtained can be patched together to yield the desired solution.

2. The linearized canonical form

In this section we will bring the linearization of (1.1) into the canonical form (1.2). Before doing this, however, we must specify at which function the linearization will be evaluated. For this we need an appropriate approximate solution, z_0. We will then search for a solution of (1.1) in the form

$$z = z_0 + \varepsilon^5 w,$$

where $\varepsilon > 0$ is a small parameter. Let $y = (y^1, y^2)$ be local coordinates in a neighborhood of the origin with $g = g_{ij} dy^i dy^j$, then we are interested in solving

$$\Phi(w) := \det \nabla_{ij} z - K|g|(1 - |\nabla_g z|^2) = 0, \tag{2.1}$$

where $|g| = \det g_{ij}$ and ∇_{ij} are covariant derivatives with respect to these coordinates. We choose

$$z_0 = \frac{1}{2}(y^1)^2 + \sum_{n=3}^{m_*} p_n(y),$$

where each p_n is a homogeneous polynomial of degree n, chosen so that

$$\partial^\alpha \Phi(0) = 0, \qquad |\alpha| \leq m_* - 2. \tag{2.2}$$

Here $\alpha = (\alpha_1, \alpha_2)$ is a multi-index, and m_* is as in Theorem 1.1. Note that such a polynomial z_0 may be found in the usual way by following the proof of the Cauchy–Kowalevski Theorem, since the line $y^2 = 0$ is noncharacteristic for (2.1) as $\nabla_{11} z_0(0) = 1$.

Upon rescaling coordinates by $y^i = \varepsilon^2 x^i$, the linearization of Φ at a function w,

$$\mathcal{L}(w)u = \frac{d}{dt}\Phi(w + tu)|_{t=0},$$

is given by

$$\varepsilon^{-1}\mathcal{L}(w)u = a^{ij} u_{;ij} + 2\varepsilon^4 K|g|\langle \nabla_g z, \nabla_g u \rangle, \tag{2.3}$$

where $u_{;ij}$ denote covariant derivatives in x^i coordinates, $\langle \cdot, \cdot \rangle$ is the inner product associated with g, and

$$(a^{ij}) = \begin{pmatrix} \nabla_{22} z & -\nabla_{12} z \\ -\nabla_{12} z & \nabla_{11} z \end{pmatrix}$$

is the cofactor matrix of $\mathrm{Hess}_g\, z$. Note that the quantity $|g|^{-1} a^{ij}$ transforms like a contravariant two-tensor. According to the assumptions of Theorem 1.1, $K^{-1}(0)$ divides a small neighborhood of the origin into domains $\{\Omega_\kappa^+\}_{\kappa=1}^{\kappa_0}$

on which $K > 0$, and $\{\Omega_\varrho^-\}_{\varrho=1}^{\varrho_0}$ on which $K < 0$ (obviously $\kappa_0 + \varrho_0 = 4$). The following lemma gives the desired canonical form. We will denote the Sobolev space of square integrable derivatives by H^m, with norm $\|\cdot\|_{H^m}$.

Lemma 2.1. *Let $g \in C^{m_*}$ and $w \in C^\infty$ with $|w|_{C^3} < 1$. Given a domain Ω_κ^+, or Ω_ϱ^-, $\varrho = 1, 2$, or Ω_ϱ^-, $\varrho = 3, 4$, and given small $\sigma, \delta > 0$, there exists a local C^{m_*-2} change of coordinates $\xi^i = \xi^i(x)$ such that*

$$\Omega_\kappa^+ \cap B_\sigma(0) = \{(\xi^1, \xi^2) \mid 0 < \xi^2 < (\tan\delta)\xi^1,\ \ |\xi| < \sigma\},$$

or

$$\Omega_\varrho^- \cap B_\sigma(0) = \{(\xi^1, \xi^2) \mid h(\xi^1) < \xi^2,\ \ |\xi| < \sigma\}, \qquad \varrho = 1, 2,$$

or

$$\Omega_\varrho^- \cap B_\sigma(0) = \{(\xi^1, \xi^2) \mid h(\xi^2) < \xi^1,\ \ |\xi| < \sigma\}, \qquad \varrho = 3, 4,$$

for some Lipschitz function $h(\xi^i)$ (not necessarily the same for different regions) satisfying $h(0) = 0$ and $|h(\xi^i) - |\xi^i||_{C^1} = O(\sigma)$. In this new coordinate system the linearization takes the form

$$\varepsilon^{-1}\mathcal{L}(w)u = a^{22}L(w)u + (a^{22})^{-1}\Phi(w)[\partial_{x^1}^2 u - \partial_{x^1}\log(a^{22}\sqrt{|g|})\partial_{x^1}u],$$

where

$$L(w)u = \partial_{\xi^1}(k\partial_{\xi^1}u) + \partial_{\xi^2}^2 u + c\partial_{\xi^1}u + d\partial_{\xi^2}u$$

with

$$\begin{aligned}
k &= K\overline{k}(x, w, \nabla w, \nabla^2 w, \nabla\xi),\\
c &= K\overline{c}(x, w, \nabla w, \nabla^2 w, \nabla^3 w, \nabla\xi, \nabla^2\xi) + (a^{22})^{-2}\partial_{x^1}\Phi(w)\partial_{x^1}\xi^1,\\
d &= \varepsilon^2\overline{d}(x, w, \nabla w, \nabla^2 w),
\end{aligned}$$

for some $\overline{k}, \overline{c}, \overline{d} \in C^{m_*-4}$ such that $\overline{k} > 1/2$ if $\varepsilon = \varepsilon(m)$ is chosen sufficiently small. Moreover there exists a constant C_m independent of ε, δ such that

$$\|\,\xi\,\|_{H^m} \le \delta^{-1}C_m(1 + \|\,w\,\|_{H^{m+4}}), \qquad m \le m_* - 2. \tag{2.4}$$

Remark 2.1. In estimate (2.4), δ is only relevant for the regions $\{\Omega_\kappa^+\}_{\kappa=1}^{\kappa_0}$. Furthermore, since the curvature K vanishes at least to second order, it may be possible to eliminate the role of δ in the arguments of the next section.

Proof. We may choose an initial coordinate system $x = (x^1, x^2)$ so that each of the elliptic and hyperbolic regions Ω_κ^+, Ω_ϱ^- are sector domains, that is, each occupies the region between two lines passing through the origin. Furthermore, we may assume that Ω_1^- (Ω_2^-) contains the positive (negative) x^2-axis. Note that according to the hypotheses of Theorem 1.1, $\partial\Omega_\kappa^+ - \{(0,0)\}$ and $\partial\Omega_\varrho^- - \{(0,0)\}$ are both C^{m_*} smooth, so that this initial transformation is also C^{m_*}. The approximate solution z_0 is chosen with respect to this initial coordinate system (recall that $y^i = \varepsilon^2 x^i$), and therefore

$$a^{22} > 0, \qquad a^{12} = O(\varepsilon^2).$$

It is now an easy exercise in linear algebra to show that for each domain Ω_κ^+, or Ω_ϱ^-, $\varrho = 1, 2$, or Ω_ϱ^-, $\varrho = 3, 4$, there exists a linear change of coordinates $\overline{x} = (\overline{x}^1, \overline{x}^2)$ such that

$$\Omega_\kappa^+ = \{(\overline{x}^1, \overline{x}^2) \mid 0 < \overline{x}^2 < \overline{x}^1, \ |\overline{x}| < 1\},$$

or

$$\Omega_\varrho^- = \{(\overline{x}^1, \overline{x}^2) \mid |\overline{x}^1| < \overline{x}^2 < 1\}, \qquad \varrho = 1, 2,$$

or

$$\Omega_\varrho^- = \{(\overline{x}^1, \overline{x}^2) \mid |\overline{x}^2| < \overline{x}^1 < 1\}, \qquad \varrho = 3, 4,$$

and such that

$$\partial^2_{\overline{y}^1}(y^1)^2 > 0, \qquad \partial_{\overline{y}^1}\partial_{\overline{y}^2}(y^1)^2 = 0.$$

Here $\overline{y}^i = \varepsilon^2\overline{x}^i$. It follows that $a^{22} > 0$ and $a^{12} = O(\varepsilon^2)$ are preserved under this linear change of coordinates. For convenience we will still denote $\overline{y}^i$ by y^i and $\overline{x}^i$ by x^i.

We may write (2.3) as

$$L_1(w)u = a_1^{ij} u_{x^i x^j} + a_1^i u_{x^i} := \varepsilon^{-1}\mathcal{L}(w)u,$$

where $a_1^{ij} = a^{ij}$,

$$a_1^l = -\varepsilon^2(a^{ij}\Gamma_{ij}^l + 2K|g|z^l) \tag{2.5}$$

with $z^l = g^{li} z_{y^i}$, and Γ_{ij}^l are Christoffel symbols in y^i coordinates. According to (2.1)

$$a_1^{11} = \nabla_{22} z = (a^{22})^{-1}[K|g|(1 - |\nabla_g z|^2) + (\nabla_{12} z)^2 + \Phi(w)].$$

We then set

$$L_2(w)u = a_2^{ij} u_{x^i x^j} + a_2^i u_{x^i} := L_1(w)u - (a^{22})^{-1}\Phi(w)u_{x^1 x^1},$$

that is $a_1^{ij} = a_2^{ij}$ and $a_1^i = a_2^i$ except for

$$a_2^{11} = (a^{22})^{-1}[K|g|(1 - |\nabla_g z|^2) + (\nabla_{12} z)^2]. \tag{2.6}$$

Also let

$$L_3(w)u = a_3^{ij} u_{x^i x^j} + a_3^i u_{x^i} := (a^{22})^{-1} L_2(w)u.$$

We now define the desired change of coordinates by

$$\xi^1 = \xi^1(x^1, x^2), \qquad \xi^2 = x^2,$$

with

$$a^{12}\xi^1_{x^1} + a^{22}\xi^1_{x^2} = 0, \tag{2.7}$$

so that if

$$L_4(w)u = a_4^{ij} u_{\xi^i \xi^j} + a_4^i u_{\xi^i} := L_3(w)u$$

then

$$a_4^{12} = a_3^{ij} \xi^1_{x^i} \xi^2_{x^j} = 0.$$

In order to obtain the correct expression in these new coordinates for the domains Ω^+_κ, or Ω^-_ϱ, $\varrho = 1, 2$, or Ω^-_ϱ, $\varrho = 3, 4$, we impose the initial conditions

$$\xi^1(x^1, x^1) = (\tan\delta)^{-1} x^1, \quad \text{or} \quad \xi^1(x^1, 0) = x^1, \quad \text{or} \quad \xi^1(x^1, 0) = x^1, \tag{2.8}$$

respectively. Note that since the curves $x^1 \mapsto (x^1, x^1)$ and $x^1 \mapsto (x^1, 0)$ are noncharacteristic for (2.7), Equation (2.7) with initial condition (2.8) has a unique C^{m_*-2} solution on some neighborhood of the origin. Furthermore, standard methods for first-order equations combined with the Gagliardo–Nirenberg inequalities (Lemma 5.2 below) yields (2.4). Note also that (ξ^1, ξ^2) forms a new coordinate system near the origin since

$$(\tan\delta)^{-1} = \xi^1_{x^1}(0,0) + \xi^1_{x^2}(0,0) = \xi^1_{x^1}(0,0)\left(1 - \frac{a^{12}}{a^{22}}(0,0)\right),$$

or

$$\xi^1_{x^1}(0,0) = 1,$$

according to the respective initial conditions given by (2.8).

We shall now calculate the coefficients of $L_4(w)$. Observe that (2.6) yields

$$\begin{aligned} a_4^{11} = a_3^{ij}\xi^1_{x^i}\xi^1_{x^j} = (\nabla_{11}z)^{-2}[K|g|(1-|\nabla_g z|^2) + (\nabla_{12}z)^2](\xi^1_{x^1})^2 \\ - 2(\nabla_{11}z)^{-1}(\nabla_{12}z)\xi^1_{x^1}\xi^1_{x^2} + (\xi^1_{x^2})^2, \end{aligned}$$

but (2.7) gives

$$\xi^1_{x^2} = (\nabla_{11}z)^{-1}(\nabla_{12}z)\xi^1_{x^1}$$

so

$$a_4^{11} = (a^{22})^{-2}K|g|(1-|\nabla_g z|^2)(\xi^1_{x^1})^2.$$

Next we examine a_4^1. By (2.7)

$$\xi^1_{x^1x^2} = -(a_3^{12})_{x^1}\xi^1_{x^1} - a_3^{12}\xi^1_{x^1x^1}, \qquad \xi^1_{x^2x^2} = -(a_3^{12})_{x^2}\xi^1_{x^1} - a_3^{12}\xi^1_{x^1x^2},$$

so (2.6) produces

$$\begin{aligned} a_4^1 &= a_3^{ij}\xi^1_{x^ix^j} + a_3^i\xi^1_{x^i} \\ &= (a^{22})^{-2}K|g|(1-|\nabla_g z|^2)\xi^1_{x^1x^1} \\ &\quad - \left[\left(\frac{a^{12}}{a^{22}}\right)\left(\frac{a^{12}}{a^{22}}\right)_{x^1} + \left(\frac{a^{12}}{a^{22}}\right)_{x^2}\right]\xi^1_{x^1} + a_3^i\xi^1_{x^i}. \end{aligned} \tag{2.9}$$

Calculating the second term on the right-hand side yields

$$\begin{aligned} (a^{22})^2\left[\left(\frac{a^{12}}{a^{22}}\right)\left(\frac{a^{12}}{a^{22}}\right)_{x^1} + \left(\frac{a^{12}}{a^{22}}\right)_{x^2}\right] &= a^{12}a^{12}_{x^1} - (a^{22})^{-1}(a^{12})^2a^{22}_{x^1} \\ &\quad + a^{22}a^{12}_{x^2} - a^{12}a^{22}_{x^2} \\ &= a^{12}a^{12}_{x^1} - a^{11}a^{22}_{x^1} + a^{22}a^{12}_{x^2} - a^{12}a^{22}_{x^2} \\ &\quad + (a^{22})^{-1}a^{22}_{x^1}(\det a^{ij}) \\ &= -a^{12}_{x^1}a^{12} + a^{11}_{x^1}a^{22} + a^{22}a^{12}_{x^2} - a^{12}a^{22}_{x^2} \\ &\quad + (a^{22})^{-1}a^{22}_{x^1}(\det a^{ij}) - (\det a^{ij})_{x^1}. \end{aligned}$$

Therefore (2.5), (2.7) and (2.9) imply that

$$\begin{aligned} a^{22}a_4^1 = -[a^{ij}_{x^j} + \varepsilon^2(a^{lj}\Gamma^i_{lj} + 2K|g|z^i)]\xi^1_{x^i} + ((a^{22})^{-1}\det a^{ij})_{x^1}\xi^1_{x^1} \\ + (a^{22})^{-1}K|g|(1-|\nabla_g z|^2)\xi^1_{x^1x^1}. \end{aligned} \tag{2.10}$$

A computation shows

$$\begin{aligned} a^{11}_{y^1} + a^{12}_{y^2} + a^{lj}\Gamma^1_{lj} &= -\Gamma^j_{j2} z_{y^1y^2} + \Gamma^j_{j1} z_{y^2y^2} + (\Gamma^i_{12,y^2} - \Gamma^i_{22,y^1} - \Gamma^1_{11}\Gamma^i_{22} \\ &\quad + 2\Gamma^1_{12}\Gamma^i_{12} - \Gamma^1_{22}\Gamma^i_{11}) z_{y^i} \\ &= \Gamma^j_{j2} a^{12} + \Gamma^j_{j1} a^{11} + (\Gamma^i_{12,y^2} - \Gamma^i_{22,y^1} - \Gamma^1_{11}\Gamma^i_{22} + 2\Gamma^1_{12}\Gamma^i_{12} \\ &\quad - \Gamma^1_{22}\Gamma^i_{11} - \Gamma^j_{j2}\Gamma^i_{12} + \Gamma^j_{j1}\Gamma^i_{22}) z_{y^i}. \end{aligned}$$

However, we see that the coefficient of z_{y^i} is in fact a curvature term. More precisely, if we denote it by χ^i then

$$\chi^i = \Gamma^i_{12,y^2} - \Gamma^i_{22,y^1} + \Gamma^j_{12}\Gamma^i_{j2} - \Gamma^j_{22}\Gamma^i_{j1} = -R^i_{212} = -g^{i1}|g|K,$$

where R^i_{jkl} is the Riemann tensor for g in y^i coordinates (recall that Γ^i_{lj} are Christoffel symbols in y^i coordinates). A similar calculation shows that

$$a^{12}_{y^1} + a^{22}_{y^2} + a^{lj}\Gamma^2_{lj} = \Gamma^j_{j1} a^{12} + \Gamma^j_{j2} a^{22} - g^{i2}|g|K z_{y^i}.$$

Therefore after solving for $\xi^1_{x^2}$ in (2.7), (2.10) becomes

$$\begin{aligned} a^{22} a^1_4 &= (a^{22})^{-1} K|g|(1 - |\nabla_g z|^2)\xi^1_{x^1x^1} - \varepsilon^2 K|g| z^i \xi^1_{x^i} \\ &\quad - [\varepsilon^2 \Gamma^j_{j1} (a^{22})^{-1} \det a^{ij} - ((a^{22})^{-1} \det a^{ij})_{x^1}]\xi^1_{x^1}. \end{aligned}$$

It now follows from

$$\det a^{ij} = \Phi(w) + K|g|(1 - |\nabla_g z|^2)$$

that we have

$$\begin{aligned} a^{22} a^1_4 &= -\varepsilon^2 K|g| z^i \xi^1_{x^i} + \partial_{x^1}[(a^{22})^{-1} K|g|(1 - |\nabla_g z|^2)\xi^1_{x^1}] \\ &\quad - [\varepsilon^2 \Gamma^j_{j1}(a^{22})^{-1} K|g|(1 - |\nabla_g z|^2) + \varepsilon^2 \Gamma^j_{j1}(a^{22})^{-1}\Phi(w) \\ &\quad - ((a^{22})^{-1}\Phi(w))_{x^1}]\xi^1_{x^1}. \end{aligned}$$

Lastly, it is trivial to calculate the remaining coefficients:

$$a^{22}_4 = 1, \qquad a^2_4 = -\varepsilon^2 (a^{22})^{-1}(a^{ij}\Gamma^2_{ij} + 2K|g|z^2).$$

Then by defining

$$L(w)u := L_4(w)u + (\varepsilon^2 \Gamma^j_{j1} + \partial_{x^1} \log a^{22})(a^{22})^{-2}\Phi(w)\xi^1_{x^1} u_{\xi^1}$$

and recalling that $\Gamma^j_{j1} = \frac{1}{2}\partial_{y^1} \log |g|$, we obtain the desired result. □

3. Linear theory in the elliptic regions

In light of Lemma 2.1, it will be sufficient for our purposes to study the question of existence and regularity for the operator $L(w)$, instead of the pure linearization $\mathcal{L}(w)$. More precisely, in this section we will study the Dirichlet problem for a modified version of $L(w)$ in an elliptic region. First note that by using polar coordinates $\xi^1 = r\cos\theta$, $\xi^2 = r\sin\theta$, we can transform the elliptic region $\Omega_\kappa^+ \cap B_\sigma(0)$ of Lemma 2.1 into a rectangle

$$\Omega = \{(r,\theta) \mid 0 < r < \sigma,\ \ 0 < \theta < \delta\}.$$

Under these coordinates we find that

$$L(w)u = \mathcal{K}u_{rr} + \mathcal{A}u_{r\theta} + \mathcal{B}u_{\theta\theta} + \mathcal{C}u_r + \mathcal{D}u_\theta,$$

where

$$\begin{aligned}
\mathcal{K} &= k\cos^2\theta + \sin^2\theta,\\
\mathcal{A} &= 2(1-k)\frac{\sin\theta\cos\theta}{r},\\
\mathcal{B} &= k\frac{\sin^2\theta}{r^2} + \frac{\cos^2\theta}{r^2},\\
\mathcal{C} &= k\frac{\sin^2\theta}{r} + \frac{\cos^2\theta}{r} + (c + \partial_{\xi^1}k)\cos\theta + d\sin\theta,\\
\mathcal{D} &= 2(k-1)\frac{\sin\theta\cos\theta}{r^2} - (c + \partial_{\xi^1}k)\frac{\sin\theta}{r} + d\frac{\cos\theta}{r}.
\end{aligned}$$

It will be convenient to cut-off these coefficients away from the origin. So let $\varphi \in C^\infty([0,\infty))$ be a nonnegative cut-off function with

$$\varphi(r) = \begin{cases} 1 & \text{if } 0 < r < \frac{1}{2}\sigma,\\ 0 & \text{if } \sigma < r,\end{cases}$$

and define

$$Lu = \overline{K}u_{rr} + \overline{A}u_{r\theta} + \overline{B}u_{\theta\theta} + \overline{C}u_r + \overline{D}u_\theta,$$

where

$$\overline{K} = \varphi^2\mathcal{K},\qquad \overline{A} = \varphi\mathcal{A},\qquad \overline{B} = \mathcal{B},\qquad \overline{C} = \varphi\mathcal{C},\qquad \overline{D} = \varphi\mathcal{D}.$$

We will study the boundary value problem:

$$(3.1)\quad Lu = f \ \text{ in } \Omega,\quad u(r,0) = u(r,\delta) = 0,\quad \partial_r^s u(0,\theta) = 0,\quad 0 \le s \le s_0,$$

for some large integer s_0. The motivation for considering this problem stems from our method for constructing solutions to the nonlinear problem (2.1) (see Section 5). Namely, we shall construct solutions in the elliptic and hyperbolic regions separately, and then show that they can be patched together. This requires certain compatibility conditions at the origin, and the boundary conditions of (3.1) guarantee that they will be satisfied. Of course, a necessary condition for solving (3.1) is that f must also vanish to a corresponding high order at the origin. It is therefore convenient to introduce the following weighted Sobolev spaces which control the amount of vanishing. Define norms

$$\| u \|^2_{(m,l,\gamma)} = \int_\Omega \sum_{\substack{0\le s\le m,\quad 0\le t\le l \\ s+t\le \max(m,l)}} \lambda^{-s} r^{-\gamma+2s} (\partial_r^s \partial_\theta^t u)^2,$$

where $\lambda, \gamma > 0$ are large parameters, and let $H^{(m,l,\gamma)}(\Omega)$ be the closure of $\overline{C}^\infty(\Omega)$ with respect to this norm, where $\overline{C}^\infty(\Omega)$ is the space of smooth functions that vanish in a neighborhood of $r = 0$. We will always denote the traditional Sobolev spaces having square integrable derivatives up to and including order m by $H^m(\Omega)$, with norm $\| \cdot \|_m$. The following simple lemma describes the boundary behavior exhibited by elements of the weighted spaces.

Lemma 3.1. *Suppose that* $u \in H^{(m,l,\gamma)}(\Omega) \cap C^{\max(m,l)-1}(\Omega)$, *then for any* $0 < r_0 < \sigma$ *we have*

$$\int_{r=r_0} (\partial_r^s \partial_\theta^t u)^2 \le r_0^{\gamma-2(s+1)} C \| u \|^2_{(m,l,\gamma)},$$
$$s \le m-1, \quad t \le l-1, \quad s+t \le \max(m,l)-1,$$

where the constant C *depends only on* $\sigma - r_0$.

Proof. When $s \le m-1, t \le l-1, s+t \le \max(m,l)-1$ we have $r^{-\gamma/2+s+1} \times \partial_r^s \partial_\theta^t u \in H^1(\Omega) \cap C^0(\Omega)$. The desired result now follows from the standard trace theorem for Sobolev spaces. □

In analogy with the theory of strictly elliptic equations in a sector domain such as Ω, the regularity of a solution to (3.1) will depend on the size of the angle forming the domain. More precisely, smaller angles yield higher regularity. According to Lemma 2.1 we are free to choose the angle δ arbitrarily small, with the only price being paid with the blow-up of estimate (2.4). This blow-up, however, can be controlled in the context of Equation (3.1) by taking $\varepsilon = \varepsilon(\delta)$ to be sufficiently small, since whenever $\xi(x)$ or its derivatives appear in the coefficients of the operator L, they are always multiplied by ε. These considerations lead to existence and regularity for (3.1), and the first step needed to establish such a result is the following basic estimate. Define

$$a_{\lambda,\gamma}(r,\theta) = \frac{\lambda\theta^2 - 1}{r^\gamma},$$

and as in Lemma 2.1 let $w \in C^\infty$ throughout this section.

Lemma 3.2. *Suppose that* $|w|_{C^4} < 1$ *and let* $u \in H^{(2,1,\gamma+2)}(\Omega) \cap C^2(\Omega)$ *with* $u(r,0) = u(r,\delta) = 0$. *If* $\delta = \delta(\lambda)$ *and* $\varepsilon = \varepsilon(\delta)$ *are sufficiently small, then*

$$\int_\Omega a_{\lambda,\gamma-2} uLu \geq C\int_\Omega \lambda r^{-\gamma}u^2 + r^{-\gamma+2}(\varphi\sin\theta u_r + r^{-1}\cos\theta u_\theta)^2$$

for some constant $C > 0$ *independent of* λ, δ, ε *and* w.

Proof. Let $0 < r_0 < \sigma$ and set $\Omega_{r_0} = \Omega \cap \{(r,\theta) \mid r_0 < r < \sigma\}$. Then for any $a \in C^\infty(\Omega_{r_0})$, integration by parts yields

$$\begin{aligned}\int_{\Omega_{r_0}} auLu = \int_{\Omega_{r_0}} &-a(\overline{K}u_r^2 + \overline{A}u_r u_\theta + \overline{B}u_\theta^2)\\ &+ \frac{1}{2}\int_{\Omega_{r_0}}[(a\overline{K})_{rr} + (a\overline{A})_{r\theta} + (a\overline{B})_{\theta\theta} - (a\overline{C})_r - (a\overline{D})_\theta]u^2\\ &+ \int_{\partial\Omega_{r_0}} a(\overline{K}uu_r\nu_1 + \overline{A}uu_r\nu_2 + \overline{B}uu_\theta\nu_2)\\ &+ \frac{1}{2}\int_{\partial\Omega_{r_0}}[-(a\overline{K})_r\nu_1 - (a\overline{A})_\theta\nu_1 - (a\overline{B})_\theta\nu_2 + a\overline{C}\nu_1 + a\overline{D}\nu_2]u^2,\end{aligned}$$

where (ν_1,ν_2) denotes the unit outer normal to $\partial\Omega_{r_0}$. By choosing $a = a_{\lambda,\gamma-2}$ and observing that

$$\begin{aligned}&|(a_{\lambda,\gamma-2}\overline{K})_{rr}| + |(a_{\lambda,\gamma-2}\overline{A})_{r\theta}| + |(a_{\lambda,\gamma-2}\overline{C})_r| + |(a_{\lambda,\gamma-2}\overline{D})_\theta|\\ &\quad = O(r^{-\gamma}), (a_{\lambda,\gamma-2}\overline{B})_{\theta\theta} = 2\lambda r^{-\gamma}\cos^2\theta(1 + O(\theta + \lambda^{-1})),\end{aligned}$$

the desired result follows since all boundary terms vanish according to Lemma 3.1 (after letting $r_0 \to 0$). Note that ε is chosen small depending on δ, in order to control the blow-up (implied by estimate (2.4)) found in the coefficients of L. □

This lemma is the main tool used to establish the basic existence result of the next theorem. Let $\widehat{C}^\infty(\Omega)$ denote the space of $\overline{C}^\infty(\Omega)$ functions v satisfying $v(r,0) = v(r,\delta) = 0$. Given $f \in H^{(m,1,\gamma)}(\Omega)$, we will refer to a function $u \in H^{(m,1,\gamma)}(\Omega)$ as a weak solution of (3.1) if

$$(u, L^*v) = (f, v) \quad \text{all} \quad v \in \widehat{C}^\infty(\Omega), \tag{3.2}$$

where $(\cdot,\cdot)$ is the $L^2(\Omega)$ inner product and L^* is the formal adjoint of L.

Theorem 3.1. *Suppose that $g \in C^{m_*}$, $|w|_{C^4} < 1$ and $f \in H^{(m,1,\gamma)}(\Omega)$. If $m \le m_* - 4$ and $\delta = \delta(m)$, $\varepsilon = \varepsilon(m,\delta)$ are sufficiently small, then there exists a weak solution $u \in H^{(m,1,\gamma)}(\Omega)$ of* (3.1).

Proof. Given $v \in \widehat{C}^\infty(\Omega)$, let $\zeta \in H^{(m,\infty,\gamma+2)}(\Omega) \cap C^\infty(\Omega)$ be the unique solution of the ordinary differential equation (ODE):

$$\begin{aligned}
&\sum_{s=0}^{m} \lambda^{-s}(-1)^s \partial_r^s (a_{\lambda,\gamma-2(s-1)} \partial_r^s \zeta) = v,\\
&\zeta(r,0) = \zeta(r,\delta) = 0, \qquad \partial_r^s \zeta(\sigma,\theta) = 0, \qquad 0 \le s \le m-1,\\
&\int_{r=r_0} (\partial_r^s \partial_\theta^l \zeta)^2 \le r_0^{\gamma-2s} C, \quad 0 \le s \le 2m-1, \qquad 0 \le l < \infty.
\end{aligned} \tag{3.3}$$

Here $r_0 > 0$ is assumed to be sufficiently small, and $C > 0$ is a constant depending on m, λ, γ and v. The proof that such a solution exists may be found in Appendix A.

Our first goal is to establish an estimate of the form

$$\left(L\zeta, r^4 \sum_{s=0}^{m} \lambda^{-s}(-1)^s \partial_r^s (a_{\lambda,\gamma-2(s-1)} \partial_r^s \zeta) \right) \ge C \parallel \zeta \parallel_{(m,1,\gamma)}^2 . \tag{3.4}$$

The boundary conditions of (3.3) allow us to integrate by parts in a manner similar to the proof of Lemma 3.2 to find

$$\begin{aligned}
&\left(L\zeta, r^4 \sum_{s=0}^{m} \lambda^{-s}(-1)^s \partial_r^s (a_{\lambda,\gamma-2(s-1)} \partial_r^s \zeta) \right)\\
&\quad = \sum_{s=0}^{m} \lambda^{-s} ([\partial_r^s, L]\zeta + L\partial_r^s \zeta, r^4 a_{\lambda,\gamma-2(s-1)} \partial_r^s \zeta)\\
&\qquad + \sum_{s=0}^{m} \lambda^{-s} \left(\sum_{l<s} \binom{s}{l} \partial_r^{s-l} r^4 \partial_r^l L\zeta, a_{\lambda,\gamma-2(s-1)} \partial_r^s \zeta \right).
\end{aligned} \tag{3.5}$$

Note that since all the coefficients of L vanish at $r=\sigma$, except $\overline{B}$, no boundary terms at $r=\sigma$ appear in the above formula. Furthermore since $r^4 a_{\lambda,\gamma-2(s-1)} = a_{\lambda,\gamma-2(s+1)}$ and $\partial_r^s\zeta \in H^{(2,1,\gamma-2(s-1))}(\Omega)$, $0 \le s \le m-2$ (for $m-1 \le s \le m$ the boundary behavior of $\partial_r^s\zeta$ given by (3.3) is also adequate), Lemma 3.2 implies that

$$
\begin{aligned}
(3.6)\qquad & \sum_{s=0}^{m}(L\partial_r^s\zeta, \lambda^{-s}r^4 a_{\lambda,\gamma-2(s-1)}\partial_r^s\zeta) \\
&\ge C\int_\Omega \sum_{s=0}^{m}\lambda^{-s}[r^{-\gamma+2(s+1)}(\varphi\sin\theta(\partial_r^s\zeta)_r + r^{-1}\cos\theta(\partial_r^s\zeta)_\theta)^2 \\
&\quad + \lambda r^{-\gamma+2s}(\partial_r^s\zeta)^2] \\
&\ge C\int_\Omega\left[\sum_{s=0}^{m-1}\lambda^{-s}r^{-\gamma+2s}(\partial_r^s\zeta_\theta)^2 + \sum_{s=0}^{m}\lambda^{1-s}r^{-\gamma+2s}(\partial_r^s\zeta)^2\right] \\
&\ge C \parallel \zeta \parallel^2_{(m,1,\gamma)}
\end{aligned}
$$

if $\delta=\delta(\lambda)$ and $\varepsilon=\varepsilon(\delta)$ are sufficiently small. Next, we calculate

$$
\begin{aligned}
[\partial_r^s, L]\zeta = \sum_{l<s}\binom{s}{l}[&\partial_r^{s-l}\overline{K}(\partial_r^l\zeta)_{rr} + \partial_r^{s-l}\overline{A}(\partial_r^l\zeta)_{r\theta} \\
&+ \partial_r^{s-l}\overline{B}(\partial_r^l\zeta)_{\theta\theta} + \partial_r^{s-l}\overline{C}(\partial_r^l\zeta)_r + \partial_r^{s-l}\overline{D}(\partial_r^l\zeta)_\theta],
\end{aligned}
$$

and observe that integrating by parts, again with the help of the boundary conditions in (3.3), produces

(3.7)

$$
\begin{aligned}
&\left|\left(\sum_{l<s}\binom{s}{l}\partial_r^{s-l}\overline{K}(\partial_r^l\zeta)_{rr}, r^4 a_{\lambda,\gamma-2(s-1)}\partial_r^s\zeta\right)\right| \le C_s\int_\Omega\sum_{l\le s} r^{-\gamma+2l}(\partial_r^l\zeta)^2, \\
&\left|\left(\sum_{l<s}\binom{s}{l}\partial_r^{s-l}\overline{A}(\partial_r^l\zeta)_{r\theta}, r^4 a_{\lambda,\gamma-2(s-1)}\partial_r^s\zeta\right)\right| \le C_s\int_\Omega\left[r^{-\gamma+2s}(\partial_r^s\zeta)^2\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left.+\sum_{l<s}r^{-\gamma+2l}(\partial_r^l\zeta_\theta)^2\right], \\
&\left|\left(\sum_{l<s}\binom{s}{l}\partial_r^{s-l}\overline{B}(\partial_r^l\zeta)_{\theta\theta}, r^4 a_{\lambda,\gamma-2(s-1)}\partial_r^s\zeta\right)\right| \le C_s\int_\Omega\left[r^{-\gamma+2s}(\partial_r^s\zeta)^2\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left.+\sum_{l<s}r^{-\gamma+2l}(\partial_r^l\zeta_\theta)^2\right],
\end{aligned}
$$

$$\left|\left(\sum_{l<s}\binom{s}{l}\partial_r^{s-l}\overline{C}(\partial_r^l\zeta)_r, r^4 a_{\lambda,\gamma-2(s-1)}\partial_r^s\zeta\right)\right| \leq C_s\int_\Omega \sum_{l\leq s} r^{-\gamma+2l}(\partial_r^l\zeta)^2,$$

$$\left|\left(\sum_{l<s}\binom{s}{l}\partial_r^{s-l}\overline{D}(\partial_r^l\zeta)_\theta, r^4 a_{\lambda,\gamma-2(s-1)}\partial_r^s\zeta\right)\right| \leq C_s\int_\Omega \Bigg[r^{-\gamma+2s}(\partial_r^s\zeta)^2 + \sum_{l<s} r^{-\gamma+2l}(\partial_r^l\zeta_\theta)^2\Bigg].$$

Also observe that

$$\sum_{l<s}\binom{s}{l}\partial_r^{s-l}r^4\partial_r^l L\zeta = \sum_{s-4\leq l<s}\binom{s}{l}\frac{4!}{(4-s+l)!}r^{4-s+l}\Bigg[\sum_{t\leq l}\binom{l}{t}(\partial_r^{l-t}\overline{K}(\partial_r^t\zeta)_{rr} + \partial_r^{l-t}\overline{A}(\partial_r^t\zeta)_{r\theta} + \partial_r^{l-t}\overline{B}(\partial_r^t\zeta)_{\theta\theta} + \partial_r^{l-t}\overline{C}(\partial_r^t\zeta)_r + \partial_r^{l-t}\overline{D}(\partial_r^t\zeta)_\theta)\Bigg],$$

so similar calculations yield

$$\left|\left(\sum_{l<s}\binom{s}{l}\partial_r^{s-l}r^4\partial_r^l L\zeta, a_{\lambda,\gamma-2(s-1)}\partial_r^s\zeta\right)\right| \leq C_s\int_\Omega\Bigg[\sum_{l\leq s} r^{-\gamma+2l}(\partial_r^l\zeta)^2 + \sum_{l<s} r^{-\gamma+2l}(\partial_r^l\zeta_\theta)^2\Bigg]. \tag{3.8}$$

Then the combination of (3.5) to (3.8) produces (3.4) for $\lambda = \lambda(m)$ sufficiently large and $\delta = \delta(\lambda)$, $\varepsilon = \varepsilon(\delta)$ sufficiently small.

We need one last estimate before proving existence. Since $r^4\eta \in H^{(m,0,\gamma+2)}(\Omega)$ whenever $\eta \in H^{(m,1,\gamma)}(\Omega)$, we have

$$\begin{aligned}\| r^4 v \|_{(-m,-1,\gamma)} &:= \sup_{\eta\in H^{(m,1,\gamma)}(\Omega)} \frac{|(\eta, r^4 v)|}{\| \eta \|_{(m,1,\gamma)}} \\ &= \sup_{\eta\in H^{(m,1,\gamma)}(\Omega)} \frac{|(r^4(\lambda\theta^2-1)\eta, \zeta)_{(m,0,\gamma+2)}|}{\| \eta \|_{(m,1,\gamma)}} \\ &\leq C \sup_{\eta\in H^{(m,1,\gamma)}(\Omega)} \frac{\| \eta \|_{(m,0,\gamma)} \| \zeta \|_{(m,0,\gamma)}}{\| \eta \|_{(m,1,\gamma)}} \\ &\leq C \| \zeta \|_{(m,1,\gamma)} .\end{aligned} \tag{3.9}$$

Here $(\cdot,\cdot)_{(m,0,\gamma+2)}$ denotes the inner product on $H^{(m,0,\gamma+2)}(\Omega)$, and the norm $\|\cdot\|_{(-m,-1,\gamma)}$ comes from the dual space $H^{(-m,-1,\gamma)}(\Omega)$ of $H^{(m,1,\gamma)}(\Omega)$, which may be obtained as the completion of $L^2(\Omega)$ in this negative norm.

Now apply (3.4) to obtain

$$\begin{aligned}\| \zeta \|_{(m,1,\gamma)} \| L^* r^4 v \|_{(-m,-1,\gamma)} &\geq (\zeta, L^* r^4 v)\\ &= (L\zeta, r^4 v)\\ &= \left(L\zeta, r^4 \sum_{s=0}^{m} \lambda^{-s}(-1)^s \partial_r^s (a_{\lambda,\gamma-2(s-1)} \partial_r^s \zeta)\right)\\ &\geq C \| \zeta \|^2_{(m,1,\gamma)},\end{aligned}$$

which when combined with (3.9) yields

$$\| r^4 v \|_{(-m,-1,\gamma)} \leq C \| L^* r^4 v \|_{(-m,-1,\gamma)} . \tag{3.10}$$

Consider the linear functional $F : X \to \mathbb{R}$ where $X = L^*(r^4 \widehat{C}^\infty(\Omega))$, given by

$$F(L^* r^4 v) = (f, r^4 v).$$

According to (3.10) we have that F is bounded on the subspace X of $H^{(-m,-1,\gamma)}(\Omega)$ since

$$\begin{aligned}|F(L^* r^4 v)| = |(f, r^4 v)| &\leq \| f \|_{(m,1,\gamma)} \| r^4 v \|_{(-m,-1,\gamma)}\\ &\leq C \| f \|_{(m,1,\gamma)} \| L^* r^4 v \|_{(-m,-1,\gamma)} .\end{aligned}$$

Note that we use $f \in H^{(m,1,\gamma)}(\Omega)$ here (i.e., f has to have this particular boundary behavior at $r = 0$). Thus, we can apply the Hahn–Banach theorem to obtain a bounded extension of F (still denoted F) defined on all of $H^{(-m,-1,\gamma)}(\Omega)$. It follows that there exists a unique $u \in H^{(m,1,\gamma)}(\Omega)$ such that

$$F(\eta) = (u, \eta) \quad \text{all} \quad \eta \in H^{(-m,-1,\gamma)}(\Omega).$$

Now restrict η back to X to obtain

$$(u, L^* r^4 v) = (f, r^4 v) \quad \text{all} \quad v \in \widehat{C}^\infty(\Omega).$$

Since every $\overline{v} \in \widehat{C}^\infty(\Omega)$ can be written as $\overline{v} = r^4 v$ for some $v \in \widehat{C}^\infty(\Omega)$, u is a weak solution of (3.1). □

If $f \in H^{(m,1,\gamma)}(\Omega) \cap C^{m_*-4}(\Omega)$, then the strict ellipticity of L on the interior of Ω shows that the solution given by Theorem 3.1 satisfies $u \in$

$H^{(m,1,\gamma)}(\Omega) \cap C^{m_*-3}(\Omega)$, since the coefficients of L will be in C^{m_*-4} when $g \in C^{m_*}$. In particular $Lu = f$ pointwise on the interior of Ω, so that

$$-u_{\theta\theta} = \overline{B}^{-1}[\overline{K}u_{rr} + \overline{A}u_{r\theta} + \overline{C}u_r + \overline{D}u_\theta - f] \quad \text{in} \quad \Omega. \tag{3.11}$$

Since $\overline{B}^{-1} = O(r^2)$ as $r \to 0$ and $u \in H^{(m,1,\gamma)}(\Omega)$ it follows that $u_{\theta\theta} \in H^{(m-2,0,\gamma)}(\Omega)$. Let $H^m_\gamma(\Omega) := H^{(m,m,\gamma)}(\Omega)$ with norm $\| \cdot \|_{(m,\gamma)}$. Then if $f \in H^m_\gamma(\Omega)$ we may continue to differentiate the expression for $u_{\theta\theta}$ to eventually obtain (by induction) that $u \in H^m_\gamma(\Omega)$.

In order to determine the boundary values for u, integrate by parts in expression (3.2) to obtain

$$\begin{aligned} 0 = \int_{\partial\Omega} (&\overline{K}vu_r\nu_1 - (\overline{K}v)_r u\nu_1 + \overline{A}vu_\theta\nu_1 - (\overline{A}v)_r u\nu_2 \\ &+ \overline{B}vu_\theta\nu_2 - (\overline{B}v)_\theta u\nu_2 + \overline{C}vu\nu_1 + \overline{D}vu\nu_2), \end{aligned}$$

for all $v \in \widehat{C}^\infty(\Omega)$. This implies that $u(r,0) = u(r,\delta) = 0$, as $\overline{B} > 0$. It also shows that we cannot arbitrarily prescribe boundary values for u at $r = \sigma$, since all the coefficients of L (except $\overline{B}$) vanish at $r = \sigma$. However it is clear that the boundary values at $r = \sigma$ are given explicitly in terms of $f(\sigma,\theta)$ according to (3.11); although we will not have need of this fact. Moreover, the boundary behavior at $r = 0$ is completely determined by the fact that $u \in H^m_\gamma(\Omega)$, so that if $m \geq s_0 + 1$ and $\gamma > 2m$ then u will vanish to the desired order s_0 at $r = 0$ by Lemma 3.1. We summarize all that we have found with the following theorem, and also give an a priori estimate needed for the Nash-Moser iteration of Section 5.

Theorem 3.2. *Suppose that $g \in C^{m_*}$ and $f \in \overline{C}^\infty(\Omega)$. If $s_0 + 1 \leq m \leq m_* - 4$, $\gamma > 2m$, $|w|_{C^6} < 1$ and $\delta = \delta(m)$, $\varepsilon = \varepsilon(m,\delta)$ are sufficiently small then there exists a unique solution $u \in H^m_\gamma(\Omega) \cap C^{m_*-3}(\Omega)$ of (3.1). Furthermore, there exists a constant C_m independent of δ and ε such that*

$$\| u \|_{(m,\gamma)} \leq C_m(\| f \|_{m+2+\gamma} + \| w \|_{m+6} \| f \|_{5+\gamma}),$$

for each $m \leq m_* - 6$.

Proof. The first half of this theorem follows from the discussion directly above, and thus it only remains to establish the estimate. From Lemma 3.2

we have that

$$\int_\Omega r^{-\gamma+2}(\lambda u^2 + u_\theta^2) \le C \int_\Omega r^{-\gamma+6}(f^2 + r^{-2}u_r^2).$$

Now differentiate Equation (3.1) to obtain

$$Lu_r = f_r - \overline{K}_r u_{rr} - \overline{A}_r u_{r\theta} - \overline{B}_r u_{\theta\theta} - \overline{C}_r u_r - \overline{D}_r u_\theta.$$

Solving for $u_{\theta\theta}$ as in (3.11) then yields

$$\begin{aligned} L_1 u_r := &\ \overline{K}(u_r)_{rr} + \overline{A}(u_{r\theta})_r + \overline{B}(u_r)_{\theta\theta} \\ &+ (\overline{C} + \overline{K}_r - \overline{B}^{-1}\overline{B}_r\overline{K})(u_r)_r + (\overline{D} + \overline{A}_r - \overline{B}^{-1}\overline{B}_r\overline{A})(u_r)_\theta \\ = &\ f_r - (\overline{C}_r - \overline{B}^{-1}\overline{B}_r\overline{C})u_r - (\overline{D}_r - \overline{B}^{-1}\overline{B}_r\overline{D})u_\theta. \end{aligned}$$

Applying the proof of Lemma 3.2 to the above equation gives

$$C\int_\Omega [\lambda r^{-\gamma+4}u_r^2 + r^{-\gamma+6}(\varphi \sin\theta u_{rr} + r^{-1}\cos\theta u_{r\theta})^2] \le (a_{\lambda,\gamma-6}u_r, L_1 u_r),$$

from which we find that

$$\int_\Omega r^{-\gamma+4}(\lambda u_r^2 + u_{r\theta}^2) \le C\int_\Omega r^{-\gamma+6}(f^2 + r^2 f_r^2 + u_{rr}^2)$$

if λ is sufficiently large. Eventually, with the help of $|w|_{C^6} < 1$ and (3.11), we obtain

$$\| u \|_{(3,3,\gamma-2)} \ \le C \| f \|_{(3,3,\gamma-6)} .$$

By repeatedly differentiating with respect to r, we can continue this procedure and apply the Gagliardo–Nirenberg inequalities (Lemma 5.2 below) in the usual way to obtain

$$\begin{aligned} (3.12) \qquad & \int_\Omega \left(\lambda \sum_{s\le m} r^{-\gamma+2s+2}(\partial_r^s u)^2 + \sum_{s<m} r^{-\gamma+2s+2}(\partial_r^s u_\theta)^2 \right) \\ & \le C_m(\| f \|^2_{(m,0,\gamma-6)} + \Lambda_m^2 \| u \|^2_{(3,3,\gamma-2)}) \\ & \le C_m(\| f \|^2_{(m,\gamma)} + \Lambda_m^2 \| f \|^2_{(3,\gamma)}), \end{aligned}$$

where

$$\Lambda_m = \ \| r^2\overline{K} \|_m + \| r^2\overline{A} \|_m + \| r^2\overline{B} \|_m + \| r^2\overline{C} \|_m + \| r^2\overline{D} \|_m \ + 1.$$

To see how this works, we will show the calculation for one representative term that appears on the right-hand side of the equation after differentiating; the remaining terms may be treated similarly. Differentiate (3.1) s-times to obtain

$$\begin{aligned} L_s\partial_r^s u := {} & \overline{K}(\partial_r^s u)_{rr} + \overline{A}(\partial_r^s u)_{r\theta} + \overline{B}(\partial_r^s u)_{\theta\theta} \\ & + (\overline{C} + s\overline{K}_r - s\overline{B}^{-1}\overline{B}_r\overline{K})(\partial_r^s u)_r + (\overline{D} + s\overline{A}_r - s\overline{B}^{-1}\overline{B}_r\overline{A})(\partial_r^s u)_\theta \\ = {} & \partial_r^s f - \partial_r^{s-1}(\overline{C}_r u_r) + \cdots . \end{aligned}$$

Since $\partial_r^s u \in H^{(2,1,\gamma-2s)}(\Omega)$ when $s \le m_* - 6$, the basic estimate yields

$$\begin{aligned} & \int_\Omega r^{-\gamma+2s+2}[\lambda(\partial_r^s u)^2 + (\partial_r^s u_\theta)^2] \\ & \qquad \le C\int_\Omega r^{-\gamma+2s+6}[(\partial_r^s f)^2 + (\partial_r^{s-1}(\overline{C}_r u_r))^2] + \cdots . \end{aligned}$$

Furthermore, observe that

$$\partial_r^{s-1}(\overline{C}_r u_r) = \sum_{l\le s-1}\binom{s-1}{l}\partial_r^{s-1-l}\overline{C}_r\partial_r^l u_r,$$

and

$$\begin{aligned} \partial_r^{s-1-l}\overline{C}_r = \partial_r^{s-1-l}(r^{-2}r^2\overline{C}_r) &= \sum_{t_1\le s-1-l} C_{t_1} r^{-2-(s-1-l-t_1)}\partial_r^{t_1}(r^2\overline{C}_r), \\ r^{-\gamma/2+s+3+[-2-(s-1-l-t_1)]}\partial_r^l u_r &= \sum_{t_2\le l} C_{t_2}\partial_r^{t_2}(r^{-\gamma/2+t_1+t_2+2}u_r), \end{aligned}$$

for some constants C_{t_1} and C_{t_2}. Therefore we may apply Lemma 5.2(i), the Sobolev embedding theorem, and $|w|_{C^6} < 1$ to find

$$\begin{aligned} & \| r^{-\gamma/2+s+3}\partial_r^{s-1}(\overline{C}_r u_r) \| \\ & \quad \le \sum_{2\le t_1+t_2\le s-1} C_{t_1t_2} \| \partial_r^{t_1}(r^2\overline{C}_r)\partial_r^{t_2}(r^{-\gamma/2+t_1+t_2+2}u_r) \| \\ & \qquad + \sum_{t_1+t_2<2} C_{t_1t_2} \| \partial_r^{t_1}(r^2\overline{C}_r)\partial_r^{t_2}(r^{-\gamma/2+t_1+t_2+2}u_r) \| \\ & \quad \le C_s(|r^2\overline{C}_r|_\infty \| u_r \|_{(s-1,0,\gamma-4)} + \| r^2\overline{C}_r \|_{s-1} |r^{-\gamma/2+4}u_r|_\infty) \\ & \qquad + C_s \| u \|_{(2,2,\gamma-2)} \\ & \quad \le C_s(\| u \|_{(s,0,\gamma-2)} + \Lambda_s \| u \|_{(3,3,\gamma-2)}), \end{aligned}$$

where $|\cdot|_\infty$ denotes the $L^\infty(\Omega)$ norm. Note that the first term of the last line above, may be absorbed into the left-hand side of (3.12) for large $\lambda(s)$.

The remaining derivatives of u involving higher orders of ∂_θ may be estimated by differentiating (3.11) and using the above estimates. Since the coefficients of L depend on the derivatives of w up to and including order 3 and the derivatives of ξ (the coordinates of Lemma 2.1) up to and including order 2, with the help of (2.4) we obtain

$$\| u \|_{(m,\gamma-2)} \leq C_m(\| f \|_{(m,\gamma)} + \| w \|_{m+6} \| f \|_{(3,\gamma)}).$$

Lastly since f vanishes to all orders at $r=0$, a little calculation shows that

$$\int_\Omega r^{-\gamma+2s}(\partial_r^s \partial_\theta^t f)^2 \leq C_s \int_\Omega (\partial_r^{s+\gamma/2+1}\partial_\theta^t f)^2,$$

from which the desired result follows. □

4. Linear theory in the hyperbolic regions

Here we shall study the question of existence and regularity for the Cauchy problem associated with the operator $L(w)$ of Lemma 2.1, in the hyperbolic regions. In the previous section concerning the elliptic regions, after cutting-off some of the coefficients away from the origin, we were able to invert $L(w)$ in the appropriate function spaces. However in the hyperbolic case, we will not necessarily be able to make such an inversion, and as a result we must consider a regularized version of $L(w)$ as we explain below. For convenience let (x,y) denote the coordinates (ξ^1,ξ^2) of Lemma 2.1, so that a portion of the given hyperbolic region $\Omega_\varrho^- \cap B_\sigma(0)$, $\varrho = 1,2$, may be written as

$$\Omega = \{(x,y) \mid h(x) < y < \sigma\}$$

for some Lipschitz function $h(x)$ satisfying $h(0)=0$ and $|h(x)-|x||_{C^1} = O(\sigma)$. Then in these coordinates $L(w)$ is given by

$$Lu = (\overline{K}u_x)_x + u_{yy} + \overline{C}u_x + \overline{D}u_y$$

with

$$\overline{K} = \overline{k}K, \qquad \overline{C} = \overline{c}K + (a^{22})^{-2}\partial_{x^1}\xi^1\partial_{x^1}\Phi(w), \qquad \overline{D} = \varepsilon^2\overline{d},$$

in the notation of Lemma 2.1. Consider the Cauchy problem

(4.1)

$$Lu = f \quad \text{in} \quad \Omega, \quad u|_{\partial\Omega_1} = \phi, \quad u_y|_{\partial\Omega_1} = \psi, \quad \partial^\alpha u(0,0) = 0, \quad 0 \leq |\alpha| \leq \alpha_0,$$

for some given data ϕ, ψ and a large integer α_0, where $\partial\Omega_1$ denotes the "bottom" portion of the boundary given by $y = h(x)$. The "top" portion of the boundary will be denoted $\partial\Omega_2$, and is given by $y = \sigma$. As in problem (3.1) the solution is required to vanish to high order at the origin, in order to satisfy certain compatibility conditions which arise when constructing solutions to the nonlinear problem (2.1) (see Section 5). Of course, a necessary condition to have such behavior is that ϕ, ψ and f must also vanish to a corresponding high order at the origin.

Equation (4.1) is degenerate hyperbolic, and as such, solvability of the Cauchy problem depends on the so-called Levi conditions. These are relations between the coefficients $\overline{K}$ and $\overline{C}$, which if satisfies would guarantee existence for the Cauchy problem (when $y = h(x)$ is smooth and noncharacteristic). A simple example of such a relation is the condition (see [14])

$$\overline{C} \leq M\sqrt{|\overline{K}|} \quad \text{in } \Omega,$$

for some constant $M > 0$. Unfortunately, the quantity $\partial_{x^1}\Phi(w)$ present in $\overline{C}$ prevents the validity of this inequality. However in the Nash iteration of the next section, $\partial_{x^1}\Phi(w)$ will be uniformly small. Therefore, it is natural to consider the regularized Cauchy problem

$$\begin{aligned} &L_\theta u = f \quad \text{in} \quad \Omega, \quad u|_{\partial\Omega_1} = \phi, \quad u_y|_{\partial\Omega_1} = \psi, \\ &\partial^\alpha u(0,0) = 0, \quad 0 \leq |\alpha| \leq \alpha_0, \end{aligned} \tag{4.2}$$

where L_θ differs from L in that $\overline{K}$ is replaced by $\overline{K}_\theta := \overline{K} - \theta$ with $\theta = |\Phi(w)|_{C^1}$. There then exists a constant $M > 0$ such that

$$\overline{C} \leq M|\overline{K}_\theta| \quad \text{in} \quad \Omega. \tag{4.3}$$

However we cannot simply apply the results of [14] to obtain the desired solution of (4.2), since the Cauchy surface $y = h(x)$ is not smooth. We will therefore prove existence by hand in what follows. Note that with regards to the Nash iteration θ is of quadratic error (see Section 5), so that the small perturbation (4.2) will not affect convergence of this procedure.

It will be convenient to first establish an existence result for (4.2) with homogeneous Cauchy data and with f vanishing to high order on all of $\partial\Omega_1$. To this end, we define $H^{(m,l)}(\Omega)$ ($H_0^{(m,l)}(\Omega)$) to be the closure of all $C^\infty(\Omega)$

functions (which vanish to all orders at $\partial\Omega_1$) in the norm

$$\| u \|^2_{(m,l)} = \int_\Omega \sum_{\substack{0\le s\le m\\ 0\le t\le l}} \lambda^{-s}(\partial_x^s\partial_y^t u)^2,$$

where $\lambda > 0$ is a large parameter. The $L^2(\Omega)$ inner product will as usual be denoted by $(\cdot,\cdot)$, and the formal adjoint of L_θ by L_θ^*. Also as in Lemma 2.1 $w \in C^\infty$ throughout this section.

Theorem 4.1. *Suppose that* $g \in C^{m_*}$, $|w|_{C^4} < 1$ *and* $f \in H_0^{(m,0)}(\Omega)$. *If* $m \le m_* - 6$ *and* ε *is sufficiently small, then there exists a weak solution* $u_\theta \in H_0^{(m,1)}(\Omega)$ *of (4.2) with* $\phi,\psi = 0$, *for each* $\theta > 0$. *That is*

$$(u_\theta, L_\theta^* v) = (f, v) \quad \text{all} \quad v \in C^\infty(\Omega) \tag{4.4}$$

with $v|_{\partial\Omega_2} = v_y|_{\partial\Omega_2} = 0$.

Proof. Set $b(x,y) = \overline{K}_\theta^{-1}(x,y)\mathrm{e}^{-\lambda y}$ and let ζ be the unique solution of

$$\begin{aligned} \sum_{s=0}^{m}(-1)^{s+1}\lambda^{-s}\partial_x^s(b\partial_x^s\zeta_y) &= v \quad \text{in} \quad \Omega, \\ \zeta|_{\partial\Omega_1} = \partial_x^s\zeta_y|_{\partial\Omega_1} &= 0, \quad 0 \le s \le m-1, \end{aligned} \tag{4.5}$$

where v is as stated in the theorem. Note that for each $y \in (0,\sigma)$, this equation may be interpreted as an ODE in ζ_y, and therefore the theory of such equations guarantees the existence of a unique solution to (4.5). Furthermore if the metric $g \in C^{m_*}$ as in Lemma 2.1, then the coefficients of (4.5) are in C^{m_*-m-4}. Thus as long as $m \le m_* - 6$, we have $\partial_x^s\zeta \in C^2(\Omega)$ for $0 \le s \le 2m-1$.

We first note that the solution ζ of (4.5) satisfies extra boundary conditions, namely

$$\partial_x^s\partial_y^t\zeta|_{\partial\Omega_1} = 0, \qquad s+t \le m, \qquad 0 \le t \le 2. \tag{4.6}$$

To see this let $\partial_T\zeta$ be differentiation along the right portion of $\partial\Omega_1$ (that is, the curve $y = h(x)$, $x > 0$), which we denote by $\partial\Omega_1^+$. Then since

$\zeta|_{\partial\Omega_1^+} = \zeta_y|_{\partial\Omega_1^+} = 0$ we have

$$0 = \partial_T \zeta = \frac{1}{\sqrt{1+h'^2}}(\zeta_x + h'\zeta_y)|_{\partial\Omega_1^+} = \frac{\zeta_x}{\sqrt{1+h'^2}}|_{\partial\Omega_1^+},$$

where $h' = dh/dx$. It follows that

$$0 = \partial_T \zeta_x = \frac{1}{\sqrt{1+h'^2}}(\zeta_{xx} + h'\zeta_{xy})|_{\partial\Omega_1^+} = \frac{\zeta_{xx}}{\sqrt{1+h'^2}}|_{\partial\Omega_1^+},$$
$$0 = \partial_T \zeta_y = \frac{1}{\sqrt{1+h'^2}}(\zeta_{xy} + h'\zeta_{yy})|_{\partial\Omega_1^+} = \frac{h'\zeta_{yy}}{\sqrt{1+h'^2}}|_{\partial\Omega_1^+}.$$

Since $|h(x) - |x||_{C^1} = O(\sigma)$ this shows (4.6) for $m = 2$, and continuing this procedure establishes the full result on $\partial\Omega_1^+$. Also the same arguments hold to yield (4.6) on $\partial\Omega_1^-$, the left portion of $\partial\Omega_1$ (that is, the curve $y = h(x)$, $x < 0$).

We will now establish the basic estimate on which the existence proof is based. More precisely, we will show that

$$\left(L_\theta \zeta, \sum_{s=0}^{m} (-1)^{s+1} \lambda^{-s} \partial_x^s (b \partial_x^s \zeta_y) \right) \geq C \parallel \zeta \parallel_{(m,1)}^2, \tag{4.7}$$

for some constant $C > 0$. Observe that the following calculations hold for $0 \leq s \leq m$ according to the boundary conditions (4.6):

$$\begin{aligned}((\overline{K}_\theta \zeta_x)_x, (-1)^{s+1}\partial_x^s(b\partial_x^s\zeta_y)) &= \int_\Omega \partial_x(b\partial_x^s\zeta_y)\partial_x^s(\overline{K}_\theta\zeta_x) \\ &= -\int_\Omega \left[\frac{1}{2}(b\overline{K}_\theta)_y(\partial_x^{s+1}\zeta)^2 - b_x\overline{K}_\theta\partial_x^s\zeta_y\partial_x^{s+1}\zeta\right] \\ &\quad - \int_\Omega b\partial_x^s\zeta_y\partial_x\left[\sum_{l=1}^{s}\binom{s}{l}\partial_x^l\overline{K}_\theta\partial_x^{s+1-l}\zeta\right] \\ &\quad + \int_{\partial\Omega}\frac{1}{2}b\overline{K}_\theta(\partial_x^{s+1}\zeta)^2\nu_2,\end{aligned}$$

$$\begin{aligned}(\zeta_{yy}, (-1)^{s+1}\partial_x^s(b\partial_x^s\zeta_y)) &= -\int_\Omega b\partial_x^s\zeta_{yy}\partial_x^s\zeta_y + \int_{\partial\Omega} b\partial_x^{s-1}\zeta_{yy}\partial_x^s\zeta_y\nu_1 \\ &= \int_\Omega \frac{1}{2}b_y(\partial_x^s\zeta_y)^2 + \int_{\partial\Omega}\Big[b\partial_x^{s-1}\zeta_{yy}\partial_x^s\zeta_y\nu_1 \\ &\qquad - \frac{1}{2}b(\partial_x^s\zeta_y)^2\nu_2\Big],\end{aligned}$$

$$
\begin{aligned}
&(\overline{C}\zeta_x, (-1)^{s+1}\partial_x^s(b\partial_x^s\zeta_y))\\
&\quad = -\int_\Omega b\partial_x^s\zeta_y\partial_x^s(\overline{C}\zeta_x)\\
&\quad = -\int_\Omega \left[b\overline{C}\partial_x^s\zeta_y\partial_x^{s+1}\zeta + b\partial_x^s\zeta_y\sum_{l=1}^{s}\binom{s}{l}\partial_x^l\overline{C}\partial_x^{s+1-l}\zeta\right],
\end{aligned}
$$

$$
\begin{aligned}
&(\overline{D}\zeta_y, (-1)^{s+1}\partial_x^s(b\partial_x^s\zeta_y))\\
&\quad = -\int_\Omega b\partial_x^s\zeta_y\partial_x^s(\overline{D}\zeta_y)\\
&\quad = -\int_\Omega \left[b\overline{D}(\partial_x^s\zeta_y)^2 + b\partial_x^s\zeta_y\sum_{l=1}^{s}\binom{s}{l}\partial_x^l\overline{D}\partial_x^{s-l}\zeta_y\right],
\end{aligned}
$$

where (ν_1, ν_2) denotes the unit outer normal to $\partial\Omega$. Next observe that if $\lambda > 0$ is sufficiently large then

$$
b_y = -\frac{1}{\overline{K}_\theta}\left(\lambda + \frac{\partial_y\overline{K}_\theta}{\overline{K}_\theta}\right)\mathrm{e}^{-\lambda y} > 0, \quad -(b\overline{K}_\theta)_y = \lambda\mathrm{e}^{-\lambda y} > 0,
$$

and in light of the Levi condition (4.3) as well as $\overline{D} = O(\varepsilon)$ we have

$$
\begin{aligned}
&-(b_y - 2b\overline{D})(b\overline{K}_\theta)_y - (b_x\overline{K}_\theta - b\overline{C})^2\\
&\quad \geq -\frac{\mathrm{e}^{-2\lambda y}}{\overline{K}_\theta^2}\left[\left(\lambda + \frac{\partial_y\overline{K}_\theta}{\overline{K}_\theta} + O(\varepsilon)\right)\lambda\overline{K}_\theta + O(\overline{K}_x^2 + \overline{K}_\theta^2)\right]\\
&\quad \geq -\frac{\mathrm{e}^{-2\lambda y}}{\overline{K}_\theta^2}\left[\frac{\lambda^2}{2}\overline{K}_\theta + \lambda\overline{K}_y + O(\overline{K}_x^2)\right]\\
&\quad > \mathrm{e}^{-2\lambda y},
\end{aligned}
$$

as $\lambda\overline{K}_y + O(\overline{K}_x^2) \leq 0$ near $\partial\Omega_1$ for large λ. It follows that for large λ depending on m,

$$
\begin{aligned}
&\left(L_\theta\zeta, \sum_{s=0}^{m}(-1)^{s+1}\lambda^{-s}\partial_x^s(b\partial_x^s\zeta_y)\right)\\
&\quad \geq C \parallel \zeta \parallel_{(m,1)}^2 + \frac{\lambda^{-m}}{2}\int_{\partial\Omega}(b\overline{K}_\theta(\partial_x^{m+1}\zeta)^2\nu_2\\
&\qquad + 2b\partial_x^{m-1}\zeta_{yy}\partial_x^m\zeta_y\nu_1 - b(\partial_x^m\zeta_y)^2\nu_2),
\end{aligned}
$$

where we have used a Poincaré type inequality to estimate $\parallel \zeta \parallel_{L^2(\Omega)}$. Note that the boundary integral has the correct sign on $\partial\Omega_2$. We claim that it also has the correct sign on $\partial\Omega_1$. To see this we use the boundary condition (4.6). That is, by (4.6) $\partial_x^{m-1}\zeta_y|_{\partial\Omega_1} = 0$ so if ∂_T is differentiation along $\partial\Omega_1 - \{(0,0)\}$ then

$$0 = \partial_T\partial_x^{m-1}\zeta_y = -\nu_2\partial_x^m\zeta_y + \nu_1\partial_x^{m-1}\zeta_{yy},$$

which yields

$$b\partial_x^{m-1}\zeta_{yy}\partial_x^m\zeta_y\nu_1 = b(\partial_x^m\zeta_y)^2\nu_2 \geq 0.$$

Moreover since $\partial_x^m\zeta|_{\partial\Omega_1} = 0$ we have

$$0 = \partial_T\partial_x^m\zeta|_{\partial\Omega_1} = -\nu_2\partial_x^{m+1}\zeta + \nu_1\partial_x^m\zeta_y,$$

which yields

$$b\overline{K}_\theta(\partial_x^{m+1}\zeta)^2\nu_2 = b\overline{K}_\theta(\partial_x^m\zeta_y)^2\frac{\nu_1^2}{\nu_2}.$$

It follows that the boundary integral on $\partial\Omega_1$ is nonnegative, and hence (4.7) holds.

We will need one last estimate before proving existence. Namely

$$\begin{aligned}(4.8)\qquad \| v \|_{(-m,0)} &:= \sup_{\eta\in H_0^{(m,0)}(\Omega)} \frac{|(\eta, v)|}{\| \eta \|_{(m,0)}} \\ &= \sup_{\eta\in H_0^{(m,0)}(\Omega)} \frac{|(\eta, \sum_{s=0}^m(-1)^{s+1}\lambda^{-s}\partial_x^s(b\partial_x^s\zeta_y))|}{\| \eta \|_{(m,0)}} \\ &\leq \theta^{-1}C \| \zeta \|_{(m,1)},\end{aligned}$$

which follows after integration by parts. Here $\| \cdot \|_{(-m,0)}$ is the norm on the dual space $H_0^{(-m,0)}(\Omega)$ of $H_0^{(m,0)}(\Omega)$. This dual space may be obtained as the completion of $L^2(\Omega)$ in the norm $\| \cdot \|_{(-m,0)}$.

To prove existence apply (4.7) to obtain

$$\begin{aligned}\| \zeta \|_{(m,1)} \| L_\theta^* v \|_{(-m,-1)} &\geq (\zeta, L_\theta^* v) = (L_\theta\zeta, v) \\ &= \left(L_\theta\zeta, \sum_{s=0}^m(-1)^{s+1}\lambda^{-s}\partial_x^s(b\partial_x^s\zeta_y)\right) \geq C \| \zeta \|_{(m,1)}^2,\end{aligned}$$

which together with (4.8) implies that

$$(4.9)\qquad \| v \|_{(-m,0)} \leq \theta^{-1}C \| L_\theta^* v \|_{(-m,-1)} \quad \text{all} \quad v \in \widehat{C}^\infty(\Omega),$$

where $\widehat{C}^\infty(\Omega)$ consists of all $C^\infty(\Omega)$ functions with $v|_{\partial\Omega_2} = v_y|_{\partial\Omega_2} = 0$. Consider the linear functional $F : X \to \mathbb{R}$, where $X = L_\theta^* \widehat{C}^\infty(\Omega)$, given by

$$F(L_\theta^* v) = (f, v).$$

According to (4.9) we have that F is bounded on the subspace X of $H_0^{(-m,-1)}(\Omega)$ since

$$|F(L_\theta^* v)| \leq \| f \|_{(m,0)} \| v \|_{(-m,0)} \leq \theta^{-1} C \| f \|_{(m,0)} \| L_\theta^* v \|_{(-m,-1)} .$$

Note that the generalized Schwarz inequality (the first inequality in the above sequence) holds because $f \in H_0^{(m,0)}(\Omega)$, that is, f vanishes appropriately on $\partial\Omega_1$. Thus we can apply the Hahn–Banach theorem to obtain a bounded extension of F defined on all of $H_0^{(-m,-1)}(\Omega)$. It follows that there exists $u_\theta \in H_0^{(m,1)}(\Omega)$ such that

$$F(\eta) = (u_\theta, \eta) \quad \text{all} \quad \eta \in H_0^{(-m,-1)}(\Omega).$$

Now restrict η back to X to obtain

$$(u_\theta, L_\theta^* v) = (f, v) \quad \text{all} \quad v \in \widehat{C}^\infty(\Omega).$$

□

In order to obtain higher regularity for the solution given by Theorem 4.1, we will utilize the following standard lemma concerning the difference quotient:

$$u^q(x, y) := \frac{u(x, y + q) - u(x, y)}{q}.$$

Lemma 4.1. (i) *Let $u \in H^{(0,1)}(\Omega)$ and $\Omega' \subset\subset \Omega$ (that is, Ω' is compactly contained in Ω). Then*

$$\| u^q \|_{L^2(\Omega')} \leq \| u_y \|_{L^2(\Omega)}$$

for all $0 < |q| < \frac{1}{2}\text{dist}(\Omega', \partial\Omega)$.

(ii) *If $u \in L^2(\Omega)$ and $\| u^q \|_{L^2(\Omega')} \leq C$ for all $0 < |q| < \frac{1}{2}\text{dist}(\Omega', \partial\Omega)$, then $u \in H^{(0,1)}(\Omega')$.*

The Sobolev space of square integrable derivatives up to and including order m will be denoted by $H^m(\Omega)$ with norm $\| \cdot \|_m$, and the completion of

$C^\infty(\Omega)$ functions which vanish to all orders at $\partial\Omega_1$ in the norm $\|\cdot\|_m$ shall be denoted by $H_0^m(\Omega)$.

Corollary 4.1. *Under the hypotheses of Theorem* 4.1, *if* $f \in H_0^m(\Omega)$ *there exists a unique solution* $u_\theta \in H_0^m(\Omega)$ *of* (4.2) *with* $\phi, \psi = 0$, *for each* $\theta > 0$.

Proof. Let $u_\theta \in H_0^{(m,1)}(\Omega)$ be the weak solution given by Theorem 4.1, so that (4.4) holds. If $m \le 1$ then this corollary follows directly from Theorem 4.1, so assume that $m \ge 2$. We may integrate by parts to obtain

$$\begin{aligned}-(\partial_y u_\theta + \overline{D}u_\theta, v_y) &= (f - \partial_x(\overline{K}_\theta \partial_x u_\theta) - \overline{C}\partial_x u_\theta + \overline{D}_y u_\theta, v) \\ &\quad + \int_{\partial\Omega} (\overline{C} v u_\theta \nu_1 - v_y u_\theta \nu_2 - \overline{K}_\theta v_x u_\theta \nu_1 + \overline{K}_\theta v \partial_x u_\theta \nu_1)\end{aligned}$$

for all $v \in \widehat{C}^\infty(\Omega)$, where (ν_1, ν_2) is the unit outer normal to $\partial\Omega$. Note that since $u_\theta, \partial_x u_\theta \in H^1(\Omega)$ both $u_\theta|_{\partial\Omega}$ and $\partial_x u_\theta|_{\partial\Omega}$ are meaningful in $L^2(\partial\Omega)$, and in particular as $u_\theta, \partial_x u_\theta \in H_0^1(\Omega)$ we have $u_\theta|_{\partial\Omega_1} = \partial_x u_\theta|_{\partial\Omega_1} = 0$ in the $L^2(\partial\Omega_1)$ sense. Thus we may write

$$(\overline{u}_\theta, v_y) = (\overline{f}, v) \quad \text{all} \quad v \in \widehat{C}^\infty(\Omega),$$

where

$$\overline{u}_\theta = -\partial_y u_\theta - \overline{D}u_\theta, \qquad \overline{f} = f - \partial_x(\overline{K}_\theta \partial_x u_\theta) - \overline{C}\partial_x u_\theta + \overline{D}_y u_\theta.$$

Furthermore

$$(\overline{u}_\theta^q, v_y) = (\overline{f}^q, v) \quad \text{all} \quad v \in C_c^\infty(\Omega),$$

so that choosing a sequence $v_i \in C_c^\infty(\Omega)$ with $v_i \to -\eta u_\theta^q$ in $H^{(0,1)}(\Omega)$ for some nonnegative $\eta \in C_c^\infty(\Omega)$, implies that

$$\begin{aligned}\| \sqrt{\eta}\overline{u}_\theta^q \|^2 &\le |(\overline{f}^q, \eta u_\theta^q)| + |(\overline{u}_\theta^q, \eta_y u_\theta^q)| + |(\overline{u}_\theta^q, \eta(\overline{D}u_\theta)^q)| \\ &\le \| \sqrt{\eta}\overline{f}^q \| \| \sqrt{\eta} u_\theta^q \| + \| \sqrt{\eta}\overline{u}_\theta^q \| \| \frac{\eta_y}{\sqrt{\eta}} u_\theta^q \| \\ &\quad + \| \sqrt{\eta}\overline{u}_\theta^q \| \| \sqrt{\eta}(\overline{D}u_\theta)^q \| .\end{aligned}$$

Then since $u_\theta, \overline{f} \in H^{(0,1)}(\Omega)$ and $|\nabla\eta|^2 \le C\eta$, Lemma 4.1(i) yields $\| \sqrt{\eta}\overline{u}_\theta^q \| \le C$ for some constant C independent of q, if $|q|$ is sufficiently small. Now Lemma 4.1(ii) shows that $\overline{u}_\theta \in H_{\text{loc}}^{(0,1)}(\Omega)$, as η was arbitrary. Hence $\partial_y^2 u_\theta \in L_{\text{loc}}^2(\Omega)$. It follows that the equation $L_\theta u_\theta = f$ holds in $L_{loc}^2(\Omega)$, and since we can solve for $\partial_y^2 u_\theta$, we can boot-strap in the usual way to obtain $u_\theta \in H^m_{\text{loc}}(\Omega)$.

Next observe that the above restrictions on η may be relaxed if $q > 0$, that is in this case η is only required to vanish in a neighborhood of $\partial\Omega_2$. Then the same procedure yields $\eta u_\theta \in H^m(\Omega)$ for all such η. Furthermore since L_θ is strictly hyperbolic, we can use the regularity theory for such operators to obtain estimates for u_θ near $\partial\Omega_2$. It follows that $u_\theta \in H^m(\Omega)$.

To show that $u_\theta \in H^m_0(\Omega)$, it is enough to observe that

$$\partial_x^s \partial_y^t u_\theta|_{\partial\Omega_1} = 0, \qquad s + t \leq m - 1, \tag{4.10}$$

where the equality is interpreted in the $L^2(\partial\Omega_1)$ sense when $s + t = m - 1$. This follows from the fact that f satisfies (4.10), in the following way. First note that as in the arguments used to establish (4.6), $u_\theta|_{\partial\Omega_1} = \partial_x u_\theta|_{\partial\Omega_1} = 0$ implies that $\partial_y u_\theta|_{\partial\Omega_1} = 0$. Furthermore using the notation of those arguments we have

$$\begin{aligned} 0 = \partial_T(\partial_x u_\theta) &= \frac{1}{\sqrt{1+h'^2}}(\partial_x^2 u_\theta + h'\partial_x\partial_y u_\theta)|_{\partial\Omega_1^+}, \\ 0 = \partial_T(\partial_y u_\theta) &= \frac{1}{\sqrt{1+h'^2}}(\partial_x\partial_y u_\theta + h'\partial_y^2 u_\theta)|_{\partial\Omega_1^+}. \end{aligned}$$

However from Equation (4.2) we find that

$$(-\theta\partial_x^2 u_\theta + \partial_y^2 u_\theta)|_{\partial\Omega_1^+} = 0,$$

hence

$$\partial_x^2 u_\theta|_{\partial\Omega_1^+} = \partial_y^2 u_\theta|_{\partial\Omega_1^+} = \partial_x\partial_y u_\theta|_{\partial\Omega_1^+} = 0.$$

The same arguments also apply to $\partial\Omega_1^-$, so by differentiating Equation (4.2) we can continue this procedure to obtain (4.10).

Lastly we note that since $u_\theta|_{\partial\Omega_1} = |\nabla u_\theta|_{\partial\Omega_1} = 0$, (4.7) with $m = 1$ yields

$$\left(L_\theta u_\theta, \sum_{s=0}^{1} (-1)^{s+1}\lambda^{-s}\partial_x^s(b\partial_x^s\partial_y u_\theta) \right) \geq C \parallel u_\theta \parallel^2_{(1,1)},$$

from which uniqueness follows. □

This corollary yields the existence of a regular solution to (4.2) when $\phi = \psi = 0$ and f vanishes to high order on $\partial\Omega_1$. However, we are interested in solving (4.2) in the general case when ϕ, ψ and f vanish to high order at the origin but are otherwise arbitrary. The next lemma will enable us to obtain the general case from Corollary 4.1. Here and below $\overline{H}_0^m(\Omega)$ will denote the completion of $\overline{C}^\infty(\Omega)$ in the norm $\|\cdot\|_m$, and $\overline{H}_0^m(\partial\Omega_1)$ will be defined similarly with respect to the $\|\cdot\|_{m,\partial\Omega_1}$ norm. Recall that $\overline{C}^\infty(\Omega)$ consists of $C^\infty(\Omega)$ functions which vanish in a neighborhood of the origin.

Lemma 4.2. *Suppose that $g \in C^{m_*}$, $|w|_{C^6} < 1$ and $\phi \in \overline{H}_0^{m+1}(\partial\Omega_1)$, $\psi \in \overline{H}_0^m(\partial\Omega_1)$, $f \in \overline{H}_0^m(\Omega)$ with $m \le m_* - 6$. Then there exists a function $\eta_\theta \in \overline{H}_0^{m+2}(\Omega)$ such that $\eta_\theta|_{\partial\Omega_1} = \phi$, $\partial_y\eta_\theta|_{\partial\Omega_1} = \psi$ and $\partial_y^t(f - L_\theta\eta_\theta)|_{\partial\Omega_1} = 0$, $0 \le t \le m-1$, with*

$$\begin{aligned}\| \eta_\theta \|_{m+2} \le C_m(\| f \|_m + \| \phi \|_{m+1,\partial\Omega_1} + \| \psi \|_{m,\partial\Omega_1} \\ + \| w \|_{m+6} (\| f \|_2 + \| \phi \|_{2,\partial\Omega_1} + \| \psi \|_{2,\partial\Omega_1})).\end{aligned}$$

Proof. We may assume that a unique solution $u_\theta \in \overline{H}_0^{m+2}(\Omega)$ of (4.2) exists, since here we shall only use its boundary values which can be explicitly determined in terms of ϕ, ψ and f, as $\partial\Omega_1$ is noncharacteristic for L_θ. Then because Ω is a Lipschitz domain, the linear restriction map $H^{m+2}(\Omega) \to H^{m+1}(\partial\Omega_1)$ is bounded and onto (see [18]). By quotienting with the kernel and applying the closed graph theorem, we obtain a bounded inverse $H^{m+1}(\partial\Omega_1) \to H^{m+2}(\Omega)/H_0^{m+2}(\Omega)$ with respect to the quotient norm. We may then use this map to obtain an extension η_θ of u_θ from $\partial\Omega_1$ to Ω with $\partial^\alpha\eta_\theta|_{\partial\Omega_1} = \partial^\alpha u_\theta|_{\partial\Omega_1}$ for all $|\alpha| \le m+1$, and

$$\| \eta_\theta \|_{m+2} \le C_m \sum_{|\alpha|\le m+1} \| \partial^\alpha u_\theta \|_{0,\partial\Omega_1} .$$

Applying the Gagliardo–Nirenberg inequalities (Lemma 5.2) to the expression for $\partial^\alpha u_\theta|_{\partial\Omega_1}$ in terms of ϕ, ψ and f yields the desired estimate.

We remark that an equivalent and perhaps more concrete way to obtain the extension is as the unique weak solution $\eta_\theta \in \overline{H}_0^{m+2}(\Omega)$ of the boundary value problem:

$$\sum_{s=0}^{m+2}(-1)^s\Delta^s\eta_\theta = 0 \quad \text{in} \quad \Omega, \quad \partial_y^s\eta_\theta|_{\partial\Omega_1} = \partial_y^s u_\theta|_{\partial\Omega_1}, \qquad 0 \le s \le m+1,$$

$$\left(\sum_{l=s}^{m+1-s}(-1)^l\partial_y\Delta^l\eta_\theta\right)_{\partial\Omega_2} = 0, \qquad 0 \le s \le \left[\frac{m+1}{2}\right],$$

$$\left(\sum_{l=s+1}^{m+1-s}(-1)^l\Delta^l\eta_\theta\right)_{\partial\Omega_2} = 0, \qquad 0 \le s \le \left[\frac{m+(-1)^m}{2}\right]. \qquad \square$$

Now in order to solve (4.2) with $\phi \in \overline{H}_0^{m+1}(\partial\Omega_1)$, $\psi \in \overline{H}_0^{m}(\partial\Omega_1)$, and $f \in \overline{H}_0^{m}(\Omega)$, we note that if $\eta_\theta \in \overline{H}_0^{m+2}(\Omega)$ is as in Lemma 4.2 and $\overline{u}_\theta \in H_0^m(\Omega)$ is given by Corollary 4.1 with $\overline{f}_\theta = f - L_\theta\eta_\theta \in H_0^m(\Omega)$, then $u_\theta = \overline{u}_\theta + \eta_\theta \in \overline{H}_0^{m}(\Omega)$ satisfies (4.2). Observe that in order for u_θ to have the desired vanishing at the origin we require $m \geq \alpha_0 + 2$.

Our next task shall be to estimate u_θ independent of θ, in a manner suitable for application to the Nash iteration of the next section. A significant difference between the following theorem and its analog for the elliptic regions (Theorem 3.2), is that the loss of derivatives here depends on the degree to which the Gaussian curvature vanishes at the origin.

Theorem 4.2. *Suppose that $g \in C^{m_*}$, $\phi, \psi \in \overline{C}^{\infty}(\partial\Omega_1)$, $f \in \overline{C}^{\infty}(\Omega)$ and $|w|_{C^{2N+4}} < 1$ where $N \leq m_* - 2$ is the largest integer such that $\partial^\alpha K(0,0) = 0$ for all $|\alpha| \leq N$. If $\alpha_0 + 2 \leq m \leq m_* - 6$ and $\varepsilon = \varepsilon(m)$ is sufficiently small, then there exists a unique solution $u_\theta \in \overline{H}_0^{m}(\Omega)$ of (4.2) for each $\theta > 0$. Furthermore, there exists a constant C_m independent of ε and θ such that*

$$\begin{aligned}\| u_\theta \|_m \leq C_m(&\| f \|_{m+N} + \| \phi \|_{m+N+1,\partial\Omega_1} + \| \psi \|_{m+N,\partial\Omega_1}\\ &+ \| w \|_{m+N+6} \left(\| f \|_{N+2} + \| \phi \|_{N+3,\partial\Omega_1} + \| \psi \|_{N+2,\partial\Omega_1}\right))\end{aligned}$$

for each $m \leq m_* - N - 8$.

Proof. The existence of a solution $u_\theta \in \overline{H}_0^{m_*-6}(\Omega)$ for each $\theta > 0$ follows directly from the discussion preceding the statement of this theorem. In order to make estimates it will be advantageous to have a zeroth-order term for L_θ. Therefore we set $v_\theta = \mathrm{e}^{-\frac{1}{2}y^2}u_\theta$ and observe that

$$\begin{aligned}\overline{L}_\theta v_\theta := \partial_x(\overline{K}_\theta \partial_x v_\theta) + \partial_y^2 v_\theta + \overline{C}\partial_x v_\theta + (2y + \overline{D})\partial_y v_\theta\\ + (1 + y^2 + y\overline{D})v_\theta = \mathrm{e}^{-\frac{1}{2}y^2} f := \overline{f}.\end{aligned}$$

With the aim of treating the x-derivatives first, we differentiate the above equation to find

$$\begin{aligned}\overline{L}_\theta^{(m)} \partial_x^m v_\theta = \partial_x^m \overline{f} - \sum_{s=3}^{m} \binom{m}{s} \partial_x^s \overline{K}_\theta \partial_x^{m-s}(\partial_x^2 v_\theta)\\ - \sum_{s=2}^{m} \binom{m}{s} \partial_x^s(\overline{C} + \partial_x \overline{K}_\theta)\partial_x^{m-s}(\partial_x v_\theta)\end{aligned}$$

$$
\begin{aligned}
&-\sum_{s=1}^{m}\binom{m}{s}\left[\partial_x^s\overline{D}\partial_x^{m-s}(\partial_y v_\theta)+y\partial_x^s\overline{D}\partial_x^{m-s}v_\theta\right]\\
&:=\partial_x^m\overline{f}+\overline{f}^{(m)}(v_\theta),
\end{aligned}
$$

where

$$
\begin{aligned}
\overline{L}_\theta^{(m)}v := {}& (\overline{K}_\theta v_x)_x+v_{yy}+(\overline{C}+m\partial_x\overline{K}_\theta)v_x+(2y+\overline{D})v_y\\
&+\left(1+y^2+y\overline{D}+m\partial_x\overline{C}+\frac{m(m+1)}{2}\partial_x^2\overline{K}_\theta\right)v.
\end{aligned}
$$

We first assume that $m\le N+1$. In this case let η_θ be given by Lemma 4.2 such that $\eta_\theta|_{\partial\Omega_1}=v_\theta|_{\partial\Omega_1}$, $\partial_y\eta_\theta|_{\partial\Omega_1}=\partial_y v_\theta|_{\partial\Omega_1}$ and

$$
\partial_y^t(\overline{f}-\overline{L}_\theta\eta_\theta)|_{\partial\Omega_1}=0,\qquad 0\le t\le m+N. \tag{4.11}
$$

Note that this implies that $\eta_\theta\in\overline{H}_0^{m+N+3}(\Omega)$, $\partial^\alpha\eta_\theta|_{\partial\Omega_1}=\partial^\alpha v_\theta|_{\partial\Omega_1}$ for all $|\alpha|\le m+N+2$, and we have the estimate

$$
\begin{aligned}
\|\,\eta_\theta\,\|_{m+N+3}\le {}& C(\|\,f\,\|_{m+N+1}+\|\,\phi\,\|_{m+N+2,\partial\Omega_1}+\|\,\psi\,\|_{m+N+1,\partial\Omega_1}\\
&+\|\,w\,\|_{m+N+7}\,(\|\,f\,\|_2+\|\,\phi\,\|_{2,\partial\Omega_1}+\|\,\psi\,\|_{2,\partial\Omega_1})).
\end{aligned}\tag{4.12}
$$

Furthermore, the function $\overline{v}_\theta:=v_\theta-\eta_\theta$ satisfies

$$
\overline{L}_\theta^{(m)}\partial_x^m\overline{v}_\theta=\partial_x^m(\overline{f}-\overline{L}_\theta\eta_\theta)+\overline{f}^{(m)}(\overline{v}_\theta).
$$

As in the proof of Theorem 4.1, we set $b=\overline{K}_\theta^{-1}\mathrm{e}^{-\lambda y}$ and integrate by parts:

$$
\begin{aligned}
&(-b\partial_y\partial_x^m\overline{v}_\theta,\overline{L}_\theta^{(m)}\partial_x^m\overline{v}_\theta)\\
&\quad=\int_\Omega\left[-\frac{1}{2}(b\overline{K}_\theta)_y(\partial_x^{m+1}\overline{v}_\theta)^2+(b_x\overline{K}_\theta-mb\partial_x\overline{K}_\theta-b\overline{C})\partial_x^{m+1}\overline{v}_\theta\partial_y\partial_x^m\overline{v}_\theta\right]\\
&\qquad+\int_\Omega\left(\frac{1}{2}b_y-b(2y+\overline{D})\right)(\partial_y\partial_x^m\overline{v}_\theta)^2\\
&\qquad+\int_\Omega\frac{1}{2}\left[b\left(1+y^2+y\overline{D}+m\partial_x\overline{C}+\frac{m(m+1)}{2}\partial_x^2\overline{K}_\theta\right)\right]_y(\partial_x^m\overline{v}_\theta)^2\\
&\qquad+\int_{\partial\Omega}\left[\frac{1}{2}b\overline{K}_\theta(\partial_x^{m+1}\overline{v}_\theta)^2\nu_2-b\overline{K}_\theta\partial_x^{m+1}\overline{v}_\theta\partial_y\partial_x^m\overline{v}_\theta\nu_1\right]
\end{aligned}\tag{4.13}
$$

$$- \int_{\partial\Omega} \left[\frac{1}{2} b (\partial_y \partial_x^m \overline{v}_\theta)^2 \nu_2 + \frac{1}{2} b \right.$$
$$\left. \times \left(1 + y^2 + y\overline{D} + m\partial_x \overline{C} + \frac{m(m+1)}{2} \partial_x^2 \overline{K}_\theta \right) (\partial_x^m \overline{v}_\theta)^2 \nu_2 \right].$$

The boundary integral along $\partial\Omega_2$ is nonnegative, and according to the choice of η_θ it vanishes along $\partial\Omega_1$. Moreover, the same calculations as in the proof of Theorem 4.1 apply to the interior integral to yield

$$\lambda(\| \partial_x^{m+1} \overline{v}_\theta \| + \| \sqrt{|b|} \partial_y \partial_x^m \overline{v}_\theta \| + \| \sqrt{|b|} \partial_x^m \overline{v}_\theta \|)$$
$$\leq C(\| \sqrt{|b|} \partial_x^m (\overline{f} - \overline{L}_\theta \eta_\theta) \| + \| \sqrt{|b|} \overline{f}^{(m)} (\overline{v}_\theta) \|).$$

We proceed to estimate each term on the right-hand side separately. First note that since $m \leq N+1$ and $|w|_{C^{N+4}} < 1$, we have

$$\| \sqrt{|b|} \overline{f}^{(m)} (\overline{v}_\theta) \|^2 \leq C_m \int_\Omega \sum_{s=0}^{m-1} \mathrm{e}^{-\lambda y} |\overline{K}_\theta|^{-1} [(\partial_x^s \overline{v}_\theta)^2 + (\partial_y \partial_x^s \overline{v}_\theta)^2].$$

Next observe that since $\overline{K}$ vanishes (at most) to order N at the origin, there exists a constant $C_0 > 0$ such that $|\overline{K}| \geq C_0^{-1} (y - h(x))^{N+1}$ in Ω. Then in light of (4.11), a little calculation shows that

$$\| \sqrt{|b|} \partial_x^m (\overline{f} - \overline{L}_\theta \eta_\theta) \|^2 \leq C_0 \int_\Omega \frac{[\partial_x^m (\overline{f} - \overline{L}_\theta \eta_\theta)]^2}{(y - h(x))^{N+1}}$$
$$\leq C_1 \int_\Omega [\partial_y^{N+1} \partial_x^m (\overline{f} - \overline{L}_\theta \eta_\theta)]^2.$$

It follows that applying (4.12) and summing from 0 to m produces

$$\text{(4.14)} \qquad \sum_{s=0}^{m+1} \| \partial_x^s \overline{v}_\theta \| + \sum_{s=0}^{m} \| \sqrt{|b|} \partial_y \partial_x^s \overline{v}_\theta \| + \sum_{s=0}^{m} \| \sqrt{|b|} \partial_x^s \overline{v}_\theta \|$$
$$\leq C_m (\| f \|_{m+N+1} + \| \phi \|_{m+N+2, \partial\Omega_1} + \| \psi \|_{m+N+1, \partial\Omega_1}$$
$$+ \| w \|_{m+N+7} (\| f \|_2 + \| \phi \|_{2, \partial\Omega_1} + \| \psi \|_{2, \partial\Omega_1})),$$

if λ is sufficiently large. From this inequality we may obtain an estimate for the x-derivatives of v_θ (and hence for u_θ), with the help of (4.12). In addition, by solving for $\partial_y^2 u_\theta$ in Equation (4.2) all remaining derivatives up to and including order m may also be estimated.

We now assume that $m \geq N+2$. In order to isolate terms involving high-order derivatives of w we break $\overline{f}^{(m)}$ into two parts. Let $v_\theta^{(s)} := \partial_x^s v_\theta - \eta_\theta^{(s)}$ with $\eta_\theta^{(s)}$ to be given below, then $\overline{f}^{(m)} = \overline{f}_1^{(m)} + \overline{f}_2^{(m)}$ where

$$\begin{aligned}
\overline{f}_1^{(m)}(v_\theta) = &- \sum_{s=N+2}^{m} \binom{m}{s} \partial_x^s \overline{K}_\theta \partial_x^{m-s+2} v_\theta \\
&- \sum_{s=N+1}^{m} \binom{m}{s} \partial_x^s (\overline{C} + \partial_x \overline{K}_\theta) \partial_x^{m-s+1} v_\theta \\
&- \sum_{s=N+1}^{m} \binom{m}{s} \partial_x^s \overline{D} \partial_y \partial_x^{m-s} v_\theta \\
&- y \sum_{s=N}^{m} \binom{m}{s} \partial_x^s \overline{D} \partial_x^{m-s} v_\theta - \sum_{s=3}^{N+1} \binom{m}{s} \partial_x^s \overline{K}_\theta \eta_\theta^{(m-s+2)} \\
&- \sum_{s=2}^{N} \binom{m}{s} \partial_x^s (\overline{C} + \partial_x \overline{K}_\theta) \eta_\theta^{(m-s+1)} \\
&- \sum_{s=1}^{N} \binom{m}{s} \partial_x^s \overline{D} \partial_y \eta_\theta^{(m-s)} - y \sum_{s=1}^{N-1} \binom{m}{s} \partial_x^s \overline{D} \eta_\theta^{(m-s)}
\end{aligned}$$

and

$$\begin{aligned}
\overline{f}_2^{(m)}(v_\theta) = &- \sum_{s=3}^{N+1} \binom{m}{s} \partial_x^s \overline{K}_\theta v_\theta^{(m-s+2)} \\
&- \sum_{s=2}^{N} \binom{m}{s} \partial_x^s (\overline{C} + \partial_x \overline{K}_\theta) v_\theta^{(m-s+1)} \\
&- \sum_{s=1}^{N} \binom{m}{s} \partial_x^s \overline{D} \partial_y v_\theta^{(m-s)} - y \sum_{s=1}^{N-1} \binom{m}{s} \partial_x^s \overline{D} v_\theta^{(m-s)}.
\end{aligned}$$

The functions $\eta_\theta^{(s)} \in \overline{H}_0^{N+3}(\Omega)$, $0 \leq s \leq m$, are defined recursively in the following way. For $0 \leq s \leq N+1$ we set $\eta_\theta^{(s)} = \partial_x^s \eta_\theta$, and if $N+2 \leq s \leq m$ we apply Lemma 4.2 to obtain $\eta_\theta^{(s)}$ such that $\eta_\theta^{(s)}|_{\partial\Omega_1} = \partial_x^s v_\theta|_{\partial\Omega_1}$, $\partial_y \eta_\theta^{(s)}|_{\partial\Omega_1} = \partial_y \partial_x^s v_\theta|_{\partial\Omega_1}$ with

$$\partial_y^t (\partial_x^s \overline{f} + \overline{f}_1^{(s)}(v_\theta) - \overline{L}_\theta^{(s)} \eta_\theta^{(s)})|_{\partial\Omega_1} = 0, \qquad 0 \leq t \leq N. \tag{4.15}$$

Note that since $\overline{f}_1^{(s)} \in \overline{H}_0^{\min(m_*-s-6,N+2)}(\Omega)$ and $\partial_x^s v_\theta|_{\partial\Omega_1} \in \overline{H}_0^{m_*-s-7}(\partial\Omega_1)$, $\partial_y\partial_x^s v_\theta|_{\partial\Omega_1} \in \overline{H}_0^{m_*-s-8}(\partial\Omega_1)$, we must have $N+1 \leq m_* - s - 8$ for the construction of $\eta_\theta^{(s)}$ to be valid. In this case the following estimate holds:

$$
\begin{aligned}
(4.16)\quad \| \eta_\theta^{(s)} \|_{N+3} \leq & C(\| \partial_x^s \overline{f} + \overline{f}_1^{(s)}(v_\theta) \|_{N+1} + \| \partial_x^s v_\theta \|_{N+2,\partial\Omega_1} \\
& + \| \partial_y \partial_x^s v_\theta \|_{N+1,\partial\Omega_1} + \| w \|_{N+7} (\| \partial_x^s \overline{f} + \overline{f}_1^{(s)}(v_\theta) \|_2 \\
& + \| \partial_x^s v_\theta \|_{2,\partial\Omega_1} + \| \partial_y \partial_x^s v_\theta \|_{2,\partial\Omega_1})) \\
\leq & C_s(\| \overline{f}_1^{(s)}(v_\theta) \|_{N+1} + \| f \|_{s+N+1} + \| \phi \|_{s+N+2,\partial\Omega_1} \\
& + \| \psi \|_{s+N+1,\partial\Omega_1} + \| w \|_{s+N+7} (\| f \|_2 + \| \phi \|_{2,\partial\Omega_1} \\
& + \| \psi \|_{2,\partial\Omega_1})),
\end{aligned}
$$

where we have used $|w|_{C^{N+7}} < 1$ and the proof of Lemma 4.2 to estimate $\partial_y^t \partial_x^s v_\theta|_{\partial\Omega_1}$, $t = 0, 1$.

Observe that $v_\theta^{(s)}$, $N+2 \leq s \leq m$, satisfies

$$
\overline{L}_\theta^{(s)} v_\theta^{(s)} = (\partial_x^s \overline{f} + \overline{f}_1^{(s)}(v_\theta) - \overline{L}_\theta^{(s)} \eta_\theta^{(s)}) + \overline{f}_2^{(s)}(v_\theta).
$$

Therefore (4.13) applies to yield

$$
\begin{aligned}
(4.17)\qquad & \lambda(\| \partial_x v_\theta^{(s)} \| + \| \sqrt{|b|}\partial_y v_\theta^{(s)} \| + \| \sqrt{|b|} v_\theta^{(s)} \|) \\
& \leq C(\| \sqrt{|b|}(\partial_x^s \overline{f} + \overline{f}_1^{(s)}(v_\theta) - \overline{L}_\theta^{(s)} \eta_\theta^{(s)}) \| + \| \sqrt{|b|} \overline{f}_2^{(s)}(v_\theta) \|).
\end{aligned}
$$

We now proceed to estimate each term on the right-hand side of (4.17) separately. First note that since $|w|_{C^{N+4}} < 1$, we have

$$
\begin{aligned}
(4.18)\quad \| \sqrt{|b|} \overline{f}_2^{(s)}(v_\theta) \|^2 & \leq C_s \int_\Omega \sum_{l=0}^{s-1} \mathrm{e}^{-\lambda y} |\overline{K}_\theta|^{-1} [(v_\theta^{(l)})^2 + (\partial_y v_\theta^{(l)})^2] \\
& = C_s \int_\Omega \sum_{l=N+2}^{s-1} \mathrm{e}^{-\lambda y} |\overline{K}_\theta|^{-1} [(v_\theta^{(l)})^2 + (\partial_y v_\theta^{(l)})^2] \\
& \quad + C_s \int_\Omega \sum_{l=0}^{N+1} \mathrm{e}^{-\lambda y} |\overline{K}_\theta|^{-1} [(\partial_x^l \overline{v}_\theta)^2 + (\partial_y \partial_x^l \overline{v}_\theta)^2].
\end{aligned}
$$

Next observe again that since $|\overline{K}| \geq C_0^{-1}(y-h(x))^{N+1}$ in Ω, (4.15) implies that

$$
\begin{aligned}
(4.19) \qquad & \| \sqrt{|b|}(\partial_x^s \overline{f} + \overline{f}_1^{(s)}(v_\theta) - \overline{L}_\theta^{(s)}\eta_\theta^{(s)}) \|^2 \\
&\leq C_0 \int_\Omega \frac{[\partial_x^s \overline{f} + \overline{f}_1^{(s)}(v_\theta) - \overline{L}_\theta^{(s)}\eta_\theta^{(s)}]^2}{(y-h(x))^{N+1}} \\
&\leq C_1 \int_\Omega [\partial_y^{N+1}(\partial_x^s \overline{f} + \overline{f}_1^{(s)}(v_\theta) - \overline{L}_\theta^{(s)}\eta_\theta^{(s)})]^2.
\end{aligned}
$$

Furthermore, applying the Gagliardo–Nirenberg inequalities (Lemma 5.2), (4.12), (4.14), (4.16), and using $|w|_{C^{2N+4}} < 1$ produces

(4.20)

$$
\begin{aligned}
\| \overline{f}_1^{(s)}(v_\theta) \|_{N+1} &\leq C_s((|\overline{K}_\theta|_{C^{N+2}} + |\overline{C}|_{C^{N+1}} + |\overline{D}|_{C^{N+1}}) \| v_\theta \|_{s+1} \\
&\quad + (\| \overline{K}_\theta \|_{s+N+3} + \| \overline{C} \|_{s+N+2} + \| \overline{D} \|_{s+N+2})|v_\theta|_{C^0}) \\
&\quad + C_s \left(\sum_{l=s-N}^{N+1} \| \eta_\theta \|_{l+N+2} + \sum_{l=N+2}^{s-1} \| \eta_\theta^{(l)} \|_{N+2} \right) \\
&\leq C_s(\| f \|_{s+N+1} + \varepsilon \| v_\theta \|_{s+1} + \| \phi \|_{s+N+2,\partial\Omega_1} \\
&\quad + \| \psi \|_{s+N+1,\partial\Omega_1} + \| w \|_{s+N+7} (\| f \|_{N+2} \\
&\quad + \| \phi \|_{N+3,\partial\Omega_1} + \| \psi \|_{N+2,\partial\Omega_1})).
\end{aligned}
$$

It follows that we may combine (4.17) to (4.20) and utilize (4.12), (4.14), as well as (4.16) to obtain

$$
\begin{aligned}
&\sum_{s=N+2}^{m} \| \partial_x v_\theta^{(s)} \| + \sum_{s=N+2}^{m} \| \sqrt{|b|}\partial_y v_\theta^{(s)} \| + \sum_{s=N+2}^{m} \| \sqrt{|b|} v_\theta^{(s)} \| \\
&\quad \leq C_m(\| f \|_{m+N+1} + \varepsilon \| u_\theta \|_{m+1} + \| \phi \|_{m+N+2,\partial\Omega_1} + \| \psi \|_{m+N+1,\partial\Omega_1} \\
&\qquad + \| w \|_{m+N+7} (\| f \|_{N+2} + \| \phi \|_{N+3,\partial\Omega_1} + \| \psi \|_{N+2,\partial\Omega_1})).
\end{aligned}
$$

Since $v_\theta^{(s)} = \partial_x^s v_\theta - \eta_\theta^{(s)}$, with the help of (4.16) the above inequality yields an estimate for the x-derivatives of u_θ. The remaining derivatives of u_θ may be estimated in the usual way, by solving for $\partial_y^2 u_\theta$ from Equation (4.2). Lastly taking $\varepsilon = \varepsilon(m)$ sufficiently small yields the desired result. □

Remark 4.1. In this section we have focused on the Cauchy problem in the domains $\Omega_\varrho^- \cap B_\sigma(0)$, $\varrho = 1, 2$. However analogous existence and regularity results, requiring only slight modifications of the arguments above, hold for the Cauchy problem in the domains $\Omega_\varrho^- \cap B_\sigma(0)$, $\varrho = 3, 4$, when the Cauchy data are prescribed on either the "upper" or "lower" parts of these domains (that is, on one of the two differentiable components of the curve $x = h(y)$).

5. The Nash–Moser iteration

In this section we will carry out Nash–Moser-type iteration procedures to obtain solutions of (2.1) in each of the elliptic, hyperbolic and mixed type regions separately. The solutions will then be patched together by choosing appropriate boundary values, to yield a solution on a full neighborhood of the origin. As a consequence of Lemma 2.1, we can assume (by a judicious choice of coordinates) that each elliptic region is given by

$$\Omega_\kappa^+ = \{(\xi^1, \xi^2) \mid 0 < \xi^2 < (\tan\delta)\xi^1,\ |\xi| < \sigma\}, \qquad 1 \le \kappa \le \kappa_0,$$

each hyperbolic region is given by

$$\Omega_\varrho^- = \{(\xi^1, \xi^2) \mid h(\xi^1) < \xi^2 < \sigma\}, \qquad \varrho = 1, 2,$$

or

$$\Omega_\varrho^- = \{(\xi^1, \xi^2) \mid h(\xi^2) < \xi^1 < \sigma\}, \qquad \varrho = 3, 4.$$

If $\overline{\partial}\Omega_\kappa^+$ denotes the portion of the boundary consisting of the curves $\xi^2 = 0$ and $\xi^2 = (\tan\delta)\xi^1$, $\overline{\partial}\Omega_\varrho^-$, $\varrho = 1, 2$, denotes the portion of the boundary given by $\xi^2 = h(\xi^1)$, and $\overline{\partial}\Omega_\varrho^-$, $\varrho = 3, 4$, denotes either the upper or lower part of the boundary curve $\xi^1 = h(\xi^2)$, then we aim to solve

$$\Phi(w_\kappa^+) = 0 \quad \text{in} \quad \Omega_\kappa^+, \qquad w_\kappa^+|_{\overline{\partial}\Omega_\kappa^+} = 0, \tag{5.1}$$

$$\partial^\alpha w_\kappa^+(0,0) = 0, \qquad |\alpha| \le \alpha_0,$$

for each $\kappa = 1, \ldots, \kappa_0$, and

$$\Phi(w_\varrho^-) = 0 \quad \text{in} \quad \Omega_\varrho^-, \qquad w_\varrho^-|_{\overline{\partial}\Omega_\varrho^-} = \phi_\varrho^-, \qquad \partial_\nu w_\varrho^-|_{\overline{\partial}\Omega_\varrho^-} = \psi_\varrho^-, \tag{5.2}$$

$$\partial^\alpha w_\varrho^-(0,0) = 0, \qquad |\alpha| \le \alpha_0,$$

for each $\varrho = 1, \ldots, \varrho_0$, where ∂_ν denotes the outward normal derivative, α_0 is a large integer and ϕ_ϱ^-, ψ_ϱ^- will be specified below.

Both of problems (5.1) and (5.2) will require slight modifications of the standard Nash–Moser procedure. This arises from the fact that instead of solving the linearized equation at each iteration, the theories developed in

Sections 3 and 4 require us to solve modified versions of the linearized equation. However the error incurred at each step by these modifications is of quadratic type and therefore does not prevent the procedure from converging to a solution. Below we shall carry out the proof in full detail for the hyperbolic regions, problem (5.2). Moreover, since the corresponding iteration for problem (5.1) differs only slightly from that of (5.2), we shall merely indicate the necessary changes required in this case.

5.1. Hyperbolic regions

In the hyperbolic regions, the linearization $\mathcal{L}(w)$ and the operator $L_\theta(w)$ that we invert in Section 4 differ by perturbation terms coming from Lemma 2.1 as well as a regularizing term involving $\theta = |\Phi(w)|_{C^1}$. More precisely, according to Lemma 2.1

$$\begin{aligned}\mathcal{L}(w)u = {} & \varepsilon a^{22}(w)L_\theta(w)u + \varepsilon\theta a^{22}(w)\partial^2_{\xi^1}u \\ & + \varepsilon(a^{22}(w))^{-1}\Phi(w)[\partial^2_{x^1}u - \partial_{x^1}\log(a^{22}(w)\sqrt{|g|})\partial_{x^1}u],\end{aligned} \tag{5.3}$$

where $\xi^i(x^1, x^2)$ are the coordinates constructed in Lemma 2.1. Furthermore, as with all Nash–Moser iterative schemes we will need smoothing operators. Since the theory of Section 4 is based on the Sobolev spaces $\overline{H}_0^m$, the smoothing operators we use should respect these spaces. For convenience, in the remainder of this section the hyperbolic region in question Ω^-_ϱ will be denoted by Ω, and the $\overline{H}_0^m(\Omega)$ norm will be denoted by $\|\cdot\|_m$.

Lemma 5.1. *Given $\mu \geq 1$ there exists a linear smoothing operator $S_\mu : L^2(\Omega) \to \overline{H}_0^\infty(\Omega)$ such that for all $l, m \in \mathbb{Z}_{\geq 0}$ and $u \in \overline{H}_0^l(\Omega)$,*

(i) $\| S_\mu u \|_m \leq C_{l,m} \| u \|_l$, $m \leq l$,

(ii) $\| S_\mu u \|_m \leq C_{l,m}\mu^{m-l} \| u \|_l$, $l \leq m$,

(iii) $\| u - S_\mu u \|_m \leq C_{l,m}\mu^{m-l} \| u \|_l$, $m \leq l$.

Furthermore, there exists a linear smoothing operator $S'_\mu : L^2(\Omega) \to H^\infty(\Omega)$ such that (i) *to* (iii) *hold whenever $u \in H^l(\Omega)$.*

Proof. See Appendix B. □

The next lemma contains the so-called Gagliardo–Nirenberg inequalities, which will be used frequently throughout this section.

Lemma 5.2. *Let $u, v \in C^k(\overline{\Omega})$.*

(i) *If α and β are multi-indices such that $|\alpha| + |\beta| = m$, then there exists a constant C_1 depending on m such that*

$$\| \partial^\alpha u \partial^\beta v \|_{L^2(\Omega)} \leq C_1(|u|_{L^\infty(\Omega)} \| v \|_{H^m(\Omega)} + \| u \|_{H^m(\Omega)} |v|_{L^\infty(\Omega)}).$$

(ii) *If $\alpha_1, \ldots, \alpha_l$ are multi-indices such that $|\alpha_1| + \cdots + |\alpha_l| = m$, then there exists a constant C_2 depending on l and m such that*

$$\begin{aligned} &\| \partial^{\alpha_1} u_1 \cdots \partial^{\alpha_l} u_l \|_{L^2(\Omega)} \\ &\quad \leq C_2 \sum_{j=1}^{l} (|u_1|_{L^\infty(\Omega)} \cdots \widehat{|u_j|}_{L^\infty(\Omega)} \cdots |u_l|_{L^\infty(\Omega)}) \| u_j \|_{H^m(\Omega)}, \end{aligned}$$

where $\widehat{|u_j|}_{L^\infty(\Omega)}$ indicates the absence of $|u_j|_{L^\infty(\Omega)}$.

(iii) *Let $\mathcal{D} \subset \mathbb{R}^l$ be compact and contain the origin, and let $G \in C^\infty(\mathcal{D})$. If $u \in H^m(\Omega, \mathcal{D}) \cap L^\infty(\Omega, \mathcal{D})$, then there exists a constant C_3 depending on m such that*

$$\| G \circ u \|_{H^m(\Omega)} \leq C_3 |u|_{L^\infty(\Omega)}(|G(0)| + \| u \|_{H^m(\Omega)}).$$

Proof. These estimates are standard consequences of the interpolation inequalities, and may be found in, for instance, [19]. □

We now set up the underlying iterative procedure. Suppose that $\phi_\varrho^- \in \overline{H}_0^{m_*-m_0+1}(\overline{\partial}\Omega)$ and $\psi_\varrho^- \in \overline{H}_0^{m_*-m_0}(\overline{\partial}\Omega)$ for some $m_0 \geq 0$. Then according to the proof of Lemma 4.2 there exists $w_0 \in \overline{H}_0^{m_*-m_0+2}(\Omega)$ such that

$$w_0|_{\overline{\partial}\Omega} = \phi_\varrho^-, \qquad \partial_\nu w_0|_{\overline{\partial}\Omega} = \psi_\varrho^-. \tag{5.4}$$

Now suppose that in addition to w_0, functions $w_1, w_2, \ldots, w_n$ have been defined on Ω, and put $v_i = S_i w_i$, $0 \leq i \leq n$, where $S_i = S_{\mu^i}$. Then define $w_{n+1} = w_n + u_n$ where u_n is the unique solution of

$$L_{\theta_n}(v_n)u_n = f_n \quad \text{in} \quad \Omega, \qquad u_n|_{\overline{\partial}\Omega} = \partial_\nu u_n|_{\overline{\partial}\Omega} = 0, \tag{5.5}$$

given by Theorem 4.2, where $\theta_n = |\Phi(v_n)|_{C^1}$ and f_n will be specified below.

Let $Q_n(w_n, u_n)$ denote the quadratic error in the Taylor expansion of Φ at w_n. Then according to (5.3) we have

$$\begin{aligned}\Phi(w_{n+1}) &= \Phi(w_n) + \mathcal{L}(w_n)u_n + Q_n(w_n, u_n) \\ &= \Phi(w_n) + \varepsilon S_n^{'} a^{22}(v_n) L_{\theta_n}(v_n)u_n + e_n,\end{aligned} \tag{5.6}$$

where

$$\begin{aligned}e_n = &(\mathcal{L}(w_n) - \mathcal{L}(v_n))u_n + \varepsilon(I - S_n^{'})a^{22}(v_n)L_{\theta_n}(v_n)u_n \\ &+ \varepsilon\theta_n a^{22}(v_n)\partial^2_{\xi^1_n} u_n + Q_n(w_n, u_n) + \varepsilon(a^{22}(v_n))^{-1}\Phi(v_n)[\partial^2_{x^1} u_n \\ &- \partial_{x^1}(\log a^{22}(v_n)\sqrt{|g|})\partial_{x^1} u_n],\end{aligned}$$

and ξ^i_n are the coordinates of Lemma 2.1 with respect to v_n.

We now define f_n. In order to solve (5.5) with the theory of Section 4, we require $f_n \in \overline{C}^\infty(\Omega)$. Furthermore, we need the right-hand side of (5.6) to tend to zero sufficiently fast, to make up for the error incurred at each step by solving (5.5) instead of solving the unmodified linearized equation. Therefore we set $E_0 = 0$, $E_n = \sum_{i=0}^{n-1} e_i$, and define

$$f_0 = -[\varepsilon S_0^{'} a^{22}(v_0)]^{-1} S_0 \Phi(w_0), \tag{5.7}$$

$$f_n = [\varepsilon S_n^{'} a^{22}(v_n)]^{-1}(S_{n-1}E_{n-1} - S_n E_n + (S_{n-1} - S_n)\Phi(w_0)).$$

It follows that

$$\begin{aligned}\Phi(w_{n+1}) &= \Phi(w_0) + \sum_{i=0}^{n} \varepsilon S_i^{'} a^{22}(v_i) f_i + E_n + e_n \\ &= (I - S_n)\Phi(w_0) + (I - S_n)E_n + e_n.\end{aligned} \tag{5.8}$$

The following theorem contains the Moser estimate for solutions of (5.5), upon which the whole iteration scheme is based.

Theorem 5.1. *Suppose that $g \in C^{m_*}$ and N is as in Theorem 4.2. If $m \le m_* - N - 8$, $|v_n|_{C^{2N+4}} < 1$ and $\varepsilon = \varepsilon(m)$ is sufficiently small then there exists a unique solution $u_n \in \overline{H}_0^m(\Omega)$ of (5.5) which satisfies the estimate*

$$\| u_n \|_m \le C_m(\| f_n \|_{m+N} + \| v_n \|_{m+N+6} \| f_n \|_{N+2}), \tag{5.9}$$

for some constant C_m independent of ε and θ_n.

Proof. This will follow from Theorem 4.2 with $\phi = \psi \equiv 0$. The only difference is that the Sobolev norms appearing in Theorem 4.2 are with respect to the coordinates $\xi_n^i(x^1, x^2)$ of Lemma 2.1. In order to obtain the current estimate from that of Theorem 4.2 we may utilize (2.4). Note that δ (of Lemma 2.1) is not chosen arbitrarily small in the hyperbolic regions, and so it does not appear in the above estimate. □

In what follows, we will show that the right-hand side of (5.8) tends to zero sufficiently fast to guarantee convergence of $\{w_n\}_{n=0}^{\infty}$ to a solution of (5.2). Let ρ be a positive number that will be chosen as large as possible, and set $\mu = \varepsilon^{-\frac{1}{2\rho}}$, $\mu_n = \mu^n$. Furthermore, note that $\Phi(w_0) \in \overline{H}_0^{m_*-m_0}(\Omega)$. The convergence of $\{w_n\}_{n=0}^{\infty}$ will follow from the following eight statements, valid for $0 \le m \le m_* - N - 8$ unless specified otherwise. These statements shall be proven by induction on n, for some constants $C_1, \dots, C_5$ independent of n and ε, but dependent on m.

$$\mathrm{I}_n\colon\ \| u_{n-1} \|_m \le \varepsilon \mu_{n-1}^{m+N+2-\rho},$$

$$\mathrm{II}_n\colon\ \| w_n \|_m \le \begin{cases} C_1\varepsilon & \text{if } m+N+2-\rho \le -1/2, \\ C_1\varepsilon\mu_n^{m+N+2-\rho} & \text{if } m+N+2-\rho \ge 1/2, \end{cases}$$

$$\mathrm{III}_n\colon\ \| w_n \|_{2N+6} \le C_1\varepsilon, \quad \| v_n \|_{2N+6} \le C_3\varepsilon,$$

$$\mathrm{IV}_n\colon\ \| w_n - v_n \|_m \le C_2\varepsilon\mu_n^{m+N+2-\rho},$$

$$\mathrm{V}_n\colon\ \| v_n \|_m \le \begin{cases} C_3\varepsilon & \text{if } m+N+2-\rho \le -1/2, \\ C_3\varepsilon\mu_n^{m+N+2-\rho} & \text{if } m+N+2-\rho \ge 1/2, \end{cases} \quad m < \infty,$$

$$\mathrm{VI}_n\colon\ \| e_{n-1} \|_m \le C_4\varepsilon^3\mu_{n-1}^{m-\rho},\ m \le \min(m_* - N - 10, m_* - m_0),$$

$$\mathrm{VII}_n\colon\ \| f_n \|_m \le C_5\varepsilon^2(1+\mu^{\rho-m})\mu_n^{m-\rho},\ m < \infty,$$

$$\mathrm{VIII}_n\colon\ \| \Phi(w_n) \|_m + \| \Phi(v_n) \|_m \le \varepsilon\mu_n^{m+N+4-\rho}, \quad m \le \min(m_* - N - 10, m_* - m_0).$$

Assume that the above eight statements hold for all nonnegative integers less than or equal to n. The next four propositions will show that they also hold for $n+1$. The case $n = 0$ will be proven shortly thereafter.

Proposition 5.1. *If $3N+8 < \rho < m_* - 6$, $0 \le m \le m_* - N - 8$ and ε is sufficiently small, then I_{n+1}, II_{n+1}, III_{n+1}, IV_{n+1} and V_{n+1} hold.*

Proof. I_{n+1}: First note that by III_n,

$$|v_n|_{C^{2N+4}} \le C \| v_n \|_{2N+6} \le CC_3\varepsilon < 1$$

for small ε. Therefore if $m \leq m_* - N - 8$ we may apply Theorem 5.1 to obtain a solution $u_n \in \overline{H}_0^m(\Omega)$ of (5.5) which satisfies estimate (5.9). When $m + 2N + 8 - \rho \geq 1/2$ this may be combined with V_n, VII_n and $\rho \geq 2N + 8$ to obtain

$$\begin{aligned}\| u_n \|_m &\leq C_m(\| f_n \|_{m+N} + \| v_n \|_{m+N+6} \| f_n \|_{N+2}) \\ &\leq C_m(C_5\varepsilon^2(1 + \mu^{\rho-m})\mu_n^{m+N-\rho} \\ &\quad + C_3C_5\varepsilon^3(1 + \mu^{\rho-N-2})\mu_n^{m+2N+8-\rho}\mu_n^{N+2-\rho}) \\ &\leq \varepsilon\mu_n^{m+N+2-\rho}\end{aligned}$$

for small ε. When $m + 2N + 8 - \rho \leq -1/2$, the estimate $\| v_n \|_{m+N+6} \leq C_3\varepsilon$ placed in the above calculation gives the desired result.

II_{n+1}: Since $w_{n+1} = \sum_{i=0}^{n} u_i$, we have

$$\| w_{n+1} \|_m \leq \sum_{i=0}^{n} \| u_i \|_m \leq \varepsilon \sum_{i=0}^{n} \mu_i^{m+N+2-\rho}.$$

Hence, if $m + N + 2 - \rho \leq -1/2$ then

$$\| w_{n+1} \|_m \leq \varepsilon \sum_{i=0}^{\infty} (\mu^i)^{-1/2} \leq \varepsilon \sum_{i=0}^{\infty} (2^i)^{-1/2} := C_1\varepsilon,$$

and if $m + N + 2 - \rho \geq 1/2$ then

$$\begin{aligned}\| w_{n+1} \|_m &\leq \varepsilon\mu_{n+1}^{m+N+2-\rho} \sum_{i=0}^{n} \left(\frac{\mu_i}{\mu_{n+1}}\right)^{m+N+2-\rho} \\ &\leq \varepsilon\mu_{n+1}^{m+N+2-\rho} \sum_{i=0}^{\infty} (\mu^{-i})^{1/2} \leq C_1\varepsilon\mu_{n+1}^{m+N+2-\rho}.\end{aligned}$$

III_{n+1}: By the largeness assumption on ρ we have $3N + 8 - \rho \leq -1/2$. Therefore II_{n+1} and V_{n+1} (proven below) imply that

$$\| w_{n+1} \|_{2N+6} \leq C_1\varepsilon \quad \text{and} \quad \| v_{n+1} \|_{2N+6} \leq C_3\varepsilon.$$

IV_{n+1}: Since $\rho < m_* - 6$ we have $(m_* - N - 8) + N + 2 - \rho \geq 1/2$. Therefore Lemma 5.1 and II_{n+1} yield

$$\begin{aligned}\| w_{n+1} - v_{n+1} \|_m &= \| (I - S_{n+1})w_{n+1} \|_m \\ &\leq C_m\mu_{n+1}^{m-(m_*-N-8)} \| w_{n+1} \|_{m_*-N-8}\end{aligned}$$

$$\leq C_m \mu_{n+1}^{m-(m_*-N-8)} C_1 \varepsilon \mu_{n+1}^{(m_*-N-8)+N+2-\rho}$$
$$:= C_2 \varepsilon \mu_{n+1}^{m+N+2-\rho}.$$

V_{n+1}: From Lemma 5.1 and $\rho < m_* - 6$ we have for all $m \geq 0$,

$$\| v_{n+1} \|_m = \| S_{n+1} w_{n+1} \|_m \leq C_m \begin{cases} \| w_{n+1} \|_{\rho-N-3} & \text{if } m+N+2-\rho \leq -1/2, \\ \mu_{n+1}^{m+N+1-\rho} \| w_{n+1} \|_{\rho-N-1} & \text{if } m+N+2-\rho \geq 1/2. \end{cases}$$

V_{n+1} now follows from II_{n+1}. □

Write $e_n = e_n^{'} + e_n^{''} + e_n^{'''}$, where

$$\begin{aligned} e_n^{'} &= (\mathcal{L}(w_n) - \mathcal{L}(v_n))u_n, \\ e_n^{''} &= \varepsilon(I - S_n^{'})a^{22}(v_n)L_{\theta_n}(v_n)u_n + \varepsilon\theta_n a^{22}(v_n)\partial^2_{\xi_n^1} u_n \\ &\quad + \varepsilon(a^{22}(v_n))^{-1}\Phi(v_n)[\partial^2_{x^1} u_n - \partial_{x^1}(\log a^{22}(v_n)\sqrt{|g|})\partial_{x^1} u_n], \\ e_n^{'''} &= Q_n(w_n, u_n). \end{aligned}$$

Proposition 5.2. *If the hypotheses of Proposition 5.1 hold in addition to $n > 0$, $\rho \geq 2N + 12$, and $0 \leq m \leq \min(m_* - N - 10, m_* - m_0)$ then VI_{n+1} holds.*

Proof. We will estimate $e_n^{'}$, $e_n^{''}$ and $e_n^{'''}$ separately. According to (2.3) we may write

$$(\mathcal{L}(w_n) - \mathcal{L}(v_n))u_n = \varepsilon \sum_{i,j} A_{ij} \partial_{x^i x^j} u_n + \varepsilon \sum_i A_i \partial_{x^i} u_n.$$

Then Lemma 5.2(i) and (iii), I_{n+1}, and IV_n show that

$$\begin{aligned} \| e_n^{'} \|_m &\leq \varepsilon C_{m,1} \left[\left(\sum_{i,j} \| A_{ij} \|_m + \sum_i \| A_i \|_m \right) \| u_n \|_4 \right. \\ &\quad \left. + \left(\sum_{i,j} \| A_{ij} \|_2 + \sum_i \| A_i \|_2 \right) \| u_n \|_{m+2} \right] \\ &\leq \varepsilon C_{m,2} (\| w_n - v_n \|_{m+2} \| u_n \|_4 + \| w_n - v_n \|_4 \| u_n \|_{m+2}) \\ &\leq C_{m,3} \varepsilon^3 \mu_n^{m+N+4-\rho} \mu_n^{N+6-\rho} \\ &\leq C_{m,3} \varepsilon^3 \mu_n^{m-\rho}. \end{aligned}$$

Note that we have used $\rho \geq 2N + 10$, as well as $m \leq m_* - N - 10$ which allows us to apply I_{n+1} and IV_n.

We now estimate $e_n^{''}$. By Lemma 5.2(i) and (iii), I_{n+1}, V_n and VIII_n,

$$\begin{aligned}\| \varepsilon\theta_n a^{22}(v_n)\partial^2_{\xi^1_n} u_n \|_m &\leq \varepsilon\theta_n C_{m,4}(\| a^{22}(v_n) \|_m \| \partial^2_{\xi^1_n} u_n \|_2 \\ &\quad + \| a^{22}(v_n) \|_2 \| \partial^2_{\xi^1_n} u_n \|_m) \\ &\leq \varepsilon\theta_n C_{m,5}[(1+ \| v_n \|_{m+2}) \| u_n \|_4 \\ &\quad + (1+ \| v_n \|_4) \| u_n \|_{m+2}] \\ &\leq \varepsilon^2 \mu_n^{3-\rho} C_{m,6}[(1+ C_3\varepsilon\mu_n^{m+N+4-\rho})\varepsilon\mu_n^{N+4-\rho} \\ &\quad + (1 + C_3\varepsilon)\varepsilon\mu_n^{m+N+4-\rho}] \\ &\leq C_{m,7}\varepsilon^3\mu_n^{m-\rho}\end{aligned}$$

if μ is large and $m + N + 4 - \rho \geq 1/2$. If $m + N + 4 - \rho \leq -1/2$ then we may use the estimate $\| v_n \|_{m+2} \leq C_3\varepsilon$ to obtain the same outcome. Another application of Lemma 5.2 gives

$$\begin{aligned}\| \varepsilon(a^{22}(v_n))^{-1}\Phi(v_n)\partial^2_{x^1} u_n \|_m &\leq \varepsilon C_{m,8}[\| \Phi(v_n) \|_m (1+ \| v_n \|_4) \| u_n \|_4 \\ &\quad + \| \Phi(v_n) \|_2 (1+ \| v_n \|_{m+2}) \| u_n \|_4 \\ &\quad + \| \Phi(v_n) \|_2 (1+ \| v_n \|_4) \| u_n \|_{m+2}] \\ &\leq C_{m,9}\varepsilon^3\mu_n^{m-\rho}\end{aligned}$$

after noting that $m \leq \min(m_* - N - 10, m_* - m_0)$ is required for VIII_n to be valid, and similar methods yield

$$\| \varepsilon(a^{22}(v_n))^{-1}\Phi(v_n)\partial_{x^1}(\log a^{22}(v_n)\sqrt{|g|})\partial_{x^1} u_n \|_m \leq C_{m,10}\varepsilon^3\mu_n^{m-\rho}.$$

Moreover, if $l = \rho + 2 \leq m_* - 2$ and $n > 0$ then we may apply Lemma 5.1 and recall that $\mu = \varepsilon^{-\frac{1}{2\rho}}$ to obtain

$$\begin{aligned}&\| \varepsilon(I - S_n^{'})a^{22}(v_n)L_{\theta_n}(v_n)u_n \|_m \\ &\quad \leq \varepsilon C_{m,11}[\| (I - S_n^{'})a^{22}(v_n) \|_m \| f_n \|_2 + \| (I - S_n^{'})a^{22}(v_n) \|_2 \| f_n \|_2] \\ &\quad \leq \varepsilon C_{m,12}[\mu_n^{m-l}(1 + \| v_n \|_{l+2})\varepsilon^2(1+ \mu^{\rho-2})\mu_n^{2-\rho} \\ &\qquad + \mu_n^{2-l}(1+ \| v_n \|_{l+2})\varepsilon^2(1+ \mu^{\rho-m})\mu_n^{m-\rho}] \\ &\quad \leq C_{m,13}\varepsilon^3\mu_n^{m-\rho}.\end{aligned}$$

Therefore

$$\| e_n^{''} \| \leq C_{m,14}\varepsilon^3\mu_n^{m-\rho}.$$

We now estimate $e_n^{'''}$. We have

$$e_n^{'''} = Q_n(w_n, u_n) = \int_0^1 (1-t)\frac{\partial^2}{\partial t^2}\Phi(w_n + tu_n)dt.$$

Apply Lemma 5.2(i) and (ii), as well as the Sobolev Lemma to obtain

$$\begin{aligned}\| e_n^{'''} \|_m &\leq \int_0^1 \sum_{|\alpha|,|\beta|\leq 2} \| \nabla_{\overline{\alpha}\overline{\beta}}\Phi(w_n + tu_n)\partial^\alpha u_n \partial^\beta u_n \|_m \, dt \\ &\leq \int_0^1 C_{m,15}(\| \nabla^2\Phi(w_n + tu_n) \|_2 \| u_n \|_4 \| u_n \|_{m+2} \\ &\quad + \| \nabla^2\Phi(w_n + tu_n) \|_m \| u_n \|_4^2) \, dt,\end{aligned}$$

where $\overline{\alpha} = \partial^\alpha(w_n + tu_n)$ and $\overline{\beta} = \partial^\beta(w_n + tu_n)$. The notation $\nabla^2\Phi$ represents the collection of second partial derivatives with respect to the variables $\overline{\alpha}$ and $\overline{\beta}$. Furthermore it is easy to see that $|\nabla^2\Phi(0)| = O(\varepsilon)$. Therefore using Lemma 5.2(iii), I_{n+1}, and II_n, we have

$$\begin{aligned}\| e_n^{'''} \|_m &\leq C_{m,16}[(\varepsilon + \| w_n \|_4 + \| u_n \|_4) \| u_n \|_4 \| u_n \|_{m+2} \\ &\quad + (\varepsilon + \| w_n \|_{m+2} + \| u_n \|_{m+2}) \| u_n \|_4^2] \\ &\leq C_{m,17}\varepsilon^3 \mu_n^{m-\rho}\end{aligned}$$

if $\rho \geq 2N + 12$. Combining the estimates of $e_n^{'}$, $e_n^{''}$ and $e_n^{'''}$ yields the desired result. □

According to the above proposition if $\rho + 1 \leq m_* - m_0$ (in addition to the other required restrictions) then $E_n \in H^{\rho+1}(\Omega)$ and the following estimate holds, which will be utilized in the next proposition:

(5.10)
$$\begin{aligned}\| E_n \|_{\rho+1} &\leq \sum_{i=0}^{n-1} \| e_i \|_{\rho+1} \leq C_4\varepsilon^3 \sum_{i=0}^{n-1} \mu_i \\ &\leq C_4\varepsilon^3 \left(\sum_{i=0}^{\infty} \mu_i^{-1}\right) \mu^n \leq C_4\varepsilon^3 \left(\sum_{i=0}^{\infty} 2^{-i}\right) \mu_n.\end{aligned}$$

Proposition 5.3. *If the hypotheses of Proposition 5.2 hold and $\rho + 1 \leq m_* - m_0$, then VII_{n+1} holds for all $0 \leq m < \infty$.*

Proof. By (5.7) as well as Lemma 5.2(i) and (iii),

$$\begin{aligned}(5.11)\quad &\| f_{n+1} \|_m \leq \varepsilon^{-1} C_m(\| S_n E_n - S_{n+1}E_{n+1} + (S_n - S_{n+1})\Phi(w_0) \|_m \\ &\qquad + \| v_{n+1} \|_{m+2} \| S_n E_n - S_{n+1}E_{n+1} + (S_n - S_{n+1})\Phi(w_0) \|_2).\end{aligned}$$

Next observe that (2.2) together with (5.14) below yields

$$\| \Phi(w_0) \|_{\rho+1} \leq C(\varepsilon^3 + \| w_0 \|_{\rho+3}) \leq C\varepsilon^3. \tag{5.12}$$

Then (5.10) implies that for all $m \geq \rho + 1$,

$$\begin{aligned}&\| S_n E_n - S_{n+1}E_{n+1} + (S_n - S_{n+1})\Phi(w_0) \|_m \\ &\leq C_m(\mu_n^{m-\rho-1} \| E_n \|_{\rho+1} + \mu_{n+1}^{m-\rho-1} \| E_{n+1} \|_{\rho+1} \\ &\quad + (\mu_n^{m-\rho-1} + \mu_{n+1}^{m-\rho-1}) \| \Phi(w_0) \|_{\rho+1}) \\ &\leq C_m \varepsilon^3 (1 + \mu^{\rho-m}) \mu_{n+1}^{m-\rho}.\end{aligned} \tag{5.13}$$

If $m < \rho + 1$, then applying similar methods along with VI_{n+1} to

$$\begin{aligned}&\| S_n E_n - S_{n+1}E_{n+1} + (S_n - S_{n+1})\Phi(w_0) \|_m \\ &\leq \| (I - S_n)E_n \|_m + \| (I - S_{n+1})E_n \|_m + \| S_{n+1} e_n \|_m \\ &\quad + \| (I - S_n)\Phi(w_0) \|_m + \| (I - S_{n+1})\Phi(w_0) \|_m,\end{aligned}$$

produces the same estimate found in (5.13). Therefore plugging into (5.11) produces

$$\begin{aligned}\| f_{n+1} \|_m &\leq C_m[\varepsilon^2(1 + \mu^{\rho-m})\mu_{n+1}^{m-\rho} + \varepsilon^3(1 + \mu^{\rho-2})\mu_{n+1}^{m+N+6-2\rho}] \\ &\leq C_m \varepsilon^2 (1 + \mu^{\rho-m}) \mu_{n+1}^{m-\rho},\end{aligned}$$

if $m + N + 4 - \rho \geq 1/2$. If $m + N + 4 - \rho \leq -1/2$ and $m \geq 2$, then using $\| v_{n+1} \|_{m+2} \leq C_3 \varepsilon$ in the estimate above gives the desired result. Moreover if $0 \leq m < 2$, then in place of (5.11) we use the estimate

$$\| f_{n+1} \|_m \leq \varepsilon^{-1} C_m \| S_n E_n - S_{n+1}E_{n+1} + (S_n - S_{n+1})\Phi(w_0) \|_m$$

combined with the above method to obtain the desired result. Lastly if $m + N + 4 - \rho = 0$, then replace $\| v_{n+1} \|_{m+2}$ in (5.11) by $\| v_{n+1} \|_{m+3}$ and follow the above method. □

Proposition 5.4. *If the hypotheses of Proposition* 5.3 *hold and* $\rho + 1 = \min(m_* - N - 10, m_* - m_0)$*, then* $VIII_{n+1}$ *holds for* $0 \leq m \leq \min(m_* - N - 10, m_* - m_0)$*.*

Proof. By (5.8) and VI_{n+1} and $m \leq \rho + 1$, we have

$$\begin{aligned}\| \Phi(w_{n+1}) \|_m &\leq \| (I - S_n)\Phi(w_0) \|_m + \| (I - S_n)E_n \|_m + \| e_n \|_m \\ &\leq C_m(\mu_n^{m-\rho-1} \| \Phi(w_0) \|_{\rho+1} \\ &\quad + \mu_n^{m-\rho-1} \| E_n \|_{\rho+1} + C_4\varepsilon^3\mu_n^{m-\rho}).\end{aligned}$$

Applying estimate (5.10) along with (5.12) and $\varepsilon\mu^{\rho-m} \leq \varepsilon^{1/2}$ produces

$$\| \Phi(w_{n+1}) \|_m \leq C_m\varepsilon^3\mu^{\rho-m}\mu_{n+1}^{m-\rho} \leq \tfrac{1}{3}\varepsilon\mu_{n+1}^{m-\rho},$$

if ε is sufficiently small. Lastly a similar estimate may be obtained for $\Phi(v_{n+1})$ by writing

$$\begin{aligned}\| \Phi(v_{n+1}) \|_m &\leq \| \Phi(w_{n+1}) \|_m + \| \Phi(v_{n+1}) - \Phi(w_{n+1}) \|_m \\ &\leq \tfrac{1}{3}\varepsilon\mu_{n+1}^{m-\rho} + \varepsilon \| v_{n+1} - w_{n+1} \|_{m+2} \\ &\leq (\tfrac{1}{3} + C_2\varepsilon^2)\mu_{n+1}^{m+N+4-\rho}.\end{aligned}$$

□

To complete the proof by induction we will now prove the case $n = 0$. Here we will assume that the initial data are appropriately small:

$$\| \phi_\varrho^- \|_{m_*-m_0+1,\bar\partial\Omega} + \| \psi_\varrho^- \|_{m_*-m_0,\bar\partial\Omega} \leq C\varepsilon^l, \qquad l \geq 3. \tag{5.14}$$

Then according to (5.4), II_0, III_0, IV_0 and V_0 are trivial as long as ε is small enough. Furthermore applying (5.12) and again taking ε to be sufficiently small yields VII_0 and VIII_0. In addition by the proof of Proposition 5.1 we obtain the following stronger version of I_1:

$$\| u_0 \|_m \leq C_0\varepsilon^2, \qquad m \leq m_* - N - 8.$$

Now the proof of Proposition 5.2 may be appropriately modified to show that VI_1 is valid. This completes the proof by induction.

In view of the hypotheses of Propositions 5.1 to 5.4, we will choose

$$\rho = \min(m_* - N - 10, m_* - m_0) - 1.$$

Since $\rho \geq 3N+9$ we must then have

$$m_* \geq \max(3N+16, 3N+m_0+10).$$

The following corollary yields a solution of (5.2) with $\alpha_0 = m_* - m_0 - N - 6$.

Corollary 5.1. *If $m_0 \geq N+10$ then under the above assumptions $w_n \to w$ in $\overline{H}_0^{m_*-m_0-N-4}(\Omega)$. Furthermore $\Phi(w_n) \to 0$ in $C^0(\Omega)$.*

Proof. When $m_0 \geq N+10$ we have $\rho - 1 = m_* - m_0 - 2$. Then for $m + N + 2 \leq \rho - 1$ and $i > j$, I_n implies that

$$\| w_i - w_j \|_m \leq \sum_{n=j}^{i-1} \| u_n \|_m \leq \varepsilon \sum_{n=j}^{i-1} \mu_n^{m+N+2-\rho} \leq \varepsilon \sum_{n=j}^{i-1} \mu^{-n}.$$

Hence, $\{w_n\}_{n=0}^{\infty}$ is Cauchy in $\overline{H}_0^m(\Omega)$ for all $m \leq m_* - m_0 - N - 4$.

Lastly by the Sobolev Lemma and VIII_n,

$$|\Phi(w_n)|_{C^0(\Omega)} \leq C \| \Phi(w_n) \|_2 \leq \varepsilon \mu_n^{N+6-\rho}.$$

The desired conclusion follows since $\rho > N+6$. □

5.2. Elliptic regions

Here we shall set up the iteration procedure for problem (5.1). For convenience we will denote the domain Ω_κ^+ by Ω. Set $w_0 = 0$ and suppose that functions $w_1, \ldots, w_n$ have been defined on Ω. If $S_i = S_{\mu^i}$ are smoothing operators given by Lemma 5.1, then we put $v_i = S_i w_i$, $0 \leq i \leq n$, and define $w_{n+1} = w_n + u_n$ where u_n is the solution of

$$L(v_n)u_n = f_n \quad \text{in} \quad \Omega, \qquad u_n|_{\overline{\partial\Omega}} = 0, \tag{5.15}$$

given by Theorem 5.2 below, $L(v_n)$ is the operator of Lemma 2.1, and f_n will also be specified below. Let $Q_n(w_n, u_n)$ again denote the quadratic error and $\mathcal{L}(w_n)$ the linearization of (5.1), then according to (5.3) (with $\theta_n = 0$)

we have

$$\begin{aligned}\Phi(w_{n+1}) &= \Phi(w_n) + \mathcal{L}(w_n)u_n + Q_n(w_n, u_n)\\ &= \Phi(w_n) + \varepsilon S_n' a^{22}(v_n)L(v_n)u_n + e_n,\end{aligned}$$

with

$$\begin{aligned}e_n = (\mathcal{L}(w_n) - \mathcal{L}(v_n))u_n + \varepsilon(I - S_n')a^{22}(v_n)L(v_n)u_n + Q_n(w_n, u_n)\\ + \varepsilon(a^{22}(v_n))^{-1}\Phi(v_n)[\partial^2_{x^1}u_n - \partial_{x^1}(\log a^{22}(v_n)\sqrt{|g|})\partial_{x^1}u_n].\end{aligned}$$

Lastly we set $E_0 = 0$, $E_n = \sum_{i=0}^{n-1} e_i$, and define f_n according to (5.7).

It is clear that similar arguments as those used for the hyperbolic regions will show that $\{w_n\}_{n=0}^{\infty}$ converges to a solution of (5.1) if a Moser estimate (like that found in Theorem 5.1) holds for the solution of (5.15). In order to establish such an estimate using the theory of Section 3, we need to extend the coefficients of $L(v_n)$ outside of Ω and cut them off. For this purpose we will use the following extension lemma.

Lemma 5.3 [18]. *Let X be a bounded convex domain in $\mathbb{R}^2$, with Lipschitz smooth boundary. Then there exists a linear operator $E_X : L^2(X) \to L^2(\mathbb{R}^2)$ such that:*

(i) $E_X(u)|_X = u$,

(ii) $E_X : H^m(X) \to H^m(\mathbb{R}^2)$ *continuously for each* $m \in \mathbb{Z}_{\geq 0}$.

Theorem 5.2. *Suppose that $g \in C^{m_*}$. If $m \leq \frac{1}{3}(m_* - 8)$, $|v_n|_{C^6} < 1$ and $\delta = \delta(m)$, $\varepsilon = \varepsilon(m, \delta)$ are sufficiently small, then there exists a solution $u_n \in \overline{H}_0^m(\Omega)$ of (5.15) which satisfies the estimate*

$$\| u_n \|_m \leq \delta^{-1} C_m \left(\| f_n \|_{m+2+\gamma} + \sum_{i+j+l \leq m+23+\gamma} (1 + \| v_n \|_i) \| v_n \|_j \| f_n \|_l\right)$$

for some constant C_m independent of δ and ε and where $2m < \gamma < m_* - m - 6$.

Proof. This will follow from Theorem 3.2. However we must first change to the coordinates $\xi_n^i(x, y)$ of Lemma 2.1, and then change to polar coordinates so that

$$\Omega = \{(r, \theta) \mid 0 < r < \sigma, 0 < \theta < \delta\}.$$

Set

$$\Omega^1 = \{(r,\theta) \mid 0 < r < \sigma + 1, 0 < \theta < \delta\}$$

and let $\varphi = \varphi(r)$ be a smooth nonnegative cut-off function with $\varphi(r) \equiv 1$ for $0 < r < \sigma$, and $\varphi(r) \equiv 0$ for $\sigma + 1 < r$. If we cut-off the coefficients of $L(E_\Omega v_n)$ as in (3.1), we may use Theorem 3.2 to solve

$$L(E_\Omega v_n)u_n = E_\Omega f_n \quad \text{in} \quad \Omega^1, \qquad u_n|_{\partial\Omega^1} = 0,$$

with

$$\| u_n \|'_{(m,\gamma),\Omega^1} \le C_m(\| E_\Omega f_n \|'_{m+2+\gamma,\Omega^1} + \| E_\Omega v_n \|'_{m+6,\Omega^1} \| E_\Omega f_n \|'_{5+\gamma,\Omega^1})$$

for $m \le m_* - 6$ where $\gamma > 2m$ and $\| \cdot \|'$ indicates that the norm is with respect to these polar coordinates. By Lemma 5.3

$$\begin{aligned} \| E_\Omega f_n \|'_{m+2+\gamma,\Omega^1} &\le C_m \| f_n \|'_{m+2+\gamma,\Omega}, \\ \| E_\Omega v_n \|'_{m+6,\Omega^1} &\le C_m \| v_n \|'_{m+6,\Omega} . \end{aligned}$$

Therefore with the help of (2.4) and Lemma 5.2 it follows that

$$\| u_n \|_{m,\Omega} \le \delta^{-1} C_m(\| f_n \|_{m+2+\gamma,\Omega} + \sum_{i+j+l \le m+23+\gamma} (1 + \| v_n \|_{i,\Omega}) \| v_n \|_{j,\Omega} \| f_n \|_{l,\Omega}).$$

The result is now obtained by noting that $\max(m+2+\gamma, 5+\gamma) \le m_* - 2$ is required to apply (2.4). □

We may now apply arguments similar to those in the hyperbolic regions to obtain a solution of (5.1). More precisely, the proofs of Propositions 5.1 to 5.4 yield the following restrictions on ρ, γ, and m_* in the elliptic regions:

$$\rho \ge 2\gamma + 54, \qquad \rho + 1 = \tfrac{1}{3}(m_* - 14).$$

Choosing the largest possible value for γ and noting that the hypothesis of Theorem 5.2 requires $2m < \gamma$, implies that we must have $m \le \frac{1}{12}(m_* - 185)$. The following corollary produces a solution of (5.1) for $\alpha_0 = \frac{1}{12}m_* - 18$.

Corollary 5.2. *If $m_* \ge 192$ then $w_n \to w$ in $\overline{H}_0^{\frac{1}{12}m_*-16}(\Omega)$ with*

$$\| w \|_{\frac{1}{12}m_*-16} \le C\varepsilon^3.$$

Furthermore $\Phi(w_n) \to 0$ in $C^0(\Omega)$.

Proof. The same arguments used for Corollary 5.1 apply. Moreover, we use the analogue of II_n to obtain the estimate for w. $\square$

5.3. Proof of Theorem 1.1

Here we shall construct a solution of (2.1) in a full neighborhood of the origin. First consider the case in which there are exactly two elliptic regions, each bordering two hyperbolic regions. Then on each elliptic region Ω_κ^+ let $w_\kappa^+ \in \overline{H}_0^{\frac{1}{12}m_*-16}(\Omega_\kappa^+)$ be the solution of (5.1) given by Corollary 5.2. On each boundary of the hyperbolic regions $\overline{\partial}\Omega_\varrho^-$ set $\phi_\varrho^- = 0$, $\psi_\varrho^- = -\partial_\nu w_{\kappa(\varrho)}^+|_{\overline{\partial}\Omega_{\kappa(\varrho)}^+} \in \overline{H}_0^{\frac{1}{12}m_*-18}(\overline{\partial}\Omega_{\kappa(\varrho)})$ where $\Omega_{\kappa(\varrho)}^+$ is the bordering elliptic region, and note that (5.14) is valid with $m_0 = \frac{11}{12}m_* + 18$. Then Corollary 5.1 yields a solution $w_\varrho^- \in \overline{H}_0^{m_*-m_0-N-4}(\Omega_\varrho^-)$ of (5.2). Under the hypotheses of Corollaries 5.1 and 5.2 we require $m_* \geq \max(192, 3N + m_0 + 10)$ or rather $m_* \geq 36(N + 10)$.

Suppose that Ω_κ^+ borders on Ω_ϱ^-. Then since the common boundary curve Υ is noncharacteristic for (2.1) (according to our original choice of approximate solution z_0), the functions w_κ^+ and w_ϱ^- agree along with their derivatives up to and including order $\frac{1}{12}m_* - N - 24$ along Υ. It follows that the individual solutions $\{w_\kappa^+\}_{\kappa=1}^{\kappa_0}$ and $\{w_\varrho^+\}_{\varrho=1}^{\varrho_0}$ combine to form a $C^{\frac{1}{12}m_*-N-24}$ solution of (2.1) on some neighborhood of the origin.

Now consider the general case in which elliptic and hyperbolic regions are allowed to border regions of the same type. If an elliptic region borders another elliptic region, they may be combined to form a single elliptic region which contains a single curve of degeneracy on the interior. By appropriately regularizing the linearized equation in this combined region to eliminate the interior degeneracy, we may apply the theory of Section 3 to obtain Theorem 5.2, and hence a solution of (5.1) in this combined region. Therefore, we may assume that each elliptic region is bordered by hyperbolic regions (unless no hyperbolic regions are present). On the other hand, if two hyperbolic regions share a common boundary, for instance Ω_1^- and Ω_3^-, then Cauchy data may be prescribed appropriately on the portion of $\partial\Omega_1^-$ which is shared with $\partial\Omega_3^-$, so that the solution on both regions may be glued together. Moreover, Cauchy data may be arbitrarily prescribed on the portion of $\partial\Omega_3^-$ emanating from the origin and which is not shared with $\partial\Omega_1^-$. It follows that in the general case, the solutions of the elliptic and hyperbolic regions may be patched together in the usual way.

6. Appendix A

The purpose of this appendix is to show existence for the ODE occurring in the proof of Theorem 3.1:

$$\text{(A.1)} \qquad \begin{aligned} &\sum_{s=0}^{m} \lambda^{-s}(-1)^s \partial_r^s (a_{\lambda,\gamma-2(s-1)}\partial_r^s \zeta) = v, \\ &\zeta(r,0) = \zeta(r,\delta) = 0, \qquad \partial_r^s \zeta(\sigma,\theta) = 0, \qquad 0 \le s \le m-1, \\ &\int_{r=r_0} (\partial_r^s \partial_\theta^l \zeta)^2 \le r_0^{\gamma-2s} C, \qquad 0 \le s \le 2m-1, \qquad 0 \le l < \infty, \end{aligned}$$

where $v \in \widehat{C}^\infty(\Omega)$, $\zeta \in H^{(m,\infty,\gamma+2)}(\Omega) \cap C^\infty(\Omega)$, r_0 is sufficiently small, and all other definitions/notation may be found in Section 3.

First note that $\eta \mapsto (\eta, (\lambda\theta^2 - 1)^{-1} v)$ is a bounded linear functional on $H^{(m,0,\gamma+2)}(\Omega)$, and thus by the Riesz representation theorem there exists a unique $\zeta \in H^{(m,0,\gamma+2)}(\Omega)$ such that

$$(\eta, \zeta)_{(m,0,\gamma+2)} = (\eta, (\lambda\theta^2 - 1)^{-1} v) \quad \text{all} \quad \eta \in H^{(m,0,\gamma+2)}(\Omega),$$

where $(\cdot,\cdot)_{(m,0,\gamma+2)}$ denotes the inner product on $H^{(m,0,\gamma+2)}(\Omega)$. It follows that ζ is a weak solution of (A.1), and according to the basic regularity theory for ODEs we have $\zeta \in C^\infty(\Omega)$. Furthermore, the desired boundary behavior of the solution at $\theta = 0, \delta$ arises from the requirement that $v(r,0) = v(r,\delta) = 0$, and the vanishing at $r = \sigma$ is a result of the trace theorem for Sobolev spaces.

Lastly we observe that since v vanishes in a neighborhood of $r = 0$, the solution ζ satisfies a version of the so-called Euler differential equation in this domain. All solutions of this equation may be written down explicitly. In particular, for r sufficiently small ζ must be a linear combination of $2m$ functions of the form: $r^\alpha (\log r)^\beta$ where $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{Z}_{\ge 0}$. However according to Lemma 3.1

$$\text{(A.2)} \quad \int_{r=r_0} (\partial_r^s \partial_\theta^l \zeta)^2 \le r_0^{\gamma-2s} C \parallel \zeta \parallel^2_{(m,l+1,\gamma+2)}, \qquad s \le m-1, \qquad 0 \le l < \infty.$$

Therefore each term in the linear combination must satisfy

$$\text{(A.3)} \qquad r^\alpha (\log r)^\beta = O(r^{\gamma/2}) \quad \text{as} \quad r \to 0.$$

The desired boundary behavior at $r = 0$ now follows from (A.2) and (A.3).

7. Appendix B

The purpose of this section is to construct the smoothing operators S_μ of Lemma 5.1. The construction will differ from the standard one for S'_μ (see [16]), in that the smoothed functions are required to vanish identically at the origin. This, of course, is only possible if the function being smoothed already vanishes in an appropriate sense at the origin.

We first construct smoothing operators on the plane, and will later restrict them back to the bounded domain Ω. Fix $\widehat{\chi} \in C_c^\infty(\mathbb{R}^2)$ such that $\widehat{\chi} \equiv 1$ on some neighborhood of the origin, and let

$$\chi(x) = \int_{\mathbb{R}^2} \mathrm{e}^{2\pi \mathrm{i}\xi\cdot x}\widehat{\chi}(\xi)\, d\xi$$

be its inverse Fourier Transform. Then χ is a Schwartz function and satisfies

$$\int_{\mathbb{R}^2} \chi(x)\, dx = 1, \qquad \int_{\mathbb{R}^2} x^\alpha \chi(x)\, dx = 0, \qquad |\alpha| > 0.$$

Furthermore, let $\eta \in C^\infty(\mathbb{R}^2)$ be a radial function vanishing to all orders at the origin, and satisfying

$$\eta(x) = \begin{cases} 1 & \text{if } |x| > 1, \\ 0 & \text{if } |x| < \frac{1}{2}. \end{cases}$$

For $\mu \geq 1$ we will write $\eta_\mu(x) = \eta(\mu x)$, $\chi_\mu(x) = \mu^2\chi(\mu x)$ and define smoothing operators $\overline{S}_\mu : L^2(\mathbb{R}^2) \to \overline{H}_0^\infty(\mathbb{R}^2)$ by

$$(\overline{S}_\mu u)(x) = \eta_\mu(x)(\chi_\mu * u)(x) = \mu^2\eta(\mu x)\int_{\mathbb{R}^2} \chi(\mu(x-y))u(y)\, dy.$$

Here the space $\overline{H}_0^l(\mathbb{R}^2)$ is the completion of $\overline{C}_c^\infty(\mathbb{R}^2)$ in the Sobolev norm $\|\cdot\|_l$, where $\overline{C}_c^\infty(\mathbb{R}^2)$ denotes all $C_c^\infty(\mathbb{R}^2)$ functions vanishing in a neighborhood of the origin.

We now proceed to show statements (i) to (iii) of Lemma 5.1 with respect to $\overline{S}_\mu$. Note that it is sufficient to prove these for $u \in \overline{C}_c^\infty(\mathbb{R}^2)$ as this space of functions is dense in $\overline{H}_0^l(\mathbb{R}^2)$. We begin with (ii). Let

$$u(y) = \sum_{|\alpha|<|\rho|} \frac{1}{|\alpha|!}\partial^\alpha u(x)(y-x)^\alpha + \frac{1}{(|\rho|-1)!} \times \sum_{|\alpha|=|\rho|} (y-x)^\alpha \int_0^1 (1-t)^{|\rho|-1}\partial^\alpha u(x+t(y-x))\,dt$$

be a Taylor expansion of u with integral remainder. Then according to the properties of χ,

(B.1)
$$(\chi_\mu * u)(x) = u(x) + \frac{\mu^2}{(|\rho|-1)!}\sum_{|\alpha|=|\rho|}\int_{\mathbb{R}^2}\int_0^1 \chi(\mu(y-x)) \times (1-t)^{|\rho|-1}(y-x)^\alpha\partial^\alpha u(x+t(y-x))\,dt\,dy.$$

Suppose that $l \leq |\sigma| \leq m$, and notice that

$$\| \partial^\sigma \overline{S}_\mu u \| \leq \sum_{\beta+\gamma=\sigma} \| \partial^\beta \eta_\mu \partial^\gamma (\chi_\mu * u) \| .$$

If $|\rho| = l - |\gamma| > 0$ we may apply (B.1) to find

$$\begin{aligned}\| \partial^\beta \eta_\mu \partial^\gamma (\chi_\mu * u) \|^2 &= \mu^{2|\beta|}\int_{\mathbb{R}^2}(\partial^\beta\eta)^2(\mu x)(\chi_\mu * \partial^\gamma u)^2(x)dx \\ &\leq C_1\mu^{2|\beta|}\int_{\mathbb{R}^2}(\partial^\beta\eta)^2(\mu x)(\partial^\gamma u)^2(x)dx \\ &\quad + C_2\mu^{2(|\beta|+|\gamma|-l)} \| u \|_l^2 .\end{aligned}$$

Under the current assumptions $|\beta| \neq 0$ which implies that supp $\partial^\beta\eta_\mu \subset \{|x| < \mu^{-1}\}$, so applying the Taylor expansion of $\partial^\gamma u$ at $x=0$ with $|\rho| = l - |\gamma|$ and recalling that u vanishes to all orders at the origin, yields

(B.2)
$$\int_{\mathbb{R}^2}(\partial^\beta\eta)^2(\mu x)(\partial^\gamma u)^2(x)\,dx \leq C_3\int_{B_{\mu^{-1}}(0)}(\partial^\gamma u)^2(x)\,dx \leq C_4\mu^{2(|\gamma|-l)} \| u \|_l^2 .$$

Moreover the case $|\gamma| \geq l$ may be treated by Young's inequality:

$$\begin{aligned}\| \partial^\beta \eta_\mu \partial^\gamma (\chi_\mu * u) \|^2 &\leq C_5\mu^{2(|\beta|+|\gamma|-|\tau|)}\int_{\mathbb{R}^2}(\partial^{\gamma-\tau}\chi_\mu * \partial^\tau u)^2(x)dx \\ &\leq C_6\mu^{2(m-l)} \| u \|_l^2,\end{aligned}$$

where $|\tau| = l$. Therefore (ii) follows once (i) is established, and (i) is established by similar arguments which will be omitted here.

We now show (iii). Let $|\alpha| = m \leq l$ and observe that

$$\begin{aligned} \| \partial^{\alpha}(u - \overline{S}_{\mu}u) \| &\leq \| \partial^{\alpha}[(1-\eta_{\mu})u] \| + \| \partial^{\alpha}[\eta_{\mu}(u - \chi_{\mu} * u)] \| \\ &\leq \sum_{\beta+\gamma=\alpha} (\| \partial^{\beta}(1-\eta_{\mu})\partial^{\gamma}u \| + \| \partial^{\beta}\eta_{\mu}\partial^{\gamma}(u - \chi_{\mu} * u) \|). \end{aligned}$$

According to the standard construction [17],

$$\begin{aligned} \| \partial^{\beta}\eta_{\mu}\partial^{\gamma}(u - \chi_{\mu} * u) \| &\leq C_7 \mu^{|\beta|} \| u - \chi_{\mu} * u \|_{|\gamma|} \\ &\leq C_8 \mu^{|\beta|} \mu^{|\gamma| - l} \| u \|_l \\ &= C_8 \mu^{m-l} \| u \|_l \,. \end{aligned}$$

Furthermore since supp $\partial^{\beta}(1-\eta_{\mu}) \subset \{|x| < \mu^{-1}\}$ we may apply the same methods used to establish (B.2) to obtain

$$\| \partial^{\beta}(1-\eta_{\mu})\partial^{\gamma}u \| \leq C_9 \mu^{m-l} \| u \|_l \,.$$

It follows that (iii) holds.

The desired smoothing operators on Ω may be obtained from $\overline{S}_{\mu}$ in the following way. If Ω is a bounded convex Lipschitz domain, then Lemma 5.3 yields an extension operator $E : H^m(\Omega) \to H^m(\mathbb{R}^2)$. We then define smoothing operators $S_{\mu} : L^2(\Omega) \to \overline{H}_0^{\infty}(\Omega)$ by $S_{\mu}u = (\overline{S}_{\mu}Eu)|_{\Omega}$. As E is bounded, it is clear that Lemma 5.1 will also hold for S_{μ}.

Acknowledgments

Q.H. acknowledges the support of NSF Grant DMS-0654261. M.K. acknowledges the support of NSF Grant DMS-1007156 and a Sloan Research Fellowship.

References

[1] Q. Han, *On the isometric embedding of surfaces with Gauss curvature changing sign cleanly*, Comm. Pure Appl. Math. **58** (2005), 285–295.

[2] Q. Han, *Local isometric embedding of surfaces with Gauss curvature changing sign stably across a curve*, Calc. Var. Partial Differential Equations **25** (2006), 79–103.

[3] Q. Han, *Smooth local isometric embedding of surfaces with Gauss curvature changing sign cleanly*, preprint.

[4] Q. Han and J.-X. Hong, *Isometric embedding of Riemannian manifolds in Euclidean spaces*, Mathematical Surveys and Monographs **130**, AMS, Providence, RI, 2006.

[5] Q. Han, J.-X. Hong and C.-S. Lin, *Local isometric embedding of surfaces with nonpositive Gaussian curvature,* J. Differential Geom. **63** (2003), 475–520.

[6] Q. Han and M. Khuri, *Smooth solutions of a class of mixed-type differential equations*, preprint.

[7] P. Hartman and A. Wintner, *Gaussian curvature and local embedding*, Amer. J. Math. **73** (1951), 876–884.

[8] M. Khuri, *The local isometric embedding in* $\mathbb{R}^3$ *of two-dimensional Riemannian manifolds with Gaussian curvature changing sign to finite order on a curve*, J. Differential Geom. **76** (2007), 249–291.

[9] M. Khuri, *Local solvability of degenerate Monge–Ampère equations and applications to geometry*, Electron. J. Differential Equations **2007**(65) (2007), 1–37.

[10] M. Khuri, *Counterexamples to the local solvability of Monge–Ampère equations in the plane,* Comm. Partial Differential Equations **32** (2007), 665–674.

[11] C.-S. Lin, *The local isometric embedding in* $\mathbb{R}^3$ *of* 2*-dimensional Riemannian manifolds with nonnegative curvature*, J. Differential Geom., **21**(2) (1985), 213–230.

[12] C.-S. Lin, *The local isometric embedding in* $\mathbb{R}^3$ *of two-dimensional Riemannian manifolds with Gaussian curvature changing sign cleanly*, Comm. Pure Appl. Math. **39**(6) (1986), 867–887.

[13] N. Nadirashvili and Y. Yuan, *Improving Pogorelov's isometric embedding couter-example,* Calc. Var. Partial Differential Equations **21**(3) (2008), 319–323.

[14] O. Oleinik, *On the Cauchy problem for weakly hyperbolic equations*, Comm. Pure Appl. Math. **23** (1970), 569–586.

[15] A. Pogorelov, *An example of a two-dimensional Riemannian metric not admitting a local realization in E_3*, Dokl. Akad. Nauk. USSR **198** (1971), 42–43.

[16] L. Schlaefli, *Nota alla memoria del Sig. Beltrami, Sugli spazii di curvatura constante,* Ann. Mat., 2e série, **5** (1873), 170–193.

[17] J. Schwartz, *Nonlinear functional analysis*, New York University, New York, 1964.

[18] E. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton, NJ, 1970.

[19] M. Taylor, *Partial differential equations* III, Springer-Verlag, New York, 1996.

[20] J. Weingarten, *Über die theorie der Aubeinander abwickelbarren Oberflächen*, Berlin, 1884.

Department of Mathematics
University of Notre Dame
Notre Dame, IN 46556
USA

and

School of Mathematical Sciences,
Peking University,
100871 Beijing,
China
E-mail address: qhan@nd.edu

Department of Mathematics
Stony Brook University
Stony Brook, NY 11794
USA
E-mail address: khuri@math.sunysb.edu

Received May 27, 2009