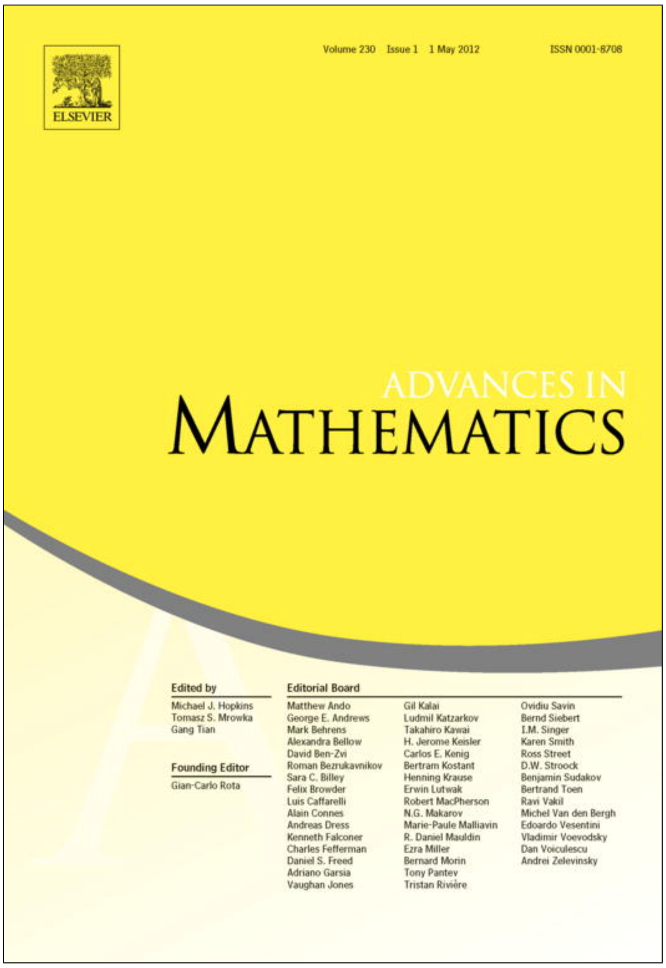
Volume 230 Issue 1 1 May 2012
ISSN 0001-8708
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Advances in Mathematics 230 (2012) 263–293

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The linearized system for isometric embeddings and its characteristic variety ☆

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Received 2 April 2011; accepted 20 November 2011

Available online 29 November 2011

Communicated by Gang Tian

Abstract

In this paper we prove a conjecture of Bryant, Griffiths, and Yang concerning the characteristic variety for the determined isometric embedding system. In particular, we show that the characteristic variety is not smooth for any dimension greater than 4. This is accomplished by introducing a smaller yet equivalent linearized system, in an appropriate way, which facilitates analysis of the characteristic variety.

Keywords: Isometric embeddings; Characteristic variety

1. Introduction

Let (M^n, g) be an n-dimensional Riemannian manifold. It is a classical problem to find an isometric embedding

$$\left(M^n, g\right) \hookrightarrow \mathbb{R}^N. \tag{1.1}$$

☆ The first author acknowledges the support of NSF Grant DMS-0654261. The second author acknowledges the support of NSF Grant DMS-1007156 and a Sloan Research Fellowship.

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0001-8708/$ – see front matter
doi:10.1016/j.aim.2011.11.011

The existence of such a global isometric embedding for some N was first proved by Nash [17]. A better N was later found by Günther [6]. In this paper we will focus exclusively on the local isometric embedding problem.

Suppose that the metric $g = g_{ij}(x)\,dx^i\,dx^j$ is given in a neighborhood of a point, say $(x^1, \dots, x^n) = 0$. Then we seek N functions $\{u^i\}_{i=1}^N$ such that

$$g = \left(du^1\right)^2 + \cdots + \left(du^N\right)^2.$$

Therefore (1.1) is equivalent to the local solvability of the following first order nonlinear system

$$\sum_{k=1}^{N} \partial_{x^i} u^k \cdot \partial_{x^j} u^k = g_{ij} \quad \text{for } 1 \leqslant i, j \leqslant n. \tag{1.2}$$

There are $n(n+1)/2$ equations and N unknowns in this system. Hence, this system is underdetermined if $N > n(n+1)/2$ and overdetermined if $N < n(n+1)/2$. In the following, we will always assume that $N = n(n+1)/2$.

For $n = 2$, the existence of local isometric embeddings of surfaces into $\mathbb{R}^3$ is equivalent to the existence of local solutions of Darboux's equation, a fully nonlinear equation of the Monge–Ampère class. The type of Darboux's equation is determined solely by the Gauss curvature. More precisely it is elliptic if the Gauss curvature is positive, hyperbolic if the Gauss curvature is negative, and degenerate if the Gauss curvature has zeroes. Under various assumptions on the Gauss curvature, the existence of local isometric embeddings was proven by Lin [14,15], Han, Hong and Lin [10], Han [7,8], Han and Khuri [11], and Khuri [12,13]. (See [9] for details.)

The situation becomes more complicated for $n \geqslant 3$. Bryant, Griffiths and Yang [4] studied the local isometric embedding problem for n-dimensional Riemannian manifolds and analyzed the structure of the characteristic variety for the linearized system (see also [2]). They proved the existence of local isometric embeddings of 3-dimensional Riemannian manifolds into $\mathbb{R}^6$ under an appropriate assumption on the curvature. Later on Nakamura and Maeda [16] (independently Goodman and Yang [5]) proved the existence of local isometric embeddings of 3-dimensional Riemannian manifolds into $\mathbb{R}^6$ when the Riemann curvature tensor does not vanish. Poole [18] has extended this result to the case in which the Riemann curvature tensor vanishes cleanly.

The difficulty in studying isometric embeddings of higher dimensional Riemannian manifolds lies with the following two related facts. First the differential system (1.2) is very large, consisting of $n(n+1)/2$ equations for $n(n+1)/2$ unknowns. Second and most importantly, it is not at all clear how the curvature determines the type of this system. Hence, a natural first step is to investigate whether this huge system can be simplified. Since (1.2) is nonlinear this requires an understanding of the linearized system. However due to its invariance under the orthogonal group, (1.2) is highly degenerate in that every direction is characteristic, so a direct study of the linearization appears to be futile. It is thus necessary to replace the linearized equations by an equivalent system which is easier to analyze. Bryant, Griffiths and Yang [4] pointed out that the linearization of (1.2) is in fact equivalent to a smaller differential system of n equations for n unknowns. One may then focus attention on the structure of the characteristic variety for this new system. For $n = 3$, they proved that the characteristic variety is smooth whenever certain parameters in the linearized equations lie in appropriate ranges. The smoothness of the characteristic variety plays an essential role in the existence results in [4,5,16,18]. For higher dimensions, they proved that the characteristic variety is smooth for $n = 4$ and not smooth for $n = 6, 10, 14, \ldots$. They also conjectured that the characteristic variety is not smooth for any $n \geqslant 5$.

In this paper, we will put this equivalent linearized system in an explicit form by introducing appropriate parameters. Based on this explicit expression, we will prove that the characteristic variety is indeed not smooth for all higher dimensions when these parameters are sufficiently small.

To motivate our study, let u be a solution of (1.2) and consider the linearization of (1.2) at u. It has the following form

$$\partial_j u \cdot \partial_i v + \partial_i u \cdot \partial_j v = f_{ij} \quad \text{for any } 1 \leqslant i, j \leqslant n. \tag{1.3}$$

To find a better equation for v, we rewrite this as

$$\partial_i(\partial_j u \cdot v) + \partial_j(\partial_i u \cdot v) - 2\partial_{ij} u \cdot v = f_{ij} \quad \text{for any } 1 \leqslant i, j \leqslant n. \tag{1.4}$$

We note that the inner product $\partial_j u \cdot v$ is a component of the projection of v into the tangent space spanned by $\{\partial_1 u, \ldots, \partial_n u\}$. It is clear from (1.4) that the derivatives are only applied to tangential components of v. This suggests that we should decompose v relative to the tangent space and normal space of the embedding u. In other words, we uncouple the system by breaking v into tangential and normal components. It turns out that the normal components of v satisfy an algebraic system which we solve first. Then the tangential components of v satisfy a differential system of first order, which consists of n equations for n unknowns. This new system is much easier to study than (1.3). Moreover, the curvature tensor of g has an explicit expression in terms of coefficients of this new system. In summary the linearized isometric embedding system, an $n(n+1)/2 \times n(n+1)/2$ system, can be reduced to an $n \times n$ system which can be put into an explicit form. We point out that (1.4) appears in [4] as (2.c.5) and (4.d.4), and that the equivalent $n \times n$ system is given by (4.d.5).

Now we describe this new $n \times n$ differential system in a more explicit way. To start with, let g be a smooth metric in a neighborhood of the origin in $\mathbb{R}^n$. For $i, j, k = 1, \ldots, n$, let $\mathbf{c} = \{c_i^{kj}\}_{k \neq j}$ be a collection of parameters with

$$c_i^{kj} = c_i^{jk} \quad \text{for any } i, j, k \text{ with } j \neq k,$$

and set

$$c_i^{ii} = 1 \quad \text{for any } i,$$
$$c_i^{jj} = 0 \quad \text{for any } i \neq j.$$

There are $n^2(n-1)/2$ elements in $\mathbf{c}$. Now define $n \times n$ matrices $A^1, \ldots, A^n$ by

$$\left(A^k\right)_{ij} = \left(c_i^{kj}\right).$$

We may then formulate a differential system in the following way

$$A^1 \partial_1 V + \cdots + A^n \partial_n V = F, \tag{1.5}$$

where V, F are vector-valued functions of n components. This system has constant coefficients which are related to the curvature of g at the origin; this will be described in detail in Section 3. An important result, stated in Lemma 3.1, asserts that (1.5) *is the equivalent linearized isometric embedding system evaluated at* $x = 0$.

As mentioned above, the main step of changing (1.3) into an equivalent $n \times n$ differential system was already observed in [4], and a version of this equivalent system was given by (4.d.5). However, the explicit form of (1.5) in this paper is new. As we will see, this explicit form has natural advantages when it comes to analyzing the characteristic variety in detail.

To better understand (1.5), it is important to study its characteristic variety. For each $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ define

$$P = P(\xi, \mathbf{c}) = \sum_{i=1}^{n} \xi_k A^k,$$

where $\mathbf{c}$ is as above. This is the *principal symbol*, and the associated *characteristic variety* is then given by

$$\Sigma(\mathbf{c}) = \bigl\{\xi \in \mathbb{R}^n \setminus \{0\} \bigm| \det P(\xi, \mathbf{c}) = 0\bigr\}.$$

In dimension 3, and under the assumption that the matrices A^k are symmetric, it was shown [4] that $\Sigma(\mathbf{c})$ is smooth in $\mathbb{R}^3 \setminus \{0\}$ except for three choices of $\mathbf{c}$. Moreover in higher dimensions, it was shown that $\Sigma(\mathbf{c})$ is generally smooth in $\mathbb{R}^n \setminus \{0\}$ for $n = 4$ but not smooth in $\mathbb{R}^n \setminus \{0\}$ for $n = 6, 10, 14, \dots$. (See Corollary 1.c.6 in [4].) The following conjecture was posed in [4].

Conjecture 1.1. $\Sigma(\mathbf{c})$ *is not smooth in* $\mathbb{R}^n \setminus \{0\}$ *for* $n \geqslant 5$.

Based on the explicit form of the principal symbol, we will give an affirmative answer to this conjecture for small $\mathbf{c}$. We will say that the parameters $\mathbf{c}$ satisfy a *generic condition* if they satisfy a finite number of (homogeneous) polynomial inequalities. For a precise statement of the following result, see Theorems 4.3 and 4.4 below.

Theorem 1.2. *For any* $n \geqslant 5$ *and any small* $\mathbf{c}$ *satisfying a generic condition,* $\Sigma(\mathbf{c})$ *is not smooth in* $\mathbb{R}^n \setminus \{0\}$.

As is shown in the proof, the generic condition will be given explicitly. In the case of dimension 4, we will show that under generic conditions and the smallness assumption the characteristic variety is smooth; a more general result of this nature has already been obtained in [4]. Our proof here is different.

Let $\Sigma_{\mathrm{sing}}(\mathbf{c})$ be the singular part of $\Sigma(\mathbf{c})$. We will prove that for $n = 5$ the set $\Sigma_{\mathrm{sing}}(\mathbf{c}) \cap \mathbb{P}^4$ generically consists of exactly $10 + \alpha + 2\beta + 3\gamma$ points when $\mathbf{c}$ is sufficiently small, where α, β, γ are nonnegative integers with $\alpha + \gamma = 10$ and $\beta \leqslant 5$, and where $\mathbb{P}^4$ denotes real projective space. These points can be located in terms of the components of $\mathbf{c}$. In the general case $n \geqslant 6$, it will be shown that $\Sigma_{\mathrm{sing}}(\mathbf{c}) \cap \mathbb{S}^{n-1}$ contains a smooth surface of dimension $n - 5$ for sufficiently small $\mathbf{c}$ (also assuming generic conditions). We believe that $\Sigma_{\mathrm{sing}}(\mathbf{c}) \cap \mathbb{S}^{n-1}$ itself consists of an algebraic variety of dimension $n - 5$, possibly under extra assumptions on $\mathbf{c}$. Such an algebraic variety may have singularities. For example for $n = 6$, $\Sigma_{\mathrm{sing}}(\mathbf{c}) \cap \mathbb{S}^5$ should consist of finitely many curves which intersect at finitely many points. We note that it would be desirable to have a stratification of $\Sigma_{\text{sing}}(\mathbf{c}) \cap \mathbb{S}^{n-1}$ for any $n \geqslant 5$. Of course, it also remains a challenge to remove the smallness assumption on $\mathbf{c}$.

This paper is organized in the following way. In Section 2 we construct appropriate approximate solutions to the isometric embedding system. In Section 3 we discuss the linearized equations and introduce our explicit equivalent $n \times n$ system. Lastly in Section 4, we examine the characteristic variety and prove Theorem 1.2.

2. Constructing approximate solutions

In this section we construct an appropriate approximate isometric embedding, which plays an important role in later discussions.

We first briefly review the theory of surfaces in Euclidean spaces. In this paper, we will exclusively discuss n-dimensional surfaces in Euclidean space of dimension s_n. Here

$$s_n = \frac{1}{2}n(n+1).$$

Hence, the codimension is

$$s_n - n = \frac{1}{2}n(n-1).$$

The Einstein summation convention will be used with respect to indices $1 \leqslant i, j, k, \ldots \leqslant n$ and $1 \leqslant \mu, \tau, \ldots \leqslant n(n-1)/2$.

Let $u : \mathbb{R}^n \to \mathbb{R}^{n(n+1)/2}$ be a smooth embedding. Denote the corresponding embedded submanifold by $\mathcal{M}^n$. Then $\{\partial_i u(x)\}_{i=1}^n$ spans $T_x\mathcal{M}^n$ for each x. Let $\{N_\mu(x)\}_{\mu=1}^{n(n-1)/2}$ span $(T_x\mathcal{M}^n)^\perp$, the orthogonal complement of $T_x\mathcal{M}^n$ in $\mathbb{R}^{n(n+1)/2}$. Denote the induced metric on $\mathcal{M}^n$ by

$$p_{ij} = \partial_i u \cdot \partial_j u.$$

Now recall the fundamental equations for the surface induced by u. Namely $\partial_{ij}u$ has a decomposition into its tangential and normal components, with respect to u, given by

$$\partial_{ij}u = \Gamma_{ij}^k \partial_k u + H_{ij}, \tag{2.1}$$

where Γ_{ij}^k are Christoffel symbols corresponding to p_{ij} and H_{ij} is the second fundamental form. Moreover we have

$$\partial_j N_\mu \cdot \partial_i u = -N_\mu \cdot \partial_{ij}u = -N_\mu \cdot H_{ij}. \tag{2.2}$$

By setting $H_{ij}^\mu = H_{ij} \cdot N_\mu$, $1 \leqslant \mu \leqslant n(n-1)/2$, we have

$$H_{ij} = \sum_{\mu=1}^{n(n-1)/2} H_{ij}^\mu N_\mu.$$

Also the Gauss equations are given by

$$R_{ijkl} = \sum_{\mu=1}^{n(n-1)/2} H_{ik}^{\mu} H_{jl}^{\mu} - H_{il}^{\mu} H_{jk}^{\mu}, \tag{2.3}$$

where R_{ijkl} is the curvature tensor associated with the metric p_{ij}.

Next we construct approximate solutions to the isometric embedding system. Let g be a metric defined in a neighborhood of the origin in $\mathbb{R}^n$. Take normal coordinates so that

$$g_{ij}(0) = \delta_{ij} \quad \text{and} \quad \partial_k g_{ij}(0) = 0 \quad \text{for any } 1 \leqslant i, j, k \leqslant n. \tag{2.4}$$

Consider a map $u = (u^1, \ldots, u^{n(n+1)/2}) : \mathbb{R}^n \to \mathbb{R}^{n(n+1)/2}$ whose components are given by

$$\begin{aligned} u^l &= x^l + \frac{1}{3!} \sum_{1 \leqslant i,j,k \leqslant n} \alpha_{ijk}^l x^i x^j x^k \quad \text{for any } l = 1, \ldots, n, \\ u^{n+\mu} &= \frac{1}{2} \sum_{1 \leqslant i,j \leqslant n} h_{ij}^{\mu} x^i x^j \quad \text{for any } \mu = 1, \ldots, n(n-1)/2, \end{aligned} \tag{2.5}$$

for some constants α_{ijk}^l and h_{ij}^{μ}, $i, j, k, l = 1, \ldots, n$ and $\mu = 1, \ldots, n(n-1)/2$. We will now investigate whether the induced metric $du \cdot du$ agrees with the given metric g up to order two at the origin.

First note that for any $i, j, k = 1, \ldots, n$ and $\mu = 1, \ldots, n(n-1)/2$, we have

$$\begin{aligned} &u(0) = 0, \\ &\partial_j u^i(0) = \delta_{ij}, \qquad \partial_j u^{n+\mu}(0) = 0, \\ &\partial_{ij} u^k(0) = 0, \qquad \partial_{ij} u^{n+\mu}(0) = h_{ij}^{\mu}, \\ &\Gamma_{ij}^k(0) = 0, \\ &N_{\mu}(0) = (0, \ldots, 0, \overset{n+\mu}{1}, 0, \ldots, 0). \end{aligned} \tag{2.6}$$

Furthermore according to (2.4), the metric induced by the embedding u agrees with the given metric g up to order one at the origin. In order for such a metric to agree with g up to order two at the origin, we must have

$$\partial_{kl} g_{ij}(0) = \partial_{ik} u(0) \cdot \partial_{lj} u(0) + \partial_{il} u(0) \cdot \partial_{jk} u(0) + \partial_j u(0) \cdot \partial_{ikl} u(0) + \partial_i u(0) \cdot \partial_{ljk} u(0). \tag{2.7}$$

Recall the expression for the curvature tensor in normal coordinates

$$R_{ijkl} = \frac{1}{2}(\partial_{il} g_{jk} + \partial_{jk} g_{il} - \partial_{ik} g_{jl} - \partial_{jl} g_{ik}). \tag{2.8}$$

Hence (2.7) implies that

$$R_{ijkl}(0) = \partial_{ik}u(0) \cdot \partial_{jl}u(0) - \partial_{il}u(0) \cdot \partial_{jk}u(0).$$

Therefore we have

$$R_{ijkl}(0) = \sum_{\mu=1}^{n(n-1)/2} h^{\mu}_{ik}h^{\mu}_{jl} - h^{\mu}_{il}h^{\mu}_{jk}. \tag{2.9}$$

These are simply the Gauss equations when h^{μ}_{ij} are interpreted as the coefficients of the second fundamental form at $x = 0$. In other words, the Gauss equations (2.9) are a necessary condition for u to be an approximate solution of the isometric embedding system up to order two. Next, we prove that it is also a sufficient condition.

Lemma 2.1. *Let g be a smooth metric defined in a neighborhood of the origin in $\mathbb{R}^n$ and let R_{ijkl} be its curvature tensor. For any constants h^{μ}_{ij} satisfying* (2.9)*, there exist constants α^{l}_{ijk} such that the map $u : \mathbb{R}^n \to \mathbb{R}^{n(n+1)/2}$ in* (2.5) *satisfies*

$$du \cdot du - g = O\big(|x|^3\big) \quad \textit{as } |x| \to 0.$$

Proof. In the following, we denote derivatives of components of u evaluated at the origin by $u^k_i = \partial_i u^k(0)$, $u^k_{ij} = \partial_{ij}u^k(0)$, etc. All quantities in the proof are evaluated at the origin. We need to find α^l_{ijk} so that (2.7) holds. We now write (2.7) in the form

$$\partial_{kl}g_{ij} = u_{ik} \cdot u_{lj} + u_{il} \cdot u_{jk} + u_j \cdot u_{ikl} + u_i \cdot u_{ljk}, \tag{2.10}$$

and treat (2.10) as a linear system for α^l_{ijk}. A simple calculation yields that the total number of equations A and unknowns B are given by

$$A = \left(\frac{n(n+1)}{2}\right)^2, \qquad B = n\sum_{i=1}^{n}\frac{i(i+1)}{2} = \frac{n^2(n+1)(n+2)}{6}.$$

Obviously $A > B$. Hence, (2.10) is an overdetermined system. Our strategy is to choose a collection of B equations to solve for α^l_{ijk} and then verify that the rest of the equations hold automatically under the assumption (2.9).

To this end, we first set

$$\tau_{ij} = (i-1)n - \frac{1}{2}i(i+1) + j \quad \text{for } 1 \leqslant i < j \leqslant n.$$

Obviously $\tau_{i\,j+1} = \tau_{ij} + 1$ and $\tau_{i+1\,i+2} = \tau_{in} + 1$. Moreover $\tau_{12} = 1$ and $\tau_{n-1\,n} = n(n-1)/2$. Hence τ_{ij} enumerates the set of integers $\{1, \dots, n(n-1)/2\}$ for $1 \leqslant i < j \leqslant n$. In fact

$$1 = \tau_{12} < \cdots < \tau_{1n} < \tau_{23} \cdots < \tau_{2n} < \cdots < \tau_{n-1\,n} = \frac{1}{2}n(n-1).$$

Now, we classify the equation for $\partial_{kl} g_{ij}$ in (2.10) according to whether the 4-tuple (i, j, k, l) satisfies the conditions

$$i < j, \quad k < l, \quad \tau_{ij} \leqslant \tau_{kl}. \tag{2.11}$$

We first solve those equations which do not satisfy (2.11). To see this, we calculate the number of equations C of this form:

$$C = \frac{1}{2} \cdot \frac{n(n-1)}{2} \cdot \left(\frac{n(n-1)}{2} + 1\right).$$

The number of degrees of freedom remaining is given by

$$B - (A - C) = \frac{n(n-1)(n-2)(n-3)}{24} \geqslant 0.$$

A further calculation shows that this value coincides with the number of equivalence classes of 4-tuples with all entries distinct and with $i < j$, $k < l$, $\tau_{ij} < \tau_{kl}$ (here we say that two tuples are equivalent if they are permutations of each other). Then in order to use up all the degrees of freedom, we choose to have one equation of each of these equivalence classes (where all entries are distinct and $i < j, k < l, \tau_{ij} < \tau_{kl}$) satisfied.

The final task is to show that all remaining equations of (2.10) follow from (2.9), which has the form

$$R_{ijkl}(0) = u_{ik} \cdot u_{jl} - u_{il} \cdot u_{jk}. \tag{2.12}$$

The remaining equations may be put into three cases. The first case occurs when $i = k$, $j = l$, $i < j$. In this case we need to prove

$$\partial_{ij} g_{ij} = u_{ii} \cdot u_{jj} + u_{ij} \cdot u_{ij} + u_j \cdot u_{iij} + u_i \cdot u_{jji}, \quad 1 \leqslant i < j \leqslant n. \tag{2.13}$$

Consider the equations obtained by permuting these indices

$$\frac{1}{2}\partial_{ii} g_{jj} = u_{ij} \cdot u_{ij} + u_j \cdot u_{iij},$$
$$\frac{1}{2}\partial_{jj} g_{ii} = u_{ij} \cdot u_{ij} + u_i \cdot u_{jji}.$$

These two equations are known to be satisfied. By a simple addition, we get

$$\begin{aligned}\frac{1}{2}\partial_{ii} g_{jj} + \frac{1}{2}\partial_{jj} g_{ii} &= 2u_{ij} \cdot u_{ij} + u_j \cdot u_{iij} + u_i \cdot u_{jji} \\ &= u_{ij} \cdot u_{ij} - u_{ii} \cdot u_{jj} + u_{ii} \cdot u_{jj} + u_{ij} \cdot u_{ij} + u_j \cdot u_{iij} + u_i \cdot u_{jji}.\end{aligned}$$

By expressing R_{ijij} in terms of (2.8) and (2.12), we have

$$\partial_{ij} g_{ij} - \frac{1}{2}\partial_{ii} g_{jj} - \frac{1}{2}\partial_{jj} g_{ii} = u_{ii} \cdot u_{jj} - u_{ij} \cdot u_{ij}.$$

A simple comparison yields (2.13).

The second case occurs when $k=j, i<j, j<l$, or $i=k, i<j, i<l$, or $j=l, i<j, k<j$. We first consider $k=j, i<j, j<l$. In this case, we need to prove

$$\partial_{jl}g_{ij}=u_{ij}\cdot u_{jl}+u_{il}\cdot u_{jj}+u_j\cdot u_{ijl}+u_i\cdot u_{jjl}. \tag{2.14}$$

Consider the equations obtained by permuting these indices

$$\begin{aligned}
\partial_{il}g_{jj}&=2u_{ij}\cdot u_{jl}+2u_j\cdot u_{ijl},\\
\partial_{jj}g_{il}&=2u_{ij}\cdot u_{jl}+u_i\cdot u_{ljj}+u_l\cdot u_{ijj},\\
\partial_{ij}g_{jl}&=u_{ij}\cdot u_{jl}+u_{jj}\cdot u_{il}+u_j\cdot u_{ijl}+u_l\cdot u_{ijj}.
\end{aligned}$$

All three of these equations are known to be satisfied. By adding the first two equations and subtracting the third, we get

$$\begin{aligned}
\partial_{il}g_{jj}+\partial_{jj}g_{il}-\partial_{ij}g_{jl}&=3u_{ij}\cdot u_{jl}-u_{jj}\cdot u_{il}+u_j\cdot u_{ijl}+u_i\cdot u_{jjl}\\
&=2u_{ij}\cdot u_{jl}-2u_{ij}\cdot u_{jl}+u_{il}\cdot u_{jj}+u_j\cdot u_{ijl}+u_i\cdot u_{jjl}.
\end{aligned}$$

By expressing R_{ijjl} in terms of (2.8) and (2.12), we have

$$\partial_{il}g_{jj}+\partial_{jj}g_{il}-\partial_{ij}g_{jl}-\partial_{jl}g_{ij}=2u_{ij}\cdot u_{jl}-2u_{ij}\cdot u_{jl}.$$

A simple comparison yields (2.14), and a similar argument may be used for the cases $i=k$, $i<j, i<l$ and $j=l, i<j, k<j$.

The third case occurs when $i<j, k<l, \tau_{ij}<\tau_{kl}$, and all are distinct. Consider the permutations

$$\begin{aligned}
\partial_{kl}g_{ij}&=u_{ik}\cdot u_{lj}+u_{il}\cdot u_{jk}+u_j\cdot u_{ikl}+u_i\cdot u_{ljk},\\
\partial_{jl}g_{ik}&=u_{ij}\cdot u_{kl}+u_{il}\cdot u_{jk}+u_i\cdot u_{jkl}+u_k\cdot u_{ijl},\\
\partial_{kj}g_{il}&=u_{ij}\cdot u_{kl}+u_{ik}\cdot u_{jl}+u_i\cdot u_{ljk}+u_l\cdot u_{ijk},\\
\partial_{ij}g_{kl}&=u_{ik}\cdot u_{lj}+u_{il}\cdot u_{jk}+u_k\cdot u_{lij}+u_l\cdot u_{kij},\\
\partial_{ik}g_{jl}&=u_{ij}\cdot u_{lk}+u_{jk}\cdot u_{il}+u_j\cdot u_{ikl}+u_l\cdot u_{ijk},\\
\partial_{il}g_{jk}&=u_{ij}\cdot u_{kl}+u_{jl}\cdot u_{ik}+u_j\cdot u_{ikl}+u_k\cdot u_{ijl}.
\end{aligned}$$

The last three of these equations are known to be satisfied, where as the first three need to be established (except for one, which is known to be satisfied since these three lie in the same equivalence class of distinct 4-tuples with $i<j, k<l$). Using the last three equations in conjunction with the Gauss equations, we have

$$\begin{aligned}
\partial_{kl}g_{ij}-\partial_{jl}g_{ik}&=-\partial_{ij}g_{kl}+\partial_{ik}g_{jl}+2R_{kjil},\\
\partial_{kl}g_{ij}-\partial_{jk}g_{il}&=-\partial_{ij}g_{kl}+\partial_{il}g_{jk}+2R_{kijl},\\
\partial_{jl}g_{ik}-\partial_{jk}g_{il}&=-\partial_{ik}g_{jl}+\partial_{il}g_{jk}+2R_{jikl},
\end{aligned}$$

and hence

$$\partial_{kl}g_{ij} - \partial_{jl}g_{ik} = u_{ik}\cdot u_{jl} - u_{kl}\cdot u_{ij} + u_j\cdot u_{lik} - u_k\cdot u_{lij},$$
$$\partial_{kl}g_{ij} - \partial_{jk}g_{il} = u_{kj}\cdot u_{il} - u_{kl}\cdot u_{ij} + u_j\cdot u_{kil} - u_l\cdot u_{kij},$$
$$\partial_{jl}g_{ik} - \partial_{jk}g_{il} = u_{jk}\cdot u_{il} - u_{jl}\cdot u_{ik} + u_k\cdot u_{ijl} - u_l\cdot u_{ijk}.$$

This may be viewed as three linear equations for the three unknown $\partial_{kl}g_{ij}$, $\partial_{jl}g_{ik}$, $\partial_{jk}g_{il}$. Upon solving this system, we find that the solution has the desired form up to addition of a vector having the form (β,β,β). However since at least one of these equations is known to be satisfied a priori, it follows that $\beta = 0$ so that all are satisfied. $\square$

Our main concern in this paper is the linearized equations of the isometric embedding. We are interested in such a linearization only at the formal isometric embedding or its nearby functions. For this purpose, an approximate isometric embedding is constructed in Lemma 2.1. The constants h^{μ}_{ij} are chosen to satisfy (2.9). In order to obtain a simple form of linearized equations, more assumptions are needed.

3. Reduction to an $n \times n$ system

In this section, we reduce the linearization of (1.2) to a first order $n \times n$ system and write the linearized equations for the isometric embedding system as a perturbation of a first order differential system with constant coefficients. The linearization is evaluated at functions which are perturbations of the approximate isometric embedding in Lemma 2.1. As we mentioned in Section 1, many arguments may be traced back to [4].

Let g be a metric defined in a neighborhood of the origin in $\mathbb{R}^n$. The metric g admits a smooth isometric embedding into $\mathbb{R}^{n(n+1)/2}$ if there exists a map $w : \Omega \to \mathbb{R}^{n(n+1)/2}$ such that

$$\partial_i w\cdot\partial_j w = g_{ij} \quad \text{for any } 1 \leqslant i, j \leqslant n,$$

where $\Omega \subset \mathbb{R}^n$ is a neighborhood of the origin. Linearizing at a map $u : \mathbb{R}^n \to \mathbb{R}^{n(n+1)/2}$ yields the following linear equation for $v : \mathbb{R}^n \to \mathbb{R}^{n(n+1)/2}$

$$\partial_i u\cdot\partial_j v + \partial_j u\cdot\partial_i v = f_{ij} \quad \text{for any } 1 \leqslant i, j \leqslant n, \tag{3.1}$$

where (f_{ij}) is some smooth symmetric matrix.

In the following, we fix a map $u : \mathbb{R}^n \to \mathbb{R}^{n(n+1)/2}$ and assume that it is an embedding. Denote by $\mathcal{M}^n$ the corresponding embedded submanifold. Then $\{\partial_i u(x)\}_{i=1}^n$ spans $T_x\mathcal{M}^n$ for each x. Let $\{N_\mu(x)\}_{\mu=1}^{n(n-1)/2}$ span $(T_x\mathcal{M}^n)^\perp$, the orthogonal complement of $T_x\mathcal{M}^n$ in $\mathbb{R}^{n(n+1)/2}$. Denote the induced metric on $\mathcal{M}^n$ by

$$p_{ij} = \partial_i u\cdot\partial_j u.$$

Then $\partial_{ij}u$ has a decomposition into its tangential and normal components with respect to u given by

$$\partial_{ij} u = \Gamma_{ij}^k \partial_k u + H_{ij}, \tag{3.2}$$

where Γ_{ij}^k are the Christoffel symbols corresponding to p_{ij} and H_{ij} is the second fundamental form. Moreover we have

$$\partial_j N_\mu \cdot \partial_i u = -N_\mu \cdot \partial_{ij} u = -N_\mu \cdot H_{ij}. \tag{3.3}$$

By setting $H_{ij}^\mu = H_{ij} \cdot N_\mu$, $1 \leqslant \mu \leqslant n(n-1)/2$, we have

$$H_{ij} = \sum_{\mu=1}^{n(n-1)/2} H_{ij}^\mu N_\mu.$$

We note that (3.2) and (3.3) are simply (2.1) and (2.2).

In the following, we will express (3.1) in another form which is easier to study. For motivation, we rewrite it as

$$\partial_i(\partial_j u \cdot v) + \partial_j(\partial_i u \cdot v) - 2\partial_{ij} u \cdot v = f_{ij} \quad \text{for any } 1 \leqslant i, j \leqslant n. \tag{3.4}$$

Note that $\partial_j u \cdot v$ is a component of the projection of v into the tangent space $T_x\mathcal{M}^n$. It is clear from (3.4) that the derivatives are only applied to tangential components of v. This suggests that we should decompose v relative to the tangent space and normal space of $\mathcal{M}^n$.

Set

$$v = v' + v'' = \sum_{k=1}^n v^k \partial_k u + \sum_{\mu=1}^{n(n-1)/2} v^{n+\mu} N_\mu, \tag{3.5}$$

where v' and v'' are the tangential and normal components of v with respect to the embedding u. We now derive an equivalent formulation of (3.1) in terms of v^k and $v^{n+\mu}$. Let $\{v_l\}_{l=1}^n$ be the coordinates of the dual 1-form to the vector field $v^l \partial_l u$, i.e.,

$$v_l = p_{lk} v^k \quad \text{and} \quad v^l = p^{lk} v_k.$$

Then

$$\begin{aligned}
\partial_i u \cdot \partial_j v &= \partial_j(\partial_i u \cdot v) - \partial_{ij} u \cdot v \\
&= \partial_j\big(p_{il} v^l\big) - \big(\Gamma_{ij}^k \partial_k u + H_{ij}\big) \cdot \big(v^l \partial_l u + v^{n+\mu} N_\mu\big) \\
&= \partial_j v_i - \Gamma_{ij}^k v^l p_{kl} - v^{n+\mu} N_\mu \cdot H_{ij} \\
&= \partial_j v_i - \Gamma_{ij}^k v_k - H_{ij}^\mu v^{n+\mu}.
\end{aligned}$$

It follows that (3.1) has the form

$$\partial_j v_i + \partial_i v_j - 2\Gamma_{ij}^k v_k - 2H_{ij}^\mu v^{n+\mu} = f_{ij} \quad \text{for any } 1 \leqslant i \leqslant j \leqslant n. \tag{3.6}$$

Moreover this equation may be written invariantly as

$$\nabla_i v_j + \nabla_j v_i - 2H_{ij}^{\mu} v^{n+\mu} = f_{ij},$$

where ∇_i denotes covariant differentiation for 1-forms; that is, if $\alpha = \alpha_j\, dx^j$ then

$$\nabla_i \alpha = \left(\partial_i \alpha_j - \alpha_k \Gamma_{ij}^k\right) dx^j.$$

Clearly, solving (3.6) for $\{v_i\}_{i=1}^n$ and $\{v^{n+\mu}\}_{\mu=1}^{n(n-1)/2}$ is equivalent to solving (3.1). This will be accomplished by solving a linear system of $n(n-1)/2$ algebraic equations for $\{v^{n+\mu}\}$ in terms of $\{v_i\}$, and then inserting this solution into the remaining n equations to obtain a first order $n \times n$ differential system in the unknowns $\{v_i\}$.

We now specify the algebraic equations used to obtain $\{v^{n+\mu}\}$. An important observation here is that no derivatives of $v^{n+\mu}$ are involved in (3.6). Consider the $n(n-1)/2$ equations corresponding to $i < j$ in (3.6)

$$H_{ij}^1 v^{n+1} + \cdots + H_{ij}^{n(n-1)/2} v^{n+n(n-1)/2} = \phi_{ij} \quad \text{for any } 1 \leqslant i < j \leqslant n, \tag{3.7}$$

where

$$\phi_{ij} = \frac{1}{2}\partial_j v_i + \frac{1}{2}\partial_i v_j - \Gamma_{ij}^k v_k - \frac{1}{2} f_{ij} \quad \text{for any } 1 \leqslant i < j \leqslant n.$$

Let

$$H(x) = \begin{pmatrix} H_{12}^1 & \cdot & \cdot & H_{12}^{n(n-1)/2} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ H_{(n-1)n}^1 & \cdot & \cdot & H_{(n-1)n}^{n(n-1)/2} \end{pmatrix} \tag{3.8}$$

be the coefficient matrix on the left-hand side of (3.7), and assume that H is invertible with the inverse

$$H^{-1} = \left(H^{\mu\tau}\right) \quad \text{for any } 1 \leqslant \mu, \tau \leqslant n(n-1)/2.$$

Note that this assumption of invertibility is not restrictive, since there always exists a solution of the Gauss equations with this property (see Lemma 3.10 on p. 98 of [3]). We now solve for $\{v^{n+\mu}\}$ from (3.7) in terms of $\{v_i\}$, $\{\nabla v_i\}$, and $\{f_{ij}\}$. With τ_{ij} defined in the proof of Lemma 2.1, we have

$$v^{n+\mu} = H^{\mu\tau_{ij}}\left(\frac{1}{2}\partial_j v_i + \frac{1}{2}\partial_i v_j - \Gamma_{ij}^k v_k - \frac{1}{2} f_{ij}\right). \tag{3.9}$$

We should emphasize that the summation on the right-hand side is taken over $1 \leqslant i < j \leqslant n$.

There are n equations in (3.6) which are absent in (3.7)

$$\partial_i v_i - \Gamma_{ii}^k v_k - H_{ii}^{\mu} v^{n+\mu} = \frac{1}{2} f_{ii} \quad \text{for any } i = 1, \ldots, n. \tag{3.10}$$

By inserting (3.9) into (3.10), we obtain

$$\partial_i v_i - \frac{1}{2} H_{ii}^{\mu} H^{\mu\tau_{kl}} (\partial_l v_k + \partial_k v_l) - \big(\Gamma_{ii}^m - H_{ii}^{\mu} H^{\mu\tau_{kl}} \Gamma_{kl}^m\big) v_m = \frac{1}{2}\big(f_{ii} - H_{ii}^{\mu} H^{\mu\tau_{kl}} f_{kl}\big).$$

We emphasize that the summation for k, l is only taken for $1 \leqslant k < l \leqslant n$, since τ_{kl} is defined only for $1 \leqslant k < l \leqslant n$. We now write this as a first order $n \times n$ system for $V = (v_1, \ldots, v_n)$ of the following form

$$A^1(x)\partial_1 V + \cdots + A^n(x)\partial_n V + B(x)V = F(x), \tag{3.11}$$

where $A^k(x) = (A_{ij}^k(x))$, $B(x) = (B_{ij}(x))$ and $F(x)$ are given by

$$A_{ij}^k(x) = \begin{cases} -H_{ii}^{\mu} H^{\mu\tau_{jk}}/2 & \text{for } j < k, \\ \delta_{ik} & \text{for } j = k, \\ -H_{ii}^{\mu} H^{\mu\tau_{kj}}/2 & \text{for } j > k, \end{cases}$$

$$B_{ij}(x) = -\Gamma_{ii}^j + \sum_{1 \leqslant k < l \leqslant n} H_{ii}^{\mu} H^{\mu\tau_{kl}} \Gamma_{kl}^j,$$

$$F_i(x) = \frac{1}{2}\bigg(f_{ii} - \sum_{1 \leqslant k < l \leqslant n} H_{ii}^{\mu} H^{\mu\tau_{kl}} f_{kl}\bigg).$$

Here i and j denote the rows and columns. Note that the Christoffel symbols are from the metric induced by u, and not from the given metric g. It is now apparent that in order to solve (3.1), it is sufficient to solve (3.11) for $\{v_i\}$ and then to find $\{v^{n+\mu}\}$ in terms of $\{v_i\}$, $\{\nabla v_i\}$ and $\{f_{ij}\}$ from (3.9). Therefore the study of the linearization for (1.2) is now reduced to a study of the $n \times n$ system (3.11), as long as the matrix H in (3.8) is invertible.

We emphasize that, in the calculations so far, u is taken to be an arbitrary embedding which is not necessarily related to g. In the following, we will choose a special u (the approximate solution) so that the coefficient matrices of (3.11), when evaluated at $x = 0$, are related to the curvature tensor of g at $x = 0$.

To proceed, we let $\{h_{ij}^{\mu}\}$ be constants satisfying (2.9). Here we emphasize that $R_{ijkl}(0)$ are the components of the curvature tensor of g at $x = 0$. We then assume that

u is the approximate embedding of g constructed in Lemma 2.1.

By checking our calculations, it is clear that H in (3.8) satisfies

$$H(0) = \begin{pmatrix} h_{12}^1 & \cdot & \cdot & h_{12}^{n(n-1)/2} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ h_{(n-1)n}^1 & \cdot & \cdot & h_{(n-1)n}^{n(n-1)/2} \end{pmatrix}. \tag{3.12}$$

Let h_{ij} be the vector in $\mathbb{R}^{n(n-1)/2}$ defined by

$$h_{ij} = \big(h_{ij}^1, \ldots, h_{ij}^{n(n-1)/2}\big).$$

We require that

$$\{h_{ij}\}_{1\leqslant i<j\leqslant n} \text{ forms a basis of } \mathbb{R}^{n(n-1)/2}. \tag{3.13}$$

Then $H(0)$ is invertible and so is $H(x)$ in (3.8) for x sufficiently small. By (3.13), we set

$$h_{kk} = -2 \sum_{1\leqslant i<j\leqslant n}^{n} c_k^{ij} h_{ij} \quad \text{for any } 1 \leqslant k \leqslant n, \tag{3.14}$$

for some constants c_k^{ij}. Therefore

$$A_{ij}^k(0) = \begin{cases} c_i^{jk} & \text{for } j<k, \\ \delta_{ik} & \text{for } j=k, \\ c_i^{kj} & \text{for } j>k, \end{cases}$$

and

$$B_{ij}(0) = -\Gamma_{ii}^j(0) + \sum_{1\leqslant k<l\leqslant n} c_i^{kl} \Gamma_{kl}^j(0),$$

$$F_i(0) = \frac{1}{2}\left(f_{ii}(0) - \sum_{1\leqslant k<l\leqslant n} c_i^{kl} f_{kl}(0)\right),$$

where $\Gamma_{ij}^k(0)$ are Christoffel symbols of g at $x=0$. Hence $B_{ij}(0)=0$ by (2.4). Lastly, we point out that coefficient matrices $A^1(0),\dots,A^n(0)$ in (3.11) are related to the curvature tensor of g at $x=0$. In fact, $\{h_{ij}^\mu\}$ defined by (3.13) and (3.14) satisfies (2.9).

To summarize, for $i,j,k=1,\dots,n$. For $i,j,k=1,\dots,n$, let $\mathbf{c}=\{c_i^{kj}\}_{k\neq j}$ be a collection of parameters with

$$c_i^{kj} = c_i^{jk} \quad \text{for any } i,j,k \text{ with } j\neq k, \tag{3.15}$$

and set

$$\begin{aligned} c_i^{ii} &= 1 \quad \text{for any } i, \\ c_i^{jj} &= 0 \quad \text{for any } i\neq j. \end{aligned} \tag{3.16}$$

Define $n\times n$ matrices $A^1,\dots,A^n$ by

$$\left(A^k\right)_{ij} = \left(c_i^{kj}\right).$$

Then we have shown

Lemma 3.1. *The linear system* (3.11) *at* $x = 0$ *is given by*

$$A^1 \partial_1 v + \cdots + A^n \partial_n v = F. \tag{3.17}$$

Remark 3.2. The matrices defined above in fact form a basis of the set F_H defined on p. 916 in [4], if the second fundamental form $\{h_{ij}^{\mu}\}$ at $x = 0$ is given by (3.13) and (3.14).

For each $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, define

$$P = P(\xi, \mathbf{c}) = \sum_{i=1}^{n} \xi_k A^k.$$

This is the *principal symbol*, whose components have expressions

$$\begin{aligned} p_{ii} &= \xi_i + \sum_{k \neq i} c_i^{ki} \xi_k, \\ p_{ij} &= \sum_{k \neq j} c_i^{kj} \xi_k \quad \text{for any } i \neq j. \end{aligned} \tag{3.18}$$

The *characteristic variety* is defined by

$$\Sigma(\mathbf{c}) = \left\{ \xi \in \mathbb{R}^n \setminus \{0\} \,\middle|\, \det P(\xi, \mathbf{c}) = 0 \right\}.$$

The next well-known fact asserts that the isometric embedding system is never elliptic beyond dimension two.

Lemma 3.3. *For* $n \geqslant 3$ *and any* $\mathbf{c}$, $\Sigma(\mathbf{c}) \neq \emptyset$.

This is the second result of Theorem B(v) in [4]. In the present setting, the proof becomes straightforward. Namely, we observe that $A^1, \ldots, A^n$ are linearly independent by examining the diagonal elements. Thus the existence of characteristics follows immediately from [1].

To end this section, we briefly discuss the principal symbol and characteristic variety for low dimensions. For dimension $n = 3$ we refer the reader to [4]. In the case of dimension $n = 4$, we will show that the characteristic variety is smooth under generic conditions on small parameters. To this end, consider the condition

$$c_j^{ik} c_i^{jl} \neq c_j^{il} c_i^{jk} \quad \text{for all } i \neq j, \tag{3.19}$$

where k and l are the remaining two elements of the set $\{1, 2, 3, 4\} \setminus \{i, j\}$, and consider the four inequalities

$$\begin{aligned} c_2^{14} c_4^{13} c_3^{12} &\neq c_4^{12} c_3^{14} c_2^{13}, \\ c_4^{21} c_3^{24} c_1^{23} &\neq c_1^{24} c_4^{23} c_3^{21}, \\ c_4^{31} c_2^{34} c_1^{32} &\neq c_1^{34} c_4^{32} c_2^{31}, \\ c_2^{41} c_1^{43} c_3^{42} &\neq c_1^{42} c_3^{41} c_2^{43}. \end{aligned} \tag{3.20}$$

These conditions arise naturally when examining the characteristic variety. In the next result, we write $t\mathbf{c} = \{tc_i^{kj}\}_{k\neq j}$ for $\mathbf{c} = \{c_i^{kj}\}_{k\neq j}$.

Theorem 3.4. *Let $n = 4$. If all elements of $\mathbf{c}$ satisfy* (3.19) *and* (3.20)*, then there exists a constant $T > 0$ (depending on $\mathbf{c}$) such that for all $t \in (0, T)$ the characteristic variety $\Sigma(t\mathbf{c})$ is smooth.*

A general result was obtained in [4] without the smallness assumption on the parameters. Here we give an alternative proof.

Proof. As is discussed in the next section, Bryant, Griffiths, and Yang have shown [4] that the singular part of the characteristic variety consists of points $\xi \in \mathbb{S}^3 \subset \mathbb{R}^4$ at which all 3×3 determinant minors of the principal symbol vanish. (See Lemma 4.1.) The principal symbol $P(\xi, t\mathbf{c}) = (p_{ij})$ is given by

$$p_{ii} = \xi_i + t\sum_{k\neq i} c_i^{ki}\xi_k,$$
$$p_{ij} = t\sum_{k\neq j} c_i^{kj}\xi_k, \quad \text{for any } i \neq j.$$

Suppose that for all sufficiently small t, singular points $\xi(t) \in \mathbb{S}^3$ exist. In other words, all 3×3 determinant minors of the principal symbol vanish on $\xi(t)$. By passing to a subsequence if necessary, we may assume

$$\xi(t) \to a = (a_1, a_2, a_3, a_4) \quad \text{as } t \to 0.$$

If P_j^i denotes the 3×3 minor obtained by deleting the ith column and jth row, then a simple calculation shows that

$$\det P_1^1 = \xi_2\xi_3\xi_4 + O(t),$$
$$\det P_2^2 = \xi_1\xi_3\xi_4 + O(t),$$
$$\det P_3^3 = \xi_1\xi_2\xi_4 + O(t),$$
$$\det P_4^4 = \xi_1\xi_2\xi_3 + O(t),$$

where we have dropped (and will continue to drop) reference to t. Thus at least two components a_i must be zero, say $a_1 = a_2 = 0$. We may assume that $a_4 \neq 0$. There are then two cases to consider, $a_3 \neq 0$ and $a_3 = 0$.

Case 1. $a_3 \neq 0$.

For each $i \neq j$ the components of the principal symbol are given by $p_{ij} = tb_{ij}$, where

$$b_{ij} = \sum_{k\neq j} c_i^{kj}\xi_k.$$

Then observe that

$$\det P_2^1 = t\big(\xi_3\xi_4 b_{21} + O(t)\big), \qquad \det P_1^2 = t\big(\xi_3\xi_4 b_{12} + O(t)\big).$$

It follows that $b_{12} \to 0$ and $b_{21} \to 0$ as $t \to 0$. Thus

$$c_2^{13} a_3 + c_2^{14} a_4 = 0, \qquad c_1^{23} a_3 + c_1^{24} a_4 = 0,$$

where we have used the symmetry $c_i^{kj} = c_i^{jk}$. As $a_3 a_4 \neq 0$, we must have

$$c_2^{13} c_1^{24} = c_1^{23} c_2^{14}. \tag{3.21}$$

Case 2. $a_3 = 0$.

Observe that

$$\begin{aligned}
\det P_3^2 &= t\xi_1\xi_4 b_{23} + t^2(b_{11}b_{23} - b_{31}b_{12})\xi_4 + O\big(t^2 \max\big\{|\xi_1|, |t|\big\}\big),\\
\det P_2^3 &= t\xi_1\xi_4 b_{32} + t^2(b_{11}b_{32} - b_{31}b_{12})\xi_4 + O\big(t^2 \max\big\{|\xi_1|, |t|\big\}\big).
\end{aligned}$$

If both of these determinants are zero, then we may multiply the first by b_{32} and the second by b_{23}, and then compare the expressions for $\xi_1\xi_4 b_{23} b_{32}$ to obtain

$$(b_{21}b_{13} - b_{11}b_{23})b_{32} = (b_{12}b_{31} - b_{11}b_{32})b_{23} + O(t).$$

Recognizing that $\lim_{t\to 0} b_{ij} = c_i^{4j}\xi_4$, we find that

$$c_2^{41} c_1^{43} c_3^{42} = c_1^{42} c_3^{41} c_2^{43}. \tag{3.22}$$

Thus if neither (3.21) nor (3.22) holds, then there cannot be a limit point $a = \lim_{t\to 0}\xi(t)$ of singular points of the characteristic variety with $a_1 = a_2 = 0$. By considering all combinations $a_i = a_j = 0$ with $i \neq j$, we obtain the desired result. □

4. The characteristic variety in higher dimensions

In this section, we study the characteristic variety of the linearized isometric embedding system in higher dimensions and prove Theorem 1.2. As in Section 3, let $\mathbf{c} = \{c_i^{kj}\}_{k\neq j}$ be a collection of parameters satisfying (3.15). For any $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, we define an $n \times n$ matrix $P = P(\xi, \mathbf{c}) = (p_{ij})$ by

$$\begin{aligned}
p_{ii} &= \xi_i + \sum_{k\neq i} c_i^{ki}\xi_k,\\
p_{ij} &= \sum_{k\neq j} c_i^{kj}\xi_k \quad \text{for any } i \neq j.
\end{aligned} \tag{4.1}$$

The matrix P is the principal symbol associated with the equivalent linearized isometric embedding system. Then the characteristic variety $\Sigma = \Sigma(\mathbf{c})$ is given by

$$\Sigma(\mathbf{c}) = \left\{\xi \in \mathbb{R}^n \setminus \{0\} \,\middle|\, \det P(\xi, \mathbf{c}) = 0\right\}.$$

We observe that $\Sigma(\mathbf{c})$ can be defined alternatively as

$$\Sigma(\mathbf{c}) = \left\{\xi \in \mathbb{R}^n \setminus \{0\} \,\middle|\, \text{the rank of } P(\xi, \mathbf{c}) \leqslant n-1\right\}.$$

Next define

$$\Sigma_{\text{sing}}(\mathbf{c}) = \left\{\xi \in \mathbb{R}^n \setminus \{0\} \,\middle|\, \text{the rank of } P(\xi, \mathbf{c}) \leqslant n-2\right\}.$$

We recall the following result.

Lemma 4.1. $\Sigma_{\text{sing}}(\mathbf{c})$ *is the singular part of the characteristic variety* $\Sigma(\mathbf{c})$.

Lemma 4.1 is proved in [4]. (See [4, Theorem B].) It is also proved [4, Corollary 1.c.6] that $\Sigma_{\text{sing}}(\mathbf{c})$ is not empty for $n = 6, 10, 14, \ldots$. Furthermore it was conjectured [4, p. 920] that $\Sigma_{\text{sing}}(\mathbf{c})$ is not empty for any $n \geqslant 5$. The goal of this section is to prove that $\Sigma_{\text{sing}}(\mathbf{c})$ is not empty for sufficiently small $\mathbf{c}$ if $n \geqslant 5$, and to estimate its size.

By Lemma 4.1 $\Sigma_{\text{sing}}(\mathbf{c})$ consists of those points ξ where all $(n-1) \times (n-1)$ minors of $P(\xi, \mathbf{c})$ have zero determinant. Although there seem to be many algebraic equations involved with this statement, as we will see, there are in fact only *four* under appropriate conditions. The main tool used to reduce the number of equations is the following result from linear algebra.

Lemma 4.2. *Let $v_1, \ldots, v_n$ be n vectors in a vector space. Assume that $v_1, \ldots, v_{n-1}$ and $v_1, \ldots, v_{n-2}, v_n$ are each linearly dependent and that $v_1, \ldots, v_{n-2}$ are linearly independent. Then any subset of $n-1$ vectors from $\{v_1, \ldots, v_n\}$ is linearly dependent.*

We note that $v_1, \ldots, v_{n-2}$ are common vectors of the two sets $\{v_1, \ldots, v_{n-1}\}$ and $\{v_1, \ldots, v_{n-2}, v_n\}$. It is crucial to assume that $v_1, \ldots, v_{n-2}$ are linearly independent. We may consider $e_1, \ldots, e_{n-1}, 0$ in $\mathbb{R}^{n-1}$, where $e_1, \ldots, e_{n-1}$ form a basis in $\mathbb{R}^{n-1}$. Obviously, $e_1, \ldots, e_{n-1}$ are linearly independent. However, replacing any e_i by the zero vector yields a linearly dependent set.

Proof. The proof is a simple argument from linear algebra. Consider $v_2, \ldots, v_n$. If either $\{v_2, \ldots, v_{n-1}\}$ or $\{v_2, \ldots, v_{n-2}, v_n\}$ is linearly dependent, so is $\{v_2, \ldots, v_n\}$. We assume that both $\{v_2, \ldots, v_{n-1}\}$ and $\{v_2, \ldots, v_{n-2}, v_n\}$ are linearly independent. Then by the linear dependence of $\{v_1, \ldots, v_{n-1}\}$ and $\{v_1, \ldots, v_{n-2}, v_n\}$, there exist constants $c_2, \ldots, c_{n-1}$ and $d_2, \ldots, d_{n-2}, d_n$ such that

$$v_1 = c_2 v_2 + \cdots + c_{n-2} v_{n-2} + c_{n-1} v_{n-1},$$
$$v_1 = d_2 v_2 + \cdots + d_{n-2} v_{n-2} + d_n v_n.$$

Note that $c_{n-1} \neq 0$ and $d_n \neq 0$, otherwise $\{v_1, \dots, v_{n-2}\}$ is linearly dependent, which contradicts the assumption. By taking a difference, we have

$$(c_2 - d_2)v_2 + \cdots + (c_{n-2} - d_{n-2})v_{n-2} + c_{n-1}v_{n-1} - d_n v_n = 0.$$

This is a nontrivial combination since $c_{n-1}d_n \neq 0$. $\square$

As an application, we discuss conditions under which all minors of a matrix have zero determinant. We will present only a simple case. For a matrix P, let P^i_j be the minor obtained by deleting the i-th row and j-th column from P.

Lemma 4.3. *Let P be an $n \times n$ matrix with $n \geqslant 2$, and suppose that the four minors P^1_1, P^2_2, P^1_2, and P^2_1 have zero determinant. If all $(n-1) \times (n-2)$ and $(n-2) \times (n-1)$ submatrices are of full rank, then all minors of P have zero determinant.*

Proof. First consider minors without the first row of P. We already have $\det P^1_1 = \det P^1_2 = 0$. The common part of P^1_1 and P^1_2 is an $(n-1) \times (n-2)$ matrix obtained by deleting the first row and the first and second column from P, and hence is of full rank. By Lemma 4.2, all minors without the first row in P have zero determinant. In applying Lemma 4.2, we treat the $(n-1) \times n$ submatrix obtained by deleting the first row from P as a collection of n column vectors. Similarly, all minors without the second row in P have zero determinant.

Now we consider other minors, say without the third column. From the discussion above, P^1_3 and P^2_3 have zero determinant. The common part of these two minors is an $(n-2) \times (n-1)$ matrix obtained by deleting the third column and the first and second rows from P, and is of full rank. Hence, all minors without the third column in P have zero determinant. Similarly all minors without the first, second, $\dots$, or the n-th column in P have zero determinant. In conclusion, all minors have zero determinant. $\square$

It is clear from the proof that the assumption that all $(n-1) \times (n-2)$ and $(n-2) \times (n-1)$ submatrices are of full rank can be relaxed. We only need certain submatrices to be of full rank. However, we point out that certain conditions are indeed necessary. For example, the 4×4 diagonal matrix $\mathrm{diag}(1, 1, 1, 0)$ has all but one minor with zero determinant. In our study of the characteristic variety later on, we will not use Lemma 4.3 directly. Instead, we will examine whether certain submatrices are of full rank so that we can apply Lemma 4.2.

Lemma 4.3 asserts that there are only four algebraic equations to satisfy, under appropriate conditions, in order that all minors of the principal symbol have zero determinant.

We will now study the characteristic variety in $\mathbb{R}^n$. In order to introduce a smallness assumption on the parameters of the system (3.17), we again let

$$t\mathbf{c} = \{tc^{kj}_i\}_{k \neq j},$$

for some small t. According to (4.1), the principal symbol $P(\xi, t\mathbf{c}) = (p_{ij})$ is then given by

$$p_{ii} = \xi_i + tb_{ii},$$

$$p_{ij} = tb_{ij}, \quad \text{for any } i \neq j,$$

where

$$b_{ij} = \sum_{k \neq j} c_i^{kj} \xi_k.$$

We begin with the case $n = 5$, where certain calculations are less formidable. Pick $i, j \in \{1, \dots, 5\}$ with $i \neq j$, and consider the following generic conditions on the parameters:

$$c_i^{kj} c_j^{li} \neq c_j^{ki} c_i^{lj} \quad \text{for all } k \neq l, \text{ with } k \neq i, j \text{ and } l \neq i, j, \tag{4.2}$$

and

$$c_p^{kI}\left(c_i^{lj} c_j^{mi} - c_j^{li} c_i^{mj}\right) + c_p^{lI}\left(c_j^{ki} c_i^{mj} - c_i^{kj} c_j^{mi}\right) + c_p^{mI}\left(c_i^{kj} c_j^{li} - c_j^{ki} c_i^{lj}\right) \neq 0, \tag{4.3}$$

for all $p \neq i, j$ where k, l, m are chosen so that $\{i, j, k, l, m\} = \{1, \dots, 5\}$ and where $I = i$ or $I = j$. In general, we will say that the parameters satisfy a *generic condition* if they satisfy a finite number of (homogeneous) polynomial inequalities.

Theorem 4.4. *Let $n = 5$. If all elements c_i^{kj} of $\mathbf{c}$ satisfy the conditions* (4.2) *and* (4.3)*, then for any sufficiently small $t > 0$, $\Sigma_{\mathrm{sing}}(t\mathbf{c}) \cap \mathbb{P}^4$ contains ten points. Moreover, if the parameters satisfy a further generic condition, then for any sufficiently small $t > 0$, $\Sigma_{\mathrm{sing}}(t\mathbf{c}) \cap \mathbb{P}^4$ consists of $10 + \alpha + 2\beta + 3\gamma$ points where α, β, γ are nonnegative integers with $\alpha + \gamma = 10$ and $\beta \leqslant 5$.*

Proof. In light of the discussion at the beginning of this section, our goal will be to construct a curve $\xi(t) \in \mathbb{S}^4 \subset \mathbb{R}^5$ such that all 4×4 minor determinants of $P(\xi(t), t\mathbf{c})$ vanish, for all t sufficiently small. If this is to occur, then as in the first paragraph of the proof of Theorem 3.4, we must have (after possibly passing to a subsequence)

$$\xi(t) \to a = (a_1, \dots, a_5),$$

where two elements of a vanish, say $a_1 = a_2 = 0$. There are then three cases to consider, namely: $a_3 a_4 a_5 \neq 0$, $a_3 = 0$ and $a_4 a_5 \neq 0$, $a_3 = a_4 = 0$ and $a_5 \neq 0$.

Case 1. $a_1 = a_2 = 0$ and $a_3 a_4 a_5 \neq 0$.

Since $a_1 = a_2 = 0$, we will write

$$\xi_1(t) = t y_1(t), \qquad \xi_2(t) = t y_2(t).$$

It follows that

$$\xi_1 + t b_{11} = t x_1 + t^2 c_1^{21} y_2, \qquad \xi_2 + t b_{22} = t x_2 + t^2 c_2^{12} y_1,$$

where

$$x_i = y_i + \sum_{k > 2} c_i^{ki} \xi_k.$$

Denote by P_j^i the minor of the principal symbol obtained by deleting the ith row and jth column. We will analyze

$$\det P_1^1 = \det P_1^2 = \det P_2^1 = \det P_2^2 = 0.$$

First,

$$\det P_1^1 = t\xi_3\xi_4\xi_5 x_2 + O\big(t^2\big), \qquad \det P_2^2 = t\xi_3\xi_4\xi_5 x_1 + O\big(t^2\big).$$

This implies $x_1 \to 0$ and $x_2 \to 0$ as $t \to 0$. Hence, we write

$$x_i(t) = tz_i(t) \quad \text{for } i = 1, 2,$$

for some z_i. Moreover we have

$$\det P_2^1 = t\xi_3\xi_4\xi_5 b_{21} + O\big(t^2\big), \qquad \det P_1^2 = t\xi_3\xi_4\xi_5 b_{12} + O\big(t^2\big).$$

Then we have $b_{12} \to 0$ and $b_{21} \to 0$ as $t \to 0$. This suggests that we write

$$\sum_{k>2} c_1^{k2}\xi_k = tz_3(t), \qquad \sum_{k>2} c_2^{k1}\xi_k = tz_4(t), \tag{4.4}$$

for some z_3 and z_4, so that

$$b_{12} = t\big(c_1^{12}y_1 + z_3\big), \qquad b_{21} = t\big(c_2^{21}y_2 + z_4\big).$$

Upon calculating the four determinants above in terms of the z_i we obtain,

$$\det P_1^1 = t^2\Bigg[\bigg(z_2 - c_2^{12}\sum_{k>2} c_1^{k1}\xi_k\bigg)\xi_3\xi_4\xi_5 - b_{23}b_{32}\xi_4\xi_5 - b_{24}b_{42}\xi_3\xi_5 - b_{25}b_{52}\xi_3\xi_4\Bigg] + O\big(t^3\big), \tag{4.5}$$

$$\det P_2^2 = t^2\Bigg[\bigg(z_1 - c_1^{21}\sum_{k>2} c_2^{k2}\xi_k\bigg)\xi_3\xi_4\xi_5 - b_{13}b_{31}\xi_4\xi_5 - b_{14}b_{41}\xi_3\xi_5 - b_{15}b_{51}\xi_3\xi_4\Bigg] + O\big(t^3\big), \tag{4.6}$$

$$\det P_2^1 = t^2\Bigg[\bigg(z_4 - c_2^{12}\sum_{k>2} c_2^{k2}\xi_k\bigg)\xi_3\xi_4\xi_5 - b_{23}b_{31}\xi_4\xi_5 - b_{24}b_{41}\xi_3\xi_5 - b_{25}b_{51}\xi_3\xi_4\Bigg] + O\big(t^3\big), \tag{4.7}$$

$$\det P_1^2 = t^2\Bigg[\bigg(z_3 - c_1^{21}\sum_{k>2} c_1^{k1}\xi_k\bigg)\xi_3\xi_4\xi_5 - b_{13}b_{32}\xi_4\xi_5 - b_{14}b_{42}\xi_3\xi_5 - b_{15}b_{52}\xi_3\xi_4\Bigg] + O\big(t^3\big). \tag{4.8}$$

Define functions G_i by

$$\det P_2^2 = t^2 G_1, \qquad \det P_1^1 = t^2 G_2, \qquad \det P_1^2 = t^2 G_3, \qquad \det P_2^1 = t^2 G_4.$$

We would like each G_i to be a function of z_i and t. To see that this is the case, we recall (4.2) with $i = 1$, $j = 2$. Since (4.2) holds with $k = 3$ and $l = 4$, we may solve Eqs. (4.4) for ξ_3 and ξ_4 in terms of ξ_5, z_3, and z_4. More precisely, for these values of k and l, fix $\xi_5(t) = a_5$, a nonzero constant. Then solve to obtain

$$\begin{aligned}
\xi_3(t) &= a_3 + t\left(c_1^{32}c_2^{41} - c_2^{31}c_1^{42}\right)^{-1}\left[\left(c_2^{41}z_3(t) - c_1^{42}z_4(t)\right) + \left(c_1^{42}c_2^{51} - c_2^{41}c_1^{52}\right)a_5\right],\\
\xi_4(t) &= a_4 + t\left(c_1^{32}c_2^{41} - c_2^{31}c_1^{42}\right)^{-1}\left[\left(c_1^{32}z_4(t) - c_2^{31}z_3(t)\right) + \left(c_2^{31}c_1^{52} - c_1^{32}c_2^{51}\right)a_5\right],
\end{aligned}$$

where a_3, a_4, a_5 satisfy

$$\sum_{k>2} c_1^{k2}a_k = 0, \qquad \sum_{k>2} c_2^{k1}a_k = 0. \tag{4.9}$$

Note that if $a_5 \neq 0$, then $a_3a_4 \neq 0$, in light of (4.2). We now have a map

$$G = (G_1, G_2, G_3, G_4) : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^4,$$

with $G(\bar{z}, 0) = 0$, where $\bar{z}$ may be determined from (4.5), (4.6), (4.7), and (4.8). Moreover a simple calculation shows that $DG(\bar{z}, 0) = a_3a_4a_5 I_4$, where I_4 denotes the 4×4 identity matrix. By the implicit function theorem there exists $z(t) = (z_1(t), z_2(t), z_3(t), z_4(t))$, such that $G(z(t), t) = 0$ for all sufficiently small t. Thus we have found a curve $\xi(t)$ such that

$$\det P_1^1(t) = \det P_2^2(t) = \det P_2^1(t) = \det P_1^2(t) = 0.$$

We now claim that all remaining minor determinants vanish as well, on the curve $\xi(t)$. Consider the last three column vectors of the principal symbol $P(\xi(t), \mathbf{c}(t))$ minus the first or second row. These are linearly independent for small t since $a_3a_4a_5 \neq 0$. If we append the first or second column (again minus the first or second row) then these four vectors are linearly dependent since $\det P_1^1(t) = \det P_2^2(t) = \det P_2^1(t) = \det P_1^2(t) = 0$. So by Lemma 4.2, all minors P_j^i with $i = 1, 2$ have zero determinant.

Now we move on to columns. Consider the first or second column, that is consider the last three row vectors minus the first or second column. These are linearly independent, since $a_3a_4a_5 \neq 0$. If we append the first or second row (again minus the first or second column), then these four vectors are linearly dependent since $\det P_1^1 = \det P_1^2 = 0$. Hence by Lemma 4.2, all minors P_j^i with $j = 1, 2$ have zero determinant.

Now consider columns 3, 4, and 5. Say column 3. The last three rows (minus the third column) are linearly independent if $b_{31} \neq 0$ or $b_{32} \neq 0$. Moreover, by appending either the first or second row (minus the third column), we obtain four linearly dependent vectors by what has been shown above. Thus Lemma 4.2 shows that all minors P_j^i with $j = 3$ have zero determinant. The same conclusion follows for $j = 4$ if $b_{41} \neq 0$ or $b_{42} \neq 0$, and for $j = 5$ if $b_{51} \neq 0$ or $b_{52} \neq 0$. Therefore all minor determinants vanish, if certain b_{kl} do not vanish at $t = 0$. This fact follows directly from (4.3) with $i=1$ and $j=2$. To see this, solve the linear algebraic equations (4.9) for a_3, a_4, and a_5, and then insert the result into b_{kl} to obtain the desired conclusion.

In summary, we have constructed a curve $\xi(t) \to a = (0, 0, a_3, a_4, a_5)$, where the (nonzero) components a_3, a_4, a_5 satisfy the linear algebraic equations (4.9) (note that there is only one such point $a \in \mathbb{P}^4$), such that all minor determinants of the principal symbol vanish. It follows that for each sufficiently small t, $\xi(t)$ lies in the singular part of the characteristic variety. By considering all possible cases of two zero components, that is $a_i = a_j = 0$, we obtain ten distinct points in $\Sigma_{\text{sing}}(t\mathbf{c}) \cap \mathbb{P}^4$. This proves the first statement of the theorem.

Case 2. $a_1 = a_2 = a_3 = 0$ *and* $a_4 a_5 \neq 0$.

Following the strategy of Case 1, we will write $\xi_i(t) = t y_i(t)$ for $i = 1, 2, 3$. It follows that

$$\begin{aligned} \xi_1 + t b_{11} &= t x_1 + t^2\big(c_1^{21} y_2 + c_1^{31} y_3\big), \\ \xi_2 + t b_{22} &= t x_2 + t^2\big(c_2^{12} y_1 + c_2^{32} y_3\big), \\ \xi_3 + t b_{33} &= t x_3 + t^2\big(c_3^{13} y_1 + c_3^{23} y_2\big), \end{aligned}$$

where

$$x_i = y_i + c_i^{4i}\xi_4 + c_i^{5i}\xi_5.$$

We will analyze

$$\det P_2^1 = \det P_1^2 = \det P_3^1 = \det P_3^2 = 0.$$

First,

$$\begin{aligned} \det P_2^1 &= t^2(b_{21}x_3 - b_{23}b_{31})\xi_4\xi_5 + O\big(t^3\big), \\ \det P_1^2 &= t^2(b_{12}x_3 - b_{13}b_{32})\xi_4\xi_5 + O\big(t^3\big), \end{aligned}$$

which motivates the following. For functions $z_i(t)$, $i = 1, 2, 3, 4$, to be determined, we will write

$$\begin{aligned} &x_1 = z_1, \qquad x_2 = z_2, \qquad x_3 = \frac{b_{23}(0) b_{31}(0)}{b_{21}(0)} + t z_3, \\ &b_{12} b_{31} b_{23} - b_{21} b_{13} b_{32} = t z_4, \end{aligned} \tag{4.10}$$

where $b_{ij}(0)$ signifies b_{ij} evaluated at $t=0$. Note that under generic conditions on the parameters c_{kij} we have that $b_{21}(0) \neq 0$ (that this is possible is a consequence of the way in which a_4 and a_5 are chosen below). We claim that knowledge of $z(t) = (z_1, \ldots, z_4)$ is equivalent to knowledge of $\xi(t)$. To see this, fix $a_5 \neq 0$ and let a_4 solve the cubic polynomial

$$\begin{aligned} &\big(c_1^{42} a_4 + c_1^{52} a_5\big)\big(c_3^{41} a_4 + c_3^{51} a_5\big)\big(c_2^{43} a_4 + c_2^{53} a_5\big) \\ &\quad = \big(c_2^{41} a_4 + c_2^{51} a_5\big)\big(c_1^{43} a_4 + c_1^{53} a_5\big)\big(c_3^{42} a_4 + c_3^{52} a_5\big), \end{aligned} \tag{4.11}$$

which is the last equation of (4.10) evaluated at $t=0$. Note that this polynomial has either one or three nonzero solutions for a_4 again under generic conditions on the parameters. We then set

$\xi_5(t) = a_5$ and $\xi_4 = a_4 + ty_4(t)$. Generic conditions also imply that y_4 may be determined in terms of z_i and t from (4.10). Thus our task of constructing $\xi(t)$ is reduced to finding $z(t)$.

We now calculate four determinants in terms of the z_i:

$$\det P_2^1 = t^3 z_3 b_{21}\xi_4\xi_5 + O\big(t^3|z_1|, t^3|z_2|, t^4\big),$$
$$\det P_1^2 = t^3\big(b_{21}^{-1} z_4 + b_{12} z_3\big)\xi_4\xi_5 + O\big(t^3|z_1|, t^3|z_2|, t^4\big),$$

and

$$\det P_3^2 = t^2(b_{32}z_1 - b_{12}b_{31})\xi_4\xi_5 + O\big(t^3\big),$$
$$\det P_3^1 = t^2(b_{21}b_{32} - b_{31}z_2)\xi_4\xi_5 + O\big(t^3\big).$$

We may then define a map

$$G = (G_1, G_2, G_3, G_4) : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^4,$$

where the components of G are functions of z and t, and are given by

$$\det P_3^2 = t^2 G_1, \qquad \det P_3^1 = t^2 G_2, \qquad \det P_2^1 = t^3 G_3, \qquad \det P_1^2 = t^3 G_4.$$

Let $\bar{z}$ be such that $G(\bar{z}, 0) = 0$, it then follows that

$$DG(\bar{z}, 0) = a_4 a_5 \begin{pmatrix} b_{32}(0) & 0 & 0 & 0 \\ 0 & -b_{31}(0) & 0 & 0 \\ * & * & b_{21}(0) & 0 \\ * & * & * & b_{21}(0)^{-1} \end{pmatrix},$$

where $*$ represents an expression which is unimportant. Generic conditions on the parameters then guarantee that this matrix is invertible. The implicit function theorem then yields $z(t)$ with $G(z(t), t) = 0$ for sufficiently small t. Thus we have constructed a curve $\xi(t)$ on which the four determinants calculated above vanish.

We now show that all other minor determinants of the principal symbol vanish on $\xi(t)$, under generic conditions on the parameters. Consider columns 1, 4, and 5, minus the first or second row. These vectors are linearly independent for small t, if $b_{21}(0) \neq 0$ and $b_{31}(0) \neq 0$. Since $\det P_2^1 = \det P_3^1 = 0$ we may apply Lemma 4.2 to show that all minors constructed from an element in the first two rows have zero determinant. Now consider the last three rows. If $b_{32}(0) \neq 0$ then these vectors are linearly independent for small t. Thus by using what has been shown above, and applying Lemma 4.2, we find that all minors constructed from an element in the first column have zero determinant. The same result holds similarly for the second and third columns. For columns 4 and 5, we obtain this result if (respectively)

$$b_{31}(0)b_{42}(0) \neq b_{41}(0)b_{32}(0), \qquad b_{31}(0)b_{52}(0) \neq b_{51}(0)b_{32}(0).$$

In conclusion, we have constructed a curve $\xi(t) \to a = (0, 0, 0, a_4, a_5)$, where the (nonzero) components a_4, a_5 satisfy Eq. (4.11) (note that under generic conditions on the parameters there is either one or three such points $a \in \mathbb{P}^4$), such that all determinant minors of the principal symbol vanish. It follows that for each sufficiently small t, $\xi(t)$ lies in the singular part of the characteristic variety. By considering all possible cases of three zero components, that is $a_i = a_j = a_k = 0$, we obtain $\alpha + 3\gamma$ distinct points in $\Sigma_{\text{sing}}(t\mathbf{c}) \cap \mathbb{P}^4$, where α and γ are nonnegative integers summing to ten.

Case 3. $a_1 = a_2 = a_3 = a_4 = 0$ *and* $a_5 \neq 0$.

We proceed in the same way as in the previous two cases. Set $\xi_i(t) = ty_i(t)$, $i = 1, 2, 3, 4$, and calculate

$$\begin{aligned}
\xi_1 + tb_{11} &= tz_1 + t^2\big(c_1^{21}y_2 + c_1^{31}y_3 + c_1^{41}y_4\big),\\
\xi_2 + tb_{22} &= tz_2 + t^2\big(c_2^{12}y_1 + c_2^{32}y_3 + c_2^{42}y_4\big),\\
\xi_3 + tb_{33} &= tz_3 + t^2\big(c_3^{13}y_1 + c_3^{23}y_2 + c_3^{43}y_4\big),\\
\xi_4 + tb_{44} &= tz_4 + t^2\big(c_4^{14}y_1 + c_4^{24}y_2 + c_4^{34}y_4\big),
\end{aligned}$$

where

$$z_i = y_i + c_i^{5i}\xi_5.$$

Now calculate four determinants in terms of the z_i:

$$\begin{aligned}
\det P_2^4 &= t^3(z_1b_{32}b_{43} + z_3b_{12}b_{41} - z_1z_3b_{42} - b_{12}b_{31}b_{43} + b_{13}b_{31}b_{42} - b_{13}b_{32}b_{41})\xi_5\\
&\quad + O\big(t^4\big),\\
\det P_2^3 &= t^3(z_1z_4b_{32} - z_1b_{34}b_{42} - z_4b_{12}b_{31} + b_{12}b_{34}b_{41} + b_{14}b_{31}b_{42} - b_{14}b_{32}b_{41})\xi_5\\
&\quad + O\big(t^4\big),\\
\det P_1^2 &= t^3(z_3z_4b_{12} - z_4b_{13}b_{32} - z_3b_{14}b_{42} - b_{12}b_{34}b_{43} + b_{13}b_{34}b_{42} + b_{14}b_{32}b_{43})\xi_5\\
&\quad + O\big(t^4\big),\\
\det P_3^4 &= t^3(z_1z_2b_{43} - z_1b_{23}b_{42} - z_2b_{13}b_{41} - b_{12}b_{21}b_{43} + b_{12}b_{23}b_{41} + b_{13}b_{21}b_{42})\xi_5\\
&\quad + O\big(t^4\big).
\end{aligned}$$

Define a map

$$G = (G_1, G_2, G_3, G_4) : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^4,$$

where the components of G are functions of z and t, and are given by

$$\det P_3^4 = t^3G_1, \qquad \det P_2^4 = t^3G_2, \qquad \det P_2^3 = t^3G_3, \qquad \det P_1^2 = t^3G_4.$$

In order to find $\bar{z}$ such that $G(\bar{z}, 0) = 0$, we proceed as follows. Solve for $\bar{z}_3$ in terms of $\bar{z}_1$ from $G_2(\bar{z}, 0) = 0$, and solve for $\bar{z}_4$ in terms of $\bar{z}_1$ from $G_3(\bar{z}, 0) = 0$. Now insert these expressions for $\bar{z}_3$ and $\bar{z}_4$ into $G_4(\bar{z}, 0) = 0$ to obtain a quadratic polynomial for $\bar{z}_1$. Under generic conditions on the parameters, this polynomial has two nonzero real solutions or no real solutions. In the case of two real solutions we may then find $\bar{z}_2$ from $G_1(\bar{z}, 0) = 0$. Furthermore

$$DG(\bar{z}, 0) = \begin{pmatrix} \bar{z}_2 b_{43} - b_{23} b_{42} & \bar{z}_1 b_{43} - b_{13} b_{41} & 0 & 0 \\ b_{32} b_{43} - \bar{z}_3 b_{42} & 0 & b_{12} b_{41} - \bar{z}_1 b_{42} & 0 \\ \bar{z}_4 b_{32} - b_{34} b_{42} & 0 & 0 & \bar{z}_1 b_{32} - b_{12} b_{31} \\ 0 & 0 & \bar{z}_4 b_{12} - b_{14} b_{42} & \bar{z}_3 b_{12} - b_{13} b_{32} \end{pmatrix},$$

which is invertible under further generic conditions. Thus the implicit function theorem gives a function $z(t)$ with $G(z(t), t) = 0$ for small t. We have then constructed a curve $\xi(t)$ on which the four relevant determinant minors vanish.

We now show that all other determinant minors vanish on $\xi(t)$, again assuming appropriate generic conditions. We will use Lemma 4.2 as usual, omitting explicit details. Use that $\det P_2^3 = \det P_2^4 = 0$ to show that all minors constructed from elements of the second row have zero determinant. With this and $\det P_1^2 = 0$ we may then show that all minors constructed from elements of the second column have zero determinant. Furthermore, all determinant minors constructed from elements of the third row are zero, by the above and $\det P_3^4 = 0$. Now that determinant minors from two rows all vanish, we may proceed exactly as in Cases 1 and 2 to complete the argument.

To conclude the proof, let us note that in Case 1 we have obtained ten points $a \in \Sigma_{\mathrm{sing}}(t\mathbf{c}) \cap \mathbb{P}^4$, and in Case 2 we have obtained $\alpha + 3\gamma$ points where α and γ are nonnegative integers summing to 10. In Case 3, we imposed generic conditions on the parameters so that the quadratic polynomial equation satisfied by $\bar{z}_1$ has two real solutions or no real solutions. Since there are five different combinations of four zero components for a, we obtain 5β points in Case 3, where β is a nonnegative integer with $\beta \leqslant 5$. Thus the total is $10 + \alpha + 2\beta + 3\gamma$. $\square$

The proof of Theorem 4.4 allows a simple characterization of the limit set $\Lambda(\mathbf{c}) \subset \mathbb{P}^4$ (as $t \to 0$) of the singular part of the characteristic variety in dimension 5. Namely, under generic conditions on the parameters, it consists of the following points:

(i) the ten points $a = [a_1, \ldots, a_5]$ with $a_i = a_j = 0$, $i \neq j$, such that

$$\sum_{k \neq i,j} c_i^{kj} a_k = 0, \qquad \sum_{k \neq i,j} c_j^{ki} a_k = 0; \tag{4.12}$$

(ii) the $\alpha + 3\gamma$ points $a = [a_1, \ldots, a_5]$ with $a_i = a_j = a_l = 0$, $i < j < l$, such that

$$\begin{aligned} &\Big(\sum_{k \neq i,j,l} c_i^{kj} a_k\Big)\Big(\sum_{k \neq i,j,l} c_j^{kl} a_k\Big)\Big(\sum_{k \neq i,j,l} c_l^{ki} a_k\Big) \\ &\quad = \Big(\sum_{k \neq i,j,l} c_j^{ki} a_k\Big)\Big(\sum_{k \neq i,j,l} c_l^{kj} a_k\Big)\Big(\sum_{k \neq i,j,l} c_i^{kl} a_k\Big); \end{aligned} \tag{4.13}$$

(iii) five points

$$[1, 0, 0, 0, 0],\ [0, 1, 0, 0, 0],\ [0, 0, 1, 0, 0],\ [0, 0, 0, 1, 0],\ [0, 0, 0, 0, 1].$$

Recall that the conditions imposed on the parameters guarantee that Eqs. (4.12) have exactly one solution, and that Eq. (4.13) has either one or three solutions. We have shown that the singular variety $\Sigma_{\text{sing}}(t\mathbf{c}) \cap \mathbb{P}^4$ lies in a neighborhood of $\Lambda(\mathbf{c})$ for t sufficiently small.

We now study the singular variety for all higher dimensions $n \geqslant 5$. Although the following result is not as precise as in the 5-dimensional result above, we are able to show that the singular variety contains a significantly large algebraic variety. We believe that a similar analysis, on a case by case basis, as was carried out in the proof of Theorem 4.4 is possible, and will lead to a characterization of the singular variety for small t. However due to the extensive calculations involved, we restrict attention to a single case below. Pick $i, j \in \{1, \dots, n\}$ with $i \neq j$, and consider the following generic conditions on the parameters:

$$c_i^{kj} c_j^{li} \neq c_j^{ki} c_i^{lj} \quad \text{for all } k \neq l, \text{ with } k \neq i, j \text{ and } l \neq i, j, \tag{4.14}$$

and for each $p \neq i, j$ there exists $k, l, m \notin \{i, j\}$ with $k < l < m$ such that

$$c_p^{kI}\left(c_i^{lj} c_j^{mi} - c_j^{li} c_i^{mj}\right) + c_p^{lI}\left(c_j^{ki} c_i^{mj} - c_i^{kj} c_j^{mi}\right) + c_p^{mI}\left(c_i^{kj} c_j^{li} - c_j^{ki} c_i^{lj}\right) \neq 0, \tag{4.15}$$

where $I = i$ or $I = j$.

Theorem 4.5. *Let $n \geqslant 5$. If all elements c_i^{kj} of $\mathbf{c}$ satisfy the conditions* (4.14) *and* (4.15)*, then for any sufficiently small $t > 0$, $\Sigma_{\text{sing}}(t\mathbf{c}) \cap \mathbb{P}^{n-1}$ contains an algebraic variety of dimension $n - 5$.*

Proof. We will follow a similar strategy as in Case 1 of the proof of Theorem 4.4. Thus, our goal will be to construct a curve $\xi(t) \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$ such that all $(n-1) \times (n-1)$ determinant minors of $P(\xi(t), t\mathbf{c})$ vanish, for all sufficiently small t. If this is to occur, then as in the first paragraph of the proof of Theorem 4.4, we must have (after possibly passing to a subsequence)

$$\xi(t) \to a = (a_1, \dots, a_n),$$

where two elements of a vanish, say $a_i = a_j = 0$. Choose the remaining components of a to satisfy the following three properties:

$$\prod_{k \neq i, j} a_k \neq 0,$$

$$\sum_{k \neq i, j} c_i^{kj} a_k = 0, \qquad \sum_{k \neq i, j} c_j^{ki} a_k = 0,$$

and for all $p \neq i, j$,

$$\sum_{k \neq i, j} c_p^{ki} a_k \neq 0 \quad \text{or} \quad \sum_{k \neq i, j} c_p^{kj} a_k \neq 0. \tag{4.16}$$

The generic conditions (4.14) and (4.15) guarantee that such an a exists. Moreover, it is clear that the set of all $a \in \mathbb{S}^{n-1}$ satisfying these properties contains an algebraic variety of dimension $n - 5$.

In what follows we will let $i = 1$ and $j = 2$ for convenience. As in the first part of the proof of Theorem 4.4, we will analyze

$$\det P_1^1 = \det P_1^2 = \det P_2^1 = \det P_2^2 = 0.$$

Set

$$\xi_1(t) = ty_1(t), \qquad \xi_2(t) = ty_2(t).$$

Then

$$\xi_1 + tb_{11} = tx_1 + t^2 c_1^{21} y_2, \qquad \xi_2 + tb_{22} = tx_2 + t^2 c_2^{12} y_1,$$

where

$$x_i = y_i + \sum_{k>2} c_i^{ki} \xi_k.$$

First,

$$\det P_1^1 = t\xi_3 \cdots \xi_n x_2 + O\big(t^2\big), \qquad \det P_2^2 = t\xi_3 \cdots \xi_n x_1 + O\big(t^2\big).$$

This motivates us to write

$$x_i(t) = tz_i(t) \quad \text{for } i = 1, 2,$$

for some z_i. Moreover we have

$$\det P_2^1 = t\xi_3 \cdots \xi_n b_{21} + O\big(t^2\big), \qquad \det P_1^2 = t\xi_3 \cdots \xi_n b_{12} + O\big(t^2\big).$$

This implies $b_{12} \to 0$ and $b_{21} \to 0$ as $t \to 0$. This suggests that we write

$$\sum_{k>2} c_1^{k2} \xi_k = tz_3(t), \qquad \sum_{k>2} c_2^{k1} \xi_k = tz_4(t), \tag{4.17}$$

for some z_3 and z_4, so that

$$b_{12} = t\big(c_1^{12} y_1 + z_3\big), \qquad b_{21} = t\big(c_2^{21} y_2 + z_4\big).$$

Upon calculating the four determinants above in terms of the z_i we obtain,

$$\begin{aligned}\det P_1^1 = t^2\Bigg[&\Bigg(z_2 - c_2^{12} \sum_{k>2} c_1^{k1} \xi_k\Bigg)\xi_3 \cdots \xi_n - b_{23}b_{32}\xi_4 \cdots \xi_n - \cdots - b_{2n}b_{n2}\xi_3 \cdots \xi_{n-1}\Bigg] \\ &+ O\big(t^3\big),\end{aligned} \tag{4.18}$$

$$\det P_2^2 = t^2\Bigg[\Bigg(z_1 - c_1^{21}\sum_{k>2} c_2^{k2}\xi_k\Bigg)\xi_3\cdots\xi_n - b_{13}b_{31}\xi_4\cdots\xi_n - \cdots - b_{1n}b_{n1}\xi_3\cdots\xi_{n-1}\Bigg] + O\big(t^3\big), \tag{4.19}$$

$$\det P_2^1 = t^2\Bigg[\Bigg(z_4 - c_2^{12}\sum_{k>2} c_2^{k2}\xi_k\Bigg)\xi_3\cdots\xi_n - b_{23}b_{31}\xi_4\cdots\xi_n - \cdots - b_{2n}b_{n1}\xi_3\cdots\xi_{n-1}\Bigg] + O\big(t^3\big), \tag{4.20}$$

$$\det P_1^2 = t^2\Bigg[\Bigg(z_3 - c_1^{21}\sum_{k>2} c_1^{k1}\xi_k\Bigg)\xi_3\cdots\xi_n - b_{13}b_{32}\xi_4\cdots\xi_n - \cdots - b_{1n}b_{n2}\xi_3\cdots\xi_{n-1}\Bigg] + O\big(t^3\big). \tag{4.21}$$

Define functions G_i by

$$\det P_2^2 = t^2 G_1, \qquad \det P_1^1 = t^2 G_2, \qquad \det P_1^2 = t^2 G_3, \qquad \det P_2^1 = t^2 G_4.$$

We would like each G_i to be a function of z_i and t. To see that this is the case, we recall (4.14) with $i = 1$, $j = 2$. Since (4.14) holds with $k = 3$ and $l = 4$, we may solve Eqs. (4.17) for ξ_3 and ξ_4 in terms of $\xi_5, \ldots, \xi_n$, z_3, and z_4. More precisely, for these values of k and l, fix $\xi_l(t) = a_l$, $l = 5, \ldots, n$. Then solve to obtain

$$\begin{aligned}
\xi_3(t) &= a_3 + t\big(c_1^{32}c_2^{41} - c_2^{31}c_1^{42}\big)^{-1}\big[\big(c_2^{41}z_3(t) - c_1^{42}z_4(t)\big) \\
&\quad + \big(c_1^{42}c_2^{51} - c_2^{41}c_1^{52}\big)a_5 + \cdots + \big(c_1^{42}c_2^{n1} - c_2^{41}c_1^{n2}\big)a_n\big], \\
\xi_4(t) &= a_4 + t\big(c_1^{32}c_2^{41} - c_2^{31}c_1^{42}\big)^{-1}\big[\big(c_1^{32}z_4(t) - c_2^{31}z_3(t)\big) \\
&\quad + \big(c_2^{31}c_1^{52} - c_1^{32}c_2^{51}\big)a_5 + \cdots + \big(c_2^{31}c_1^{n2} - c_1^{32}c_2^{n1}\big)a_n\big].
\end{aligned}$$

We now have a map

$$G = (G_1, G_2, G_3, G_4) : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^4,$$

with $G(\bar{z}, 0) = 0$, where $\bar{z}$ may be determined from (4.18), (4.19), (4.20) and (4.21). Moreover a simple calculation shows that $DG(\bar{z}, 0) = a_3\cdots a_n I_4$, where I_4 is the 4×4 identity matrix. By the implicit function theorem there exists $z(t)$ such that $G(z(t), t) = 0$ for all sufficiently small t. Thus we have found a curve $\xi(t)$ such that $\det P_1^1(t) = \det P_2^2(t) = \det P_2^1(t) = \det P_1^2(t) = 0$.

The fact that all remaining determinant minors vanish as well on the curve $\xi(t)$, follows from the same arguments at the end of Case 1 in the proof of Theorem 4.4. Here one must use that $b_{k1} \neq 0$ or $b_{k2} \neq 0$ for all $k \neq 1, 2$, which follows from (4.16). □

Now we discuss the limit set of $\Sigma_{\mathrm{sing}}(t\mathbf{c}) \cap \mathbb{S}^{n-1}$ as $t \to 0$. In the proof of Theorem 4.5 for $n \geqslant 5$, we constructed an $(n-5)$-dimensional surface in $\Sigma_{\mathrm{sing}}(t\mathbf{c}) \cap \mathbb{S}^{n-1}$ which can be viewed as a perturbation of

$$\{\xi \mid \xi_1 = \xi_2 = 0\}.$$

This corresponds to Case 1 in the proof of Theorem 4.4. Similarly, we can construct an $(n-5)$-dimensional surface in $\Sigma_{\mathrm{sing}}(t\mathbf{c}) \cap \mathbb{S}^{n-1}$ corresponding to Cases 2 and 3 in the proof of Theorem 4.4. By making appropriate permutations, we can write down the limit set $\Lambda(\mathbf{c})$ of $\Sigma_{\mathrm{sing}}(t\mathbf{c}) \cap \mathbb{P}^{n-1}$ as $t \to 0$. In fact, under appropriate generic conditions on the parameters, it consists of all points $a = [a_1, \ldots, a_n] \in \mathbb{P}^{n-1}$ satisfying one of the following three sets of equations:

(i) $a_i = a_j = a_k = a_l = 0$ for $i < j < k < l$;
(ii) $a_i = a_j = a_l = 0$ for $i < j < l$ such that

$$\Bigl(\sum_{k \neq i,j,l} c_i^{kj} a_k\Bigr)\Bigl(\sum_{k \neq i,j,l} c_j^{kl} a_k\Bigr)\Bigl(\sum_{k \neq i,j,l} c_l^{ki} a_k\Bigr) = \Bigl(\sum_{k \neq i,j,l} c_j^{ki} a_k\Bigr)\Bigl(\sum_{k \neq i,j,l} c_l^{kj} a_k\Bigr)\Bigl(\sum_{k \neq i,j,l} c_i^{kl} a_k\Bigr);$$

(iii) $a_i = a_j = 0$ for $i < j$ such that

$$\sum_{k \neq i,j} c_i^{kj} a_k = \sum_{k \neq i,j} c_j^{ki} a_k = 0.$$

In terms of the rescaled principal symbol $\widetilde{P} = (\widetilde{p}_{ij})$ given by

$$\widetilde{p}_{ij} = \lim_{t \to 0} t^{-1+\delta_{ij}} p_{ij},$$

we can express $\Lambda(\mathbf{c})$ alternatively by

$$\begin{aligned}\Lambda(\mathbf{c}) = {} & \Bigl(\bigcup_{i<j} \{\widetilde{p}_{ii} = \widetilde{p}_{jj} = \widetilde{p}_{ij} = \widetilde{p}_{ji} = 0\}\Bigr) \\ & \cup \Bigl(\bigcup_{i<j<l} \{\widetilde{p}_{ii} = \widetilde{p}_{jj} = \widetilde{p}_{ll} = \widetilde{p}_{ij}\widetilde{p}_{jl}\widetilde{p}_{li} - \widetilde{p}_{ji}\widetilde{p}_{lj}\widetilde{p}_{il} = 0\}\Bigr) \\ & \cup \Bigl(\bigcup_{i<j<k<l} \{\widetilde{p}_{ii} = \widetilde{p}_{jj} = \widetilde{p}_{kk} = \widetilde{p}_{ll} = 0\}\Bigr).\end{aligned}$$

An important observation here is that $\Lambda(\mathbf{c}) \cap \mathbb{S}^{n-1}$ is smooth except along intersections of any two subsets in this expression for $\Lambda(\mathbf{c})$. In other words, these intersections constitute the singular part of $\Lambda(\mathbf{c})$.

Remark 4.6. To conclude this paper, we make a remark concerning Theorem 4.4 and Theorem 4.5. When characteristic varieties are smooth, the corresponding linear differential systems are of the principal type. Symmetrizers can be constructed and solutions can be proven to exist. However, when the characteristic varieties are not smooth, a general existence theory of solutions is not available. It is believed that symmetrizers can still be constructed if the singular sets of the characteristic varieties enjoy a simple geometry, as illustrated by these theorems.

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