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LOCAL SOLVABILITY OF DEGENERATE MONGE-AMPÈRE EQUATIONS AND APPLICATIONS TO GEOMETRY

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ABSTRACT. We consider two natural problems arising in geometry which are equivalent to the local solvability of specific equations of Monge-Ampère type. These are: the problem of locally prescribed Gaussian curvature for surfaces in $\mathbb{R}^3$, and the local isometric embedding problem for two-dimensional Riemannian manifolds. We prove a general local existence result for a large class of degenerate Monge-Ampère equations in the plane, and obtain as corollaries the existence of regular solutions to both problems, in the case that the Gaussian curvature vanishes and possesses a nonvanishing Hessian matrix at a critical point.

1. INTRODUCTION

Let $K(u,v)$ be a function defined in a neighborhood of a point in $\mathbb{R}^2$, say $(u,v) = 0$. A well-known problem is to ask, when does there exist a piece of a surface $z = z(u,v)$ in $\mathbb{R}^3$ having Gaussian curvature K?

The classical results on this problem may be found in [10, 19, 20]. They show that a solution always exists when K is analytic or K does not vanish at the origin. In the case that $K \geq 0$ and is sufficiently smooth, or $K(0) = 0$ and $|\nabla K(0)| \neq 0$, Lin provides an affirmative answer in [15, 16] (see [4] for a simplified proof of [16]). When $K \leq 0$ and ∇K possesses a certain nondegeneracy, Han, Hong, and Lin [8] show that a solution always exists. Furthermore, if K degenerates to arbitrary finite order on a single smooth curve, then Han and the author independently provide an affirmative answer in [5, 11] (see also [6] for improved regularity). For an excellent survey of these results and related topics, see [7]. In this paper we prove the following,

Theorem 1.1. *Suppose that $K(0) = |\nabla K(0)| = 0$, $\nabla^2 K(0)$ has at least one negative eigenvalue, and $K \in C^l$, $l \geq 100$. Then there exists a piece of a C^{l-98} surface in $\mathbb{R}^3$ with Gaussian curvature K.*

If a surface in $\mathbb{R}^3$ is given by $z = z(u,v)$, then its Gaussian curvature is given by

$$z_{uu}z_{vv} - z_{uv}^2 = K(1 + |\nabla z|^2)^2. \tag{1.1}$$

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Therefore our problem is equivalent to the local solvability of the above equation.

Another well-known and related problem, is that of the local isometric embedding of surfaces into $\mathbb{R}^3$. That is, if (M^2, ds^2) is a two-dimensional Riemannian manifold, when can one realize this, locally, as a small piece of a surface in $\mathbb{R}^3$? Suppose that $ds^2 = Edu^2 + 2Fdudv + Gdv^2$ is given in the neighborhood of a point, say $(u, v) = 0$. Then we must find three function $x(u, v)$, $y(u, v)$, $z(u, v)$, such that $ds^2 = dx^2 + dy^2 + dz^2$. The following strategy was first used by Weingarten [25]. We search for a function $z(u, v)$, with $|\nabla z|$ sufficiently small, such that $ds^2 - dz^2$ is flat in a neighborhood of the origin. Suppose that such a function exists, then since any Riemannian manifold of zero curvature is locally isometric to Euclidean space (via the exponential map), there exists a smooth change of coordinates $x(u, v)$, $y(u, v)$ such that $dx^2 + dy^2 = ds^2 - dz^2$. Therefore, our problem is reduced to finding $z(u, v)$ such that $ds^2 - dz^2$ is flat in a neighborhood of the origin. A computation shows that this is equivalent to the local solvability of the equation

$$(z_{11}-\Gamma_{11}^i z_i)(z_{22}-\Gamma_{22}^i z_i)-(z_{12}-\Gamma_{12}^i z_i)^2 = K(EG-F^2-Ez_2^2-Gz_1^2+2Fz_1z_2), \quad (1.2)$$

where $z_1 = \partial z/\partial u$, $z_2 = \partial z/\partial v$, z_{ij} are second partial derivatives of z, and Γ_{jk}^i are Christoffel symbols. For this problem we obtain a similar result to that of Theorem 1.1.

Theorem 1.2. *Suppose that $K(0) = |\nabla K(0)| = 0$, $\nabla^2 K(0)$ has at least one negative eigenvalue, and $ds^2 \in C^l$, $l \geq 102$. Then there exists a C^{l-100} local isometric embedding into $\mathbb{R}^3$.*

We note that Pogorelov has constructed a $C^{2,1}$ metric with no C^2 isometric embedding in $\mathbb{R}^3$. Other examples of metrics with low regularity not admitting a local isometric embedding have also been proposed by Nadirashvili and Yuan [17]. Furthermore, an alternate method for obtaining *smooth* examples of local nonsolvability, for equations with similar structure, may be found in [12].

Equations (1.1) and (1.2) are both two-dimensional Monge-Ampère equations. With the goal of treating both problems simultaneously, we will study the local solvability of the following general Monge-Ampère equation

$$\det(z_{ij} + a_{ij}(u, v, z, \nabla z)) = Kf(u, v, z, \nabla z), \quad (1.3)$$

where $a_{ij}(u, v, p, q)$ and $f(u, v, p, q)$ are smooth functions of p and q, $f > 0$, and $a_{ij}(0, 0, p, q) = \partial^\alpha a_{ij}(0, 0, 0, 0) = 0$, for any multi-index α in the variables (u, v) satisfying $|\alpha| \leq 2$. Clearly (1.1) is of the form (1.3), and (1.2) is of the form (1.3) if $\Gamma_{jk}^i(0) = 0$, which we assume without loss of generality. We will prove

Theorem 1.3. *Suppose that $K(0) = |\nabla K(0)| = 0$, $\nabla^2 K(0)$ has at least one negative eigenvalue, and K, a_{ij}, $f \in C^l$, $l \geq 100$. Then there exists a C^{l-98} local solution of (1.3).*

Remark. (1) The methods carried out below may be slightly modified to yield the same result for the case when $\nabla^2 K(0)$ has at least one positive eigenvalue; and therefore ultimately include the case of genuine second order vanishing, that is, when $K(0) = |\nabla K(0)| = 0$ and $|\nabla^2 K(0)| \neq 0$. It is conjectured that local solutions exist whenever K vanishes to finite order and the a_{ij} vanish to an order greater than or equal to half that of K.

(2) Recently Han and the author [9] have shown that local solutions exist for the isometric embedding problem, whenever K vanishes to finite order and the

zero set $K^{-1}(0)$ consists of Lipschitz curves intersecting transversely at the origin. Unfortunately the methods of [9] breakdown when the transversality assumption is removed. Therefore Theorem 1.3 (which allows tangential intersections) and the methods used to prove it, may be considered as a first step towards the general conjecture.

Equation (1.3) is elliptic if $K > 0$, hyperbolic if $K < 0$, and of mixed type if K changes sign in a neighborhood of the origin. Furthermore, the order to which K vanishes determines how (1.3) changes type in the following way. If $K(0) = 0$ and $|\nabla K(0)| \neq 0$ [16], then (1.3) is a nonlinear perturbation of the Tricomi equation:

$$vz_{uu} + z_{vv} = 0.$$

In our case, assuming that the origin is a critical point for which the Hessian matrix of K does not vanish, (1.3) is a nonlinear perturbation of Gallerstedt's equation [3]:

$$\pm v^2 z_{uu} + z_{vv} = 0.$$

Therefore, if sufficiently small linear perturbation terms are added to the above two equations, then the first (second) partial v-derivative of the z_{uu} coefficient will not vanish for the Tricomi (Gallerstedt) equation. It is this fact, which allows one to obtain appropriate estimates for the linearized equation of (1.3) in both cases. This observation, Lemma 2.3 below, is the key to our approach.

From now on we only consider the case when $\nabla^2 K(0)$ has at least one negative eigenvalue. Therefore, we can assume without loss of generality that

$$Kf(u, v, z, \nabla z) = -v^2 + O(|u|^2 + |v|^3 + |z|^2 + |\nabla z|^2).$$

Let ε be a small parameter and set $u = \varepsilon^4 x$, $v = \varepsilon^2 y$, $z = u^2/2 - v^4/12 + \varepsilon^9 w$. Then substituting into (1.3) and cancelling ε^5 on both sides yields

$$-y^2 w_{xx} + w_{yy} + \varepsilon \widetilde{F}(\varepsilon, x, y, w, \nabla w, \nabla^2 w) = 0, \tag{1.4}$$

where $\widetilde{F}(\varepsilon, x, y, p, q, r)$ is smooth with respect to ε, p, q, and r. Choose x_0, $y_0 > 0$ and define the rectangle $X = \{(x, y) : |x| < x_0, |y| < y_0\}$. Let $\psi \in C^\infty(X)$ be a cut-off function such that

$$\psi(x, y) = \begin{cases} 1 & \text{if } |x| \leq \frac{x_0}{2} \text{ and } |y| \leq \frac{y_0}{2}, \\ 0 & \text{if } |x| \geq \frac{3x_0}{4} \text{ or } |y| \geq \frac{3y_0}{4}, \end{cases}$$

and cut-off the nonlinear term of (1.4) by $F(\varepsilon, x, y, w, \nabla w, \nabla^2 w) = \psi \widetilde{F}$. Then solving

$$\Phi(w) = -y^2 w_{xx} + w_{yy} + \varepsilon F(\varepsilon, x, y, w, \nabla w, \nabla^2 w) = 0 \quad \text{in } X, \tag{1.5}$$

is equivalent to solving (1.3) locally at the origin.

In the next sections, we shall study the linearization of (1.5) about some function w. The linearized equation is a small perturbation of Gallerstedt's equation, which as mentioned above admits certain estimates. These estimates are sufficient for the existence of weak solutions, however the perturbation terms cause some difficulty in proving higher regularity. To avoid this problem, we will regularize the equation by appending a suitably small fourth order operator. In section §2 we shall prove the existence of weak solutions for a boundary value problem associated to this modified linearized equation. Regularity will be obtained in section §3. In section §4 we make the appropriate estimates in preparation for the Nash-Moser iteration procedure. Finally, in §5 we apply a modified version of the Nash-Moser procedure and obtain a solution of (1.5).

2. Linear Existence Theory

In this section we will prove the existence of weak solutions for a small perturbation of the linearized equation for (1.5). Fix a constant $\Lambda > 0$, and for all $i, j = 1, 2$ let b_{ij}, b_i, $b \in C^r(\mathbb{R}^2)$ be such that:

(i) The supports of b_{ij}, b_i, and b are contained in X, and
(ii) $\sum |b_{ij}|_{C^{10}} + |b_i|_{C^{10}} + |b|_{C^{10}} \leq \Lambda$.

We will study the following generalization of the linearization for (1.5),

$$L = \sum_{i,j} a_{ij}\partial_{x_i x_j} + \sum_i a_i \partial_{x_i} + a \tag{2.1}$$

where $x_1 = x$, $x_2 = y$ and $a_{11} = -y^2 + \varepsilon b_{11}$, $a_{12} = \varepsilon b_{12}$, $a_{22} = 1 + \varepsilon b_{22}$, $a_1 = \varepsilon b_1$, $a_2 = \varepsilon b_2$, $a = \varepsilon b$.

To simplify (2.1), we shall make a change of variables that will eliminate the mixed second derivative term. In constructing this change of variables we will make use of the following lemma from ordinary differential equations.

Lemma 2.1 ([1]). *Let $G(x,t)$ be a smooth real valued function in the closed rectangle $|x - s| \leq T_1$, $|t| \leq T_2$. Let $M = \sup|G(x,t)|$ in this domain. Then the initial-value problem $dx/dt = G(x,t)$, $x(0) = s$, has a unique smooth solution defined on the interval $|t| \leq \min(T_2, T_1/M)$.*

We now construct the desired change of variables.

Lemma 2.2. *For ε sufficiently small, there exists a C^r diffeomorphism*

$$\xi = \xi(x,y), \quad \eta = y,$$

of X onto itself, such that in the new variables (ξ, η)

$$L = \sum_{i,j} \overline{a}_{ij}\partial_{x_i x_j} + \sum_i \overline{a}_i \partial_{x_i} + \overline{a},$$

where $x_1 = \xi$, $x_2 = \eta$, $\overline{a}_{11} = -\eta^2 + \varepsilon\overline{b}_{11}$, $\overline{a}_{12} \equiv 0$, $\overline{a}_{22} = 1 + \varepsilon\overline{b}_{22}$, $\overline{a}_1 = \varepsilon\overline{b}_1$, $\overline{a}_2 = \varepsilon\overline{b}_2$, $\overline{a} = \varepsilon\overline{b}$, and $\overline{b}_{ij}$, $\overline{b}_i$, $\overline{b}$ satisfy:

(i) $\overline{b}_{ij}, \overline{b}_i, \overline{b} \in C^{r-2}(\overline{X})$,
(ii) $\overline{b}_{ij}$, $\overline{b}_i$, *and* $\overline{b}$ *vanish in a neighborhood of the lines* $\xi = \pm x_0$, *and*
(iii) $\sum |\overline{b}_{ij}|_{C^8(\overline{X})} + |\overline{b}_i|_{C^8(\overline{X})} + |\overline{b}|_{C^8(\overline{X})} \leq \Lambda'$,

for some fixed Λ'.

Proof. Using the chain rule we find that $\overline{a}_{12} = a_{12}\xi_x + a_{22}\xi_y$. Therefore, we seek a smooth function $\xi(x,y)$ such that

$$a_{12}\xi_x + a_{22}\xi_y = 0 \quad \text{in } X, \quad \xi(x,0) = x, \quad \xi(\pm x_0, y) = \pm x_0. \tag{2.2}$$

The boundary condition $\xi(\pm x_0, y) = \pm x_0$ states that the vertical sides of ∂X will be mapped identically onto themselves under the transformation (ξ, η). Moreover, the horizontal portion of ∂X will be mapped identically onto itself since $\eta = y$. Thus, (ξ, η) will act as the identity map on ∂X.

Since $a_{12} = \varepsilon b_{12}$ and $a_{22} = 1 + \varepsilon b_{22}$, by property (ii) if ε is sufficiently small the line $y = 0$ will be non-characteristic for (2.2). Then by the theory of first order partial differential equations, (2.2) is reduced to the following system of first order ODE:

$$\dot{x} = \frac{a_{12}}{a_{22}}, \quad x(0) = s, \quad -x_0 \le s \le x_0,$$
$$\dot{y} = 1, \quad y(0) = 0,$$
$$\dot{\xi} = 0, \quad \xi(0) = s, \quad \xi(\pm x_0, y) = \pm x_0,$$

where $x = x(t)$, $y = y(t)$, $\xi(t) = \xi(x(t), y(t))$ and $\dot{x}$, $\dot{y}$, $\dot{\xi}$ are derivatives with respect to t.

We first show that the characteristic curves, given parametrically by $(x, y) = (x(t), t)$, exist globally for $-y_0 \le t \le y_0$. We apply Lemma 2.1 with $T_1 = 2x_0$ and $T_2 = y_0$ to the initial-value problem $\dot{x} = \frac{a_{12}}{a_{22}}$, $x(0) = s$. By property (ii) for the b_{ij}

$$M \le \sup_X |\frac{a_{12}}{a_{22}}| = \varepsilon \sup_X |\frac{b_{12}}{1 + \varepsilon b_{22}}| \le \varepsilon C_0,$$

so for ε small, $M \le \frac{2x_0}{y_0}$. Thus $\min(T_2, T_1/M) = y_0$, and Lemma 2.1 gives the desired global existence.

We observe that $\xi = s$ is constant along each characteristic. In particular, since $\frac{a_{12}}{a_{22}}|_{(\pm x_0, y)} = 0$ the characteristics passing through $(\pm x_0, 0)$ are the vertical lines $(\pm x_0, t)$, so that $\xi(\pm x_0, y) = \pm x_0$ is satisfied.

We now show that the map $\rho : X \to X$ given by

$$(s, t) \mapsto (x(s, t), y(s, t)) = (x(s, t), t)$$

is a diffeomorphism, from which we will conclude that $\xi = s(x, y)$ is a smooth function of (x, y). To show that ρ is 1-1, suppose that $\rho(s_1, t_1) = \rho(s_2, t_2)$. Then $t_1 = t_2$ and $x(s_1, t_1) = x(s_2, t_2)$, which implies that $s_1 = s_2$ by uniqueness for the initial-value problem for ordinary differential equations. To show that ρ is onto, take an arbitrary point $(x_1, y_1) \in X$, then we will show that there exists $s \in [-x_0, x_0]$ such that $\rho(s, y_1) = (x(s, y_1), y_1) = (x_1, y_1)$. Since the map $x(s, \cdot) : [-x_0, x_0] \to [-x_0, x_0]$ is continuous and $x(\pm x_0, \cdot) = \pm x_0$, the intermediate value theorem guarantees that there is $s \in [-x_0, x_0]$ with $x(s, y_1) = x_1$, showing that ρ is onto. Therefore, ρ has a well-defined inverse.

To show that ρ^{-1} is smooth it is sufficient, by the inverse function theorem, to show that the Jacobian of ρ does not vanish at each point of X. Since

$$D\rho = \begin{pmatrix} x_s & x_t \\ 0 & 1 \end{pmatrix},$$

this is equivalent to showing that x_s does not vanish in X. Differentiate the equation for x with respect to s to obtain, $\frac{d}{dt}(x_s) = (\frac{a_{12}}{a_{22}})_x x_s$, $x_s(0) = 1$. Then by the mean value theorem,

$$|x_s(s, t) - 1| = |x_s(s, t) - x_s(s, 0)| \le y_0 \sup_X |(\frac{a_{12}}{a_{22}})_x| \sup_X |x_s|$$

for all $(s, t) \in X$. Thus by property (ii) for the b_{ij},

$$1 - \varepsilon C_1 y_0 \sup_X |x_s| \le x_s(s, t) \le \varepsilon C_1 y_0 \sup_X |x_s| + 1$$

for all $(s, t) \in X$. Hence for ε sufficiently small, $x_s(s, t) > 0$ in X. We have now shown that ρ is a diffeomorphism. Moreover, by Lemma 2.1 and the inverse function theorem, we have $\rho, \rho^{-1} \in C^r$.

Lastly we calculate $\overline{a}_{11}$, $\overline{a}_{22}$, $\overline{a}_1$, $\overline{a}_2$, and show that they possess the desired properties. It will first be necessary to estimate the derivatives of ξ. By differentiating (2.2) with respect to x, we obtain

$$(\frac{a_{12}}{a_{22}})(\xi_x)_x + (\xi_x)_y = -(\frac{a_{12}}{a_{22}})_x\xi_x, \quad \xi_x(x,0) = 1.$$

As above, let $(x(t), y(t))$ be the parameterization of an arbitrary characteristic, then $\xi_x(t) = \xi_x(x(t), y(t))$ satisfies $\dot{\xi}_x = -(\frac{a_{12}}{a_{22}})_x\xi_x$, $\xi_x(0) = 1$. By the mean value theorem,

$$|\xi_x(t) - 1| = |\xi_x(t) - \xi_x(0)| \le y_0 \sup_X |(\frac{a_{12}}{a_{22}})_x| \sup_X |\xi_x|.$$

By property (ii) for the b_{ij},

$$1 - \varepsilon C_1 y_0 \sup_X |\xi_x| \le \xi_x(t) \le \varepsilon C_1 y_0 \sup_X |\xi_x| + 1.$$

Since this holds for any characteristic, we obtain

$$\sup_X |\xi_x| \le \frac{1}{1 - \varepsilon C_1 y_0} := C_2.$$

It follows from (2.2) that

$$\sup_X |\xi_y| \le C_3,$$

where C_2, C_3 are independent of ε and b_{ij}. In order to estimate ξ_{xx}, differentiate (2.2) two times with respect to x:

$$(\frac{a_{12}}{a_{22}})(\xi_{xx})_x + (\xi_{xx})_y = -2(\frac{a_{12}}{a_{22}})_x\xi_{xx} - (\frac{a_{12}}{a_{22}})_{xx}\xi_x, \quad \xi_{xx}(x,0) = 0.$$

Then the same procedure as above yields

$$\sup_X |\xi_{xx}| \le \varepsilon C_4 y_0 \sup_X |\xi_{xx}| + \varepsilon C_5 y_0,$$

implying that

$$\sup_X |\xi_{xx}| \le \frac{\varepsilon C_5 y_0}{1 - \varepsilon C_4 y_0} := \varepsilon C_6.$$

Furthermore, using the above estimates we can differentiate (2.2) to obtain

$$\sup_X |\xi_{xy}| \le \varepsilon C_7, \quad \sup_X |\xi_{yy}| \le \varepsilon C_8,$$

for some constants C_7, C_8 independent of ε and b_{ij}. This procedure may be continued to yield

$$|\partial^\alpha \xi| \le \varepsilon C_9,$$

for any multi-index α satisfying $2 \le |\alpha| \le 10$.

We now show that $\overline{a}_{11}$, $\overline{a}_{22}$, $\overline{a}_1$, $\overline{a}_2$ satisfy properties (i), (ii), (iii) and have the desired form. Calculation shows that,

$$\overline{a}_{11} = a_{11}\xi_x^2 + 2a_{12}\xi_x\xi_y + a_{22}\xi_y^2, \quad \overline{a}_1 = a_{11}\xi_{xx} + 2a_{12}\xi_{xy} + a_{22}\xi_{yy} + a_1\xi_x + a_2\xi_y.$$

Furthermore, according to the above estimates and the fact that the b_{ij} vanish in a neighborhood of ∂X, we may write

$$\xi_x = 1 + \varepsilon\chi,$$

where $\chi \in C^{r-1}(\overline{X})$ vanishes in a neighborhood of the lines $x = \pm x_0$. It follows that

$$\overline{a}_{11} = -\eta^2 + \varepsilon\bar{b}_{11}, \quad \overline{a}_1 = \varepsilon\bar{b}_1,$$

where $\bar{b}_{11}$ and $\bar{b}_1$ satisfy properties (i), (ii), (iii). Moreover since $\overline{a}_{22} = a_{22}$ and $\overline{a}_2 = a_2$, properties (i), (ii), (iii) hold for these coefficients as well. $\square$

For the remainder of this section and section §3, (ξ, η) will be the coordinates of the plane. For simplicity of notation we put $x = \xi$, $y = \eta$, and $a_{ij} = \overline{a}_{ij}$, $a_i = \overline{a}_i$, $a = \overline{a}$, $b_{ij} = \bar{b}_{ij}$, $b_i = \bar{b}_i$, $b = \bar{b}$.

To obtain a well-posed boundary value problem, we will study a regularization of L in the infinite strip $\Omega = \{(x, y) : |x| < x_0\}$. More precisely define the operator

$$L'_\theta = -\theta\partial_{xxyy} + L,$$

where $\theta > 0$ is a small constant that will tend to zero in the Nash-Moser iteration procedure. Furthermore, we will need to modify some of the coefficients of L away from X as follows. First cut b_{ij}, b_i, and b off near the lines $y = \pm y_0$, so that by property (ii) of Lemma 2.2 these functions vanish in a neighborhood of ∂X, and the coefficients a_{ij}, a_i, and a are now defined on all of Ω. Choose values y_1, y_2, and y_3 such that $y_0 < y_1 < y_2 < y_3$, and let $\delta > 0$ be a small constant that depends on $y_2 - y_1$ and $y_3 - y_2$. Then redefine the coefficient a in the domain $\Omega - X$ so that:

(i) $a \in C^{r-2}(\overline{\Omega})$,
(ii) $a \equiv 1$ if $|y| \geq y_1$,
(iii) $a \geq 0$ for $|y| \geq y_0$,
(iv) $\partial_y a \geq 0$ if $y \geq y_0$, and $\partial_y a \leq 0$ if $y \leq -y_0$.

Redefine a_{11} in $\Omega - X$ and near $\partial\Omega$ so that:

(i) $a_{11} \in C^{r-2}(\overline{\Omega})$,
(ii) $a_{11} = \begin{cases} -y^2 & \text{if } y_0 \leq |y| \leq y_1, \\ -(\frac{y_1+y_2}{2})^2 & \text{if } |y| \geq y_2, \end{cases}$
(iii) $\partial_y a_{11} < 0$ if $y \geq y_0$, and $\partial_y a_{11} > 0$ if $y \leq -y_0$,
(iv) $\sup_\Omega \partial_{yy} a_{11} \leq \delta$,
(v) $a_{11}|_{\partial\Omega} \leq -\theta$, $\partial_x^\alpha a_{11}|_{\partial\Omega} = 0$, $\alpha \leq r - 2$, and $\sup_\Omega |\partial_x^\beta a_{11}| \leq \varepsilon\Lambda'$, $1 \leq \beta \leq 8$.

Lastly, redefine a_2 in $\Omega - X$ so that:

(i) $a_2 \in C^{r-2}(\overline{\Omega})$,
(ii) $a_2 = \begin{cases} 0 & \text{if } y_0 \leq |y| \leq y_2, \\ -\delta y + \delta(\frac{y_2+y_3}{2}) & \text{if } y \geq y_3, \\ -\delta y - \delta(\frac{y_2+y_3}{2}) & \text{if } y \leq -y_3, \end{cases}$
(iii) $a_2 \leq 0$ if $y \geq y_2$, and $a_2 \geq 0$ if $y \leq -y_2$,
(iv) $\sup_{|y|\geq y_2} |\partial_y a_2| \leq \delta$.

Denote the operator L with coefficients modified as above by L', and define

$$L_\theta = -\theta\partial_{xxyy} + L'.$$

Note that since we are studying a local problem, as stated in the introduction, we may modify the coefficients of the linearization away from a fixed neighborhood of the origin. This will become clear in the final section, where a modified version of the Nash-Moser iteration scheme is used.

Consider the following boundary value problems

$$L_\theta u = f \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0; \tag{2.3}$$

$$L_\theta u = f \quad \text{in } \Omega, \quad u_x|_{\partial\Omega} = 0, \tag{2.4}$$

and the corresponding adjoint problems

$$L_\theta^* v = g \quad \text{in } \Omega, \quad v|_{\partial\Omega} = 0; \tag{2.5}$$

$$L_\theta^* v = g \quad \text{in } \Omega, \quad v_x|_{\partial\Omega} = 0, \tag{2.6}$$

where L_θ^* is the formal adjoint of L_θ. The main result of this section is to obtain weak solutions for all four problems.

We will make extensive use of the following function spaces. For $m, n \in \mathbb{Z}_{\geq 0}$ let

$$\begin{aligned} C^{(m,n)}(\overline{\Omega}) &= \{u : \Omega \to \mathbb{R} : \partial_x^\alpha \partial_y^\beta u \in C^0(\overline{\Omega}),\ \alpha \leq m,\ \beta \leq n\}, \\ \widetilde{C}^{(m,n)}(\overline{\Omega}) &= \{u \in C^{(m,n)}(\overline{\Omega}) : u|_{\partial\Omega} = 0,\ u \text{ has bounded support}\}, \\ \widetilde{C}_x^{(m,n)}(\overline{\Omega}) &= \{u \in C^{(m,n)}(\overline{\Omega}) : u_x|_{\partial\Omega} = 0,\ u \text{ has bounded support}\}. \end{aligned}$$

Define the norm

$$\|u\|_{(m,n)} = \Big(\sum_{\alpha \leq m, \beta \leq n} \|\partial_x^\alpha \partial_y^\beta u\|_{L^2(\Omega)}^2 \Big)^{1/2},$$

and let $\widetilde{H}^{(m,n)}(\Omega)$ and $\widetilde{H}_x^{(m,n)}(\Omega)$ be the respective closures of $\widetilde{C}^{(m,n)}(\overline{\Omega})$ and $\widetilde{C}_x^{(m,n)}(\overline{\Omega})$ in the norm $\|\cdot\|_{(m,n)}$. Furthermore, let $H^m(\Omega)$ denote the Sobolev space of square integrable derivatives up to and including order m, with norm $\|\cdot\|_m$. Denote the $L^2(\Omega)$ inner product and norm by $(\cdot,\cdot)$ and $\|\cdot\|$ respectively, and define the negative norm

$$\|u\|_{(-m,-n)} = \sup_{v \in \widetilde{H}^{(m,n)}(\Omega)} \frac{|(u,v)|}{\|v\|_{(m,n)}}.$$

Let $\widetilde{H}^{(-m,-n)}(\Omega)$ be the closure of $L^2(\Omega)$ in the norm $\|\cdot\|_{(-m,-n)}$, then $\widetilde{H}^{(-m,-n)}(\Omega)$ is the dual space of $\widetilde{H}^{(m,n)}(\Omega)$. The dual space of $\widetilde{H}_x^{(m,n)}(\Omega)$ is defined similarly.

Let $f \in L^2(\Omega)$. A function $u \in L^2(\Omega)$ is said to be a weak solution of (2.3) (respectively (2.4)) if

$$(u, L_\theta^* v) = (f, v), \text{ for all } v \in \widetilde{C}^\infty(\overline{\Omega}) \text{ (for all } v \in \widetilde{C}_x^\infty(\overline{\Omega})).$$

We shall employ the energy integral method, developed by K. O. Friedrichs and others, to prove the existence of weak solutions for (2.3) and (2.4). The first step is to establish an a priori estimate.

Lemma 2.3 (Basic Estimate)**.** *If ε, θ, and δ are sufficiently small, then there exist constants $C_1, C_2 > 0$ independent of ε, θ, δ, and functions $A, B, C, D, E \in C^\infty(\overline{\Omega})$ where $E > 0$ and $E = O(|y|)$ as $|y| \to \infty$, such that:*

$$\begin{aligned} &(Au + Bu_x + Cu_y + Du_{yy}, L_\theta u) \\ &\geq C_1[\|u\|^2 + \|Eu_y\|^2 + \theta(\|u_x\|^2 + \|u_{xy}\|^2 + \|u_{yy}\|^2 + \theta\|u_{xyy}\|^2)], \end{aligned}$$

for all $u \in C^\infty(\overline{\Omega})$ with bounded support such that $u_x(-x_0, y) = 0$, and either $u(x_0, y) = 0$ or $u_x(x_0, y) = 0$. Furthermore,

$$\|u\| + \|u_y\| + \sqrt{\theta}(\|u_x\| + \|u_{xy}\| + \|u_{yy}\| + \sqrt{\theta}\|u_{xyy}\|) \leq C_2\|L_\theta u\|,$$

for all $u \in \widetilde{C}^\infty(\overline{\Omega})$ and for all $u \in \widetilde{C}_x^\infty(\overline{\Omega})$.

Proof. We first define the functions A, B, C and D. Let μ be a positive constant such that $\frac{1}{4}\mu + a_{11} \geq 1$ throughout Ω, and let $\gamma \in C^\infty([-x_0, x_0])$ be such that

$$\gamma(x) = \begin{cases} 1 & \text{if } -x_0 \leq x \leq \frac{x_0}{2}, \\ 0 & \text{if } x = x_0, \end{cases}$$

with $\gamma(x) > 0$ except at $x = x_0$, and $\gamma' \leq 0$. Define

$$A = \frac{1}{2}\partial_y C - a_{11}, \quad B = -\theta\gamma,$$
$$C = \begin{cases} \mu\partial_y a_{11} & \text{if } |y| < y_0, \\ -2\mu y & \text{if } |y| \geq y_0, \end{cases} \quad D = \theta,$$

and note that $A, B, C, D \in C^\infty(\overline{\Omega})$.

We now prove the first estimate. Let $u \in C^\infty(\overline{\Omega})$ satisfy the given hypotheses. Let (n_1, n_2) denote the unit outward normal to $\partial\Omega$. Then integrate by parts to obtain:

$$\begin{aligned} &(Au + Bu_x + Cu_y + Du_{yy}, L_\theta u) \\ &= \iint_\Omega I_1 u_{xyy}^2 + I_2 u_{yy}^2 + 2I_3 u_{yy}u_{xy} + I_4 u_{xy}^2 + 2I_5 u_{xy}u_{xx} \\ &\quad + 2I_6 u_{xy}u_y + I_7 u_x^2 + 2I_8 u_x u_y + I_9 u_y^2 + I_{10}u^2 \\ &\quad + \int_{\partial\Omega} J_1 u_{xy}^2 + J_2 u_{xy}u_x + J_3 u_x^2 + J_4 u_y^2 + J_5 u^2; \end{aligned}$$

where

$$\begin{gathered} J_1 = \frac{1}{2}\theta B n_1, \quad J_2 = \theta B_y n_1, \quad J_3 = \frac{1}{2}Ba_{11}n_1, \\ 2J_4 = -\theta A_x n_1 - \theta C_{xy} n_1 + (Da_{11})_x n_1 - Da_1 n_1, \\ 2J_5 = -(Aa_{11})_x n_1 + Aa_1 n_1 + Ban_1 + \theta A_{xyy} n_1, \end{gathered}$$

and the remaining $I_1, \dots, I_{10}$ will be given below as each term is estimated. First note that $J_2|_{\partial\Omega} = J_4|_{\partial\Omega} \equiv 0$. Furthermore $J_1 = \cdots = J_5 \equiv 0$ on the portion of the boundary $x = x_0$, since $\gamma(x_0) = 0$. Whereas on the other half of the boundary $x = -x_0$, we have $u_x(-x_0, y) = 0$ and $J_5 = \frac{1}{2}Ban_1 \geq 0$. It follows that the entire boundary integral is nonnegative.

We now proceed to estimate the integral over Ω, beginning with I_1, I_5, and I_{10}, which are given by

$$\begin{gathered} I_1 = \theta D, \quad I_5 = -\frac{1}{2}\theta B_y, \\ 2I_{10} = (Aa_{11})_{xx} + (Aa_{22})_{yy} - (Aa_1)_x - (Aa_2)_y \\ +2Aa - (Ca)_y - (Ba)_x - \theta A_{xxyy} + (Da)_{yy}. \end{gathered}$$

Since B is a function of x alone, $I_5 \equiv 0$, and by definition of D, $I_1 = \theta^2$. It will now be shown that $I_{10} \geq M_1$ in Ω, for some constant $M_1 > 0$ independent of ε and θ. In order to accomplish this we shall treat the regions $|y| \leq y_0$, $y_0 \leq |y| \leq y_1$, $y_1 \leq |y| \leq y_2$, and $|y| \geq y_2$ separately. Moreover throughout this proof M_i, $i = 1, 2, \dots,$ will always denote positive constants independent of ε and θ. A computation yields,

$$I_{10} = -a_{22}\partial_{yy}a_{11} - a_{11}a - \frac{1}{2}C\partial_y a - \frac{1}{2}(Aa_2)_y + O(\varepsilon + \theta).$$

In the region $|y| \leq y_0$ we have $a, \partial_y a, a_2, \partial_y a_2 = O(\varepsilon)$, $a_{22} = 1 + O(\varepsilon)$, and $\partial_{yy} a_{11} = -2 + O(\varepsilon)$, so that here $I_{10} \geq M_2$. If $y_0 \leq |y| \leq y_1$, the conditions placed on a guarantee that

$$-a_{11} a - \frac{1}{2} C \partial_y a \geq 0;$$

furthermore a_{22}, a_{11}, and a_2 have the same properties in this region as in the previous. Hence, $I_{10} \geq M_3$ when $y_0 \leq |y| \leq y_1$. If $y_1 \leq |y| \leq y_2$ then

$$-a_{22} \partial_{yy} a_{11} = O(\delta), \quad -a_{11} a \geq y_1^2, \quad a_2 = \partial_y a \equiv 0,$$

showing that $I_{10} \geq M_4$ in this region. Lastly, when $|y| \geq y_2$ we have $I_{10} \geq M_5$ since

$$\partial_{yy} a_{11} = \partial_y a \equiv 0, \quad -a_{11} a = (\frac{y_1 + y_2}{2})^2, \quad -\frac{1}{2}(A a_2)_y = O(\delta).$$

The desired conclusion now follows by combining the above estimates.

Next we show that

$$\iint_\Omega I_2 u_{yy}^2 + 2 I_3 u_{yy} u_{xy} + I_4 u_{xy}^2 \geq M_6 \theta (\|u_{yy}\|^2 + \|u_{xy}\|^2),$$

where

$$I_2 = -\frac{1}{2} \theta D_{xx} + D a_{22}, \quad I_3 = -\frac{1}{2} \theta C_x, \quad I_4 = -\frac{1}{2} \theta C_y - \frac{1}{2} \theta B_x - \theta A + D a_{11}.$$

This will follow if $I_2 \geq M_7 \theta$, $I_4 \geq M_8 \theta$, and $I_2 I_4 - I_3^2 > 0$. A calculation shows that

$$\begin{gathered}
I_2 = \theta a_{22} = \theta(1 + O(\varepsilon)), \quad I_3 = O(\varepsilon\theta), \\
I_4 = 2\theta(a_{11} - \frac{1}{2} C_y) + O(\varepsilon\theta) = 2\theta(\mu + a_{11} + O(\varepsilon)).
\end{gathered}$$

Therefore since μ was chosen so that $\mu + a_{11} \geq 1$ in Ω, the desired conclusion follows if ε is sufficiently small.

We now show that

$$\iint_\Omega I_7 u_x^2 + 2 I_8 u_x u_y + I_9 u_y^2 \geq M_9 (\theta \|u_x\|^2 + \|E u_y\|^2),$$

where

$$\begin{gathered}
2I_7 = -2A a_{11} - (B a_{11})_x + 2B a_1 + (C a_{11})_y + \theta B_{xyy} + \theta A_{yy} - (D a_{11})_{yy}, \\
2I_8 = -(B a_{22})_y + B a_2 - (C a_{11})_x + C a_1 + \theta A_{xy} + (D a_{11})_{xy} - (D a_1)_y, \\
2I_9 = -2A a_{22} - (C a_{22})_y + 2C a_2 + \theta C_{xxy} + \theta A_{xx} \\
- (D a_{11})_{xx} - (D a_2)_y + (D a_1)_x - 2Da.
\end{gathered}$$

Again this will follow if $I_7 \geq M_{10} \theta$, $I_9 \geq M_{11} E^2$, and $I_7 I_9 - I_8^2 > 0$. A calculation shows that

$$\begin{aligned}
I_7 &= a_{11}^2 + \frac{1}{2} C \partial_y a_{11} + \theta(-\partial_{yy} a_{11} + \frac{1}{2} \gamma_x a_{11} + O(\varepsilon)), \\
I_8 &= -\frac{1}{2} C_x a_{11} - \frac{1}{2} C \partial_x a_{11} + \frac{1}{2} C a_1 + \frac{1}{2} B a_2 + O(\theta), \\
I_9 &= (a_{11} - C_y) a_{22} + C a_2 + O(\varepsilon + \theta) \\
&= (2\mu + a_{11} + O(\varepsilon))(1 + O(\varepsilon)) + C a_2 + O(\varepsilon + \theta).
\end{aligned}$$

Then $I_9 \geq M_{11}E^2$ immediately follows since $Ca_2 = O(\varepsilon)$ if $|y| \leq y_0$, $Ca_2 \geq 0$ if $|y| \geq y_0$, $Ca_2 = O(|y|^2)$ as $|y| \to \infty$, and $2\mu + a_{11} \geq 1$. To show that $I_7 \geq M_{10}\theta$, we consider the regions $|y| \leq y_0$ and $|y| \geq y_0$ separately. If $|y| \leq y_0$ then

$$C\partial_y a_{11} = \mu(\partial_y a_{11})^2 \geq 0, \quad -\partial_{yy} a_{11} = 2 + O(\varepsilon), \quad \gamma_x a_{11} \geq -O(\varepsilon),$$

so that here $I_7 \geq 2\theta + O(\varepsilon\theta)$. Furthermore, when $|y| \geq y_0$ we have $I_7 \geq y_0^4 + O(\theta)$ since

$$a_{11}^2 \geq y_0^4, \quad C\partial_y a_{11} \geq 0.$$

Finally, $I_7 I_9 - I_8^2 > 0$ follows from the next calculation. If $|y| \leq y_0$ then

$$\begin{aligned} I_7 I_9 - I_8^2 &\geq (a_{11}^2 + \frac{\mu}{2}(\partial_y a_{11})^2 + 2\theta + O(\varepsilon\theta))(1 + O(\varepsilon + \theta)) \\ &\quad - \frac{1}{4}O(\varepsilon^2)a_{11}^2 - \frac{1}{4}O(\varepsilon^2)(\partial_y a_{11})^2 - O(\varepsilon\theta + \theta^2), \end{aligned}$$

whereas if $|y| \geq y_0$ then

$$I_7 I_9 - I_8^2 \geq (y_0^4 + O(\theta))(1 + O(\delta y^2)) - O(\theta^2 y^2).$$

Lastly we deal with the term $2I_6 u_{xy} u_y$. Consider the quadratic form:

$$M_6 \theta u_{xy}^2 + 2I_6 u_{xy} u_y + M_9 E^2 u_y^2,$$

where $I_6 = -\frac{1}{2}Ba_{22}$. Since

$$(M_2\theta)(M_3 E^2) - I_6^2 \geq M_{11}\theta - M_{12}\theta^2(1 + O(\varepsilon))$$

for some M_{11}, M_{12}, we obtain

$$M_6 \theta u_{xy}^2 + 2I_6 u_{xy} u_y + M_9 E^2 u_y^2 \geq M_{13}(\theta u_{xy}^2 + E^2 u_y^2).$$

This completes the proof of the first estimate.

To obtain the second estimate we need only observe that the above arguments hold if $B \equiv 0$ and $u \in \widetilde{C}^\infty(\overline{\Omega})$ or $u \in \widetilde{C}_x^\infty(\overline{\Omega})$. Then an application of Cauchy's inequality ($ab \leq \lambda a^2 + \frac{1}{4\lambda}b^2$, $\lambda > 0$) yields the desired result. The reason for including B in the first estimate will soon become clear. $\square$

Having established the basic estimate, our goal shall now be to establish dual inequalities of the form:

$$\begin{aligned} \|v\| &\leq C_1 \|L_\theta^* v\|_{(-1,-2)} \quad \text{for all } v \in \widetilde{C}^\infty(\overline{\Omega}), \\ \|v\| &\leq C_2 \|L_\theta^* v\|_{(-1,-2)} \quad \text{for all } v \in \widetilde{C}_x^\infty(\overline{\Omega}). \end{aligned}$$

The existence of weak solutions to problems (2.3) and (2.4) will then easily follow from these two dual estimates, respectively. In order to establish the dual estimates, we will need the following lemma. Let P denote the differential operator

$$P = D\partial_y^2 + B\partial_x + C\partial_y + A,$$

where A, B, C, and D are defined in Lemma 2.3. Note that P is parabolic in Ω, away from the portion of the boundary, $x = x_0$. This is the reason for including B in the first estimate of Lemma 2.3.

Lemma 2.4. *For every $v \in \widetilde{C}^\infty(\overline{\Omega})$ there exists a unique solution $u \in C^\infty(\Omega) \cap H^4(\Omega) \subset C^\infty(\Omega) \cap C^2(\overline{\Omega})$ of*

$$Pu = v \quad \text{in } \Omega, \quad u(-x_0, y) = u_x(-x_0, y) = 0, \quad u(x_0, y) = 0.$$

Furthermore, for every $v \in \widetilde{C}^{\infty}_{x}(\overline{\Omega})$ *there exists a unique solution* $u \in C^{\infty}(\Omega) \cap H^4(\Omega) \subset C^{\infty}(\Omega) \cap C^2(\overline{\Omega})$ *of*

$$Pu = v \quad \text{in } \Omega, \quad u_x(-x_0, y) = 0, \quad u_x(x_0, y) = 0.$$

Proof. Let $\tau > 0$ be a small parameter, and define the subdomains

$$\Omega_\tau = \{(x, y) : -x_0 < x < x_0 - \tau\}.$$

Then P is parabolic in $\overline{\Omega}_\tau$ for each τ. We now consider the case when $v \in \widetilde{C}^{\infty}(\overline{\Omega})$. The parabolicity of P guarantees the existence (see [13]) of a unique solution to the Cauchy problem

$$Pu = v \quad \text{in } \Omega, \quad u(-x_0, y) = 0,$$

such that $u \in H^{\infty}(\Omega_\tau)$ for every τ. Furthermore, $u_x(-x_0, y) = 0$ since

$$Bu_x|_{(-x_0,y)} = Pu|_{(-x_0,y)} = v(-x_0, y) = 0.$$

We shall now show that $u \in H^4(\Omega)$. This will be accomplished by estimating the $H^4(\Omega_\tau)$ norm of u in terms of the $H^4(\Omega)$ norm of v, independent of τ. To facilitate the estimates, we first construct an appropriate approximating sequence $\{u^k\}_{k=1}^{\infty}$, for u. Define functions $\nu_k \in C^{\infty}(\mathbb{R})$ by

$$\nu_k(y) = \begin{cases} 1 & \text{if } |y| \le k, \\ 0 & \text{if } |y| \ge 3k, \end{cases} \tag{2.7}$$

such that $0 \le \nu_k \le 1$, $\sup |\nu_k'| \le \frac{1}{k}$, and $|\nu_k|_{C^4(\overline{\Omega})} \le M$ for some constant M independent of k. Let $u^k = \nu_k u$, then

(i) $u^k \in C^{\infty}(\overline{\Omega}_\tau)$ for all τ,
(ii) u^k has bounded support and $u^k(-x_0, y) = u^k_x(-x_0, y) = 0$,
(iii) $\|u - u^k\|_{4,\Omega_\tau} \to 0$ as $k \to \infty$,
(iv) $\|Cu_y - Cu^k_y\|_{\Omega_\tau} \to 0$ as $k \to \infty$,

where C was defined in Lemma 2.3. All of the above properties are evident except for (iv), and (iv) follows from the following calculation. Let

$$\Omega_\tau^{(k_1,k_2)} = \{(x, y) \in \Omega_\tau : k_1 \le |y| \le k_2\},$$

then

$$\begin{aligned} \|Cu_y - Cu^k_y\|^2_{\Omega_\tau} &\le \|C(u_y - \nu_k u_y)\|^2 + \|C\nu_k' u\|^2 \\ &\le \iint_{\Omega_\tau^{(k,\infty)}} C^2 u_y^2 + \iint_{\Omega_\tau^{(k,3k)}} (C\nu_k')^2 u^2 \\ &\le \iint_{\Omega_\tau^{(k,\infty)}} C^2 u_y^2 + \iint_{\Omega_\tau^{(k,3k)}} (6\mu k)^2 (\frac{1}{k})^2 u^2, \end{aligned}$$

where μ was defined in the proof of Lemma 2.3. By solving for Cu_y in the equation $Pu = v$, we have

$$Cu_y = v - Du_{yy} - Bu_x - Au \in L^2(\Omega_\tau).$$

Therefore

$$\iint_{\Omega_\tau^{(k,\infty)}} C^2 u_y^2 \to 0 \quad \text{as } k \to \infty.$$

Furthermore

$$\iint_{\Omega_\tau^{(k,3k)}} 36\mu^2 u^2 \to 0 \quad \text{as } k \to \infty$$

since $u \in L^2(\Omega_\tau)$. This proves (iv).

We now proceed to estimate the $H^4(\Omega_\tau)$ norm of u. Let $\zeta = \zeta(y) \in C^\infty(\mathbb{R})$ be such that $\zeta < 0$, $\zeta(y) = -|y|^{-1/2}$ if $|y| \geq y_1$, $\zeta'(y) \geq 0$ if $y \geq 0$, and $\zeta'(y) \leq 0$ if $y \leq 0$. Then set $\kappa = 2\sup|\zeta a_{11}|$, and integrate by parts to obtain

$$\begin{aligned}\iint_{\Omega_\tau}(\kappa u^k_{yy}+\zeta u^k)Pu^k &= \iint_{\Omega_\tau}[\kappa D](u^k_{yy})^2 + [-D\zeta + \kappa(\frac{1}{2}B_x - \frac{1}{2}C_y - A)](u^k_y)^2\\ &\quad + [\frac{1}{2}\kappa A_{yy} + \frac{1}{2}(D\zeta)_{yy} - \frac{1}{2}(B\zeta)_x - \frac{1}{2}(C\zeta)_y + \zeta A](u^k)^2\\ &\quad + \int_{\partial\Omega_\tau}[-\frac{1}{2}\kappa B n_1](u^k_y)^2 + [\frac{1}{2}B\zeta n_1](u^k)^2.\end{aligned}$$

The boundary integral is nonnegative since $u^k(-x_0,y) = u^k_y(-x_0,y) = 0$, and $-\kappa B n_1|_{(x_0-\tau,y)}, B\zeta n_1|_{(x_0-\tau,y)} > 0$. Also $\kappa D > 0$,

$$-D\zeta + \kappa(\frac{1}{2}B_x - \frac{1}{2}C_y - A) \geq \kappa(2\mu + a_{11} + O(\varepsilon+\theta)) \geq \kappa,$$

and

$$\begin{aligned}&\frac{1}{2}\kappa A_{yy} + \frac{1}{2}(D\zeta)_{yy} - \frac{1}{2}(B\zeta)_x - \frac{1}{2}(C\zeta)_y + \zeta A\\ &= -\frac{1}{2}\kappa\partial_{yy}a_{11} - \frac{1}{2}C\zeta_y - \zeta a_{11} + \frac{1}{2}(D\zeta)_{yy} - \frac{1}{2}(B\zeta)_x + O(\varepsilon)\\ &\geq \begin{cases}\kappa - \zeta a_{11} + O(\varepsilon+\theta) & \text{if } |y| \leq y_1,\\ |y|^{-1/2}[\frac{1}{2}\mu + a_{11} + O(\theta)] + O(\kappa\delta) & \text{if } y_1 \leq |y| \leq y_2,\\ |y|^{-1/2}[\frac{1}{2}\mu + a_{11} + O(\theta)] & \text{if } |y| \geq y_2.\end{cases}\end{aligned}$$

Therefore if ε, θ, and δ are sufficiently small, we may apply the Schwarz inequality followed by Cauchy's inequality to obtain

$$\|\sqrt{-\zeta}u^k\|_{\Omega_\tau} + \|u^k_y\|_{\Omega_\tau} + \|u^k_{yy}\|_{\Omega_\tau} \leq M_1\|Pu^k\|_{\Omega_\tau},$$

for some constant M_1 independent of τ. The properties of u^k guarantee that by letting $k \to \infty$, we obtain

$$\|\sqrt{-\zeta}u\|_{\Omega_\tau} + \|u_y\|_{\Omega_\tau} + \|u_{yy}\|_{\Omega_\tau} \leq M_1\|Pu\|_{\Omega_\tau} = M_1\|v\|_{\Omega_\tau} \leq M_1\|v\|.$$

We now estimate $\partial^\alpha_x\partial^\beta_y u$ for $\alpha = 1,\dots,4$, and $\beta = 0,1,2$. Differentiate $Pu = v$ with respect to x:

$$D(u_x)_{yy} + B(u_x)_x + C(u_x)_y + (A + B_x)u_x = v_x - C_x u_y - A_x u. \tag{2.8}$$

Since $u_x(-x_0,y) = 0$ and A_x, C_x vanish outside a compact set, we can apply the same procedure as above to obtain

$$\begin{aligned}\|\sqrt{-\zeta}u_x\|_{\Omega_\tau} + \|u_{xy}\|_{\Omega_\tau} + \|u_{xyy}\|_{\Omega_\tau} &\leq M_1\|v_x - C_x u_y - A_x u\|_{\Omega_\tau}\\ &\leq M_2(\|v_x\|_{\Omega_\tau} + \|u_y\|_{\Omega_\tau} + \|u\|_{\Omega_\tau})\\ &\leq M_3(\|v\| + \|v_x\|).\end{aligned}$$

Differentiating (2.8) with respect to x produces

$$\begin{aligned}&D(u_{xx})_{yy} + B(u_{xx})_x + C(u_{xx})_y + (A + 2B_x)u_{xx}\\ &= v_{xx} - \partial_x(C_x u_y + A_x u) - C_x u_{xy} - (A_x + B_{xx})u_x := v_1.\end{aligned}$$

Again we apply the same method. However since $u_{xx}(-x_0,y)=B^{-1}v_x|_{(-x_0,y)}$ from (2.8), we now have

$$\begin{aligned}\|\sqrt{-\zeta}u_{xx}\|_{\Omega_\tau}+\|u_{xxy}\|_{\Omega_\tau}+\|u_{xxyy}\|_{\Omega_\tau} &\leq M_1\|v_1\|_{\Omega_\tau}+M_4\\ &\leq M_5(\|v\|+\|v_x\|+\|v_{xx}\|)+M_4,\end{aligned}$$

where $M_4=\kappa|B|^{-1}(\int_{x=-x_0}v_{xy}^2+v_x^2)^{1/2}$ which is independent of τ. We can estimate $\|\sqrt{-\zeta}\partial_x^\alpha u\|_{\Omega_\tau}$, $\alpha=3,4$, and $\|\partial_x^\alpha\partial_y^\beta u\|_{\Omega_\tau}$, $\alpha=3,4$, $\beta=1,2$, in a similar manner.

To estimate u_{yyy}, differentiate $Pu=v$ with respect to y:

$$D(u_y)_{yy}+B(u_y)_x+C(u_y)_y+(A+C_y)u_y=v_y-A_yu. \tag{2.9}$$

Since $u_y(-x_0,y)=0$, $C_y<0$, and A_y vanishes outside a compact set, the same method as above yields

$$\begin{aligned}\|\sqrt{-\zeta}u_y\|_{\Omega_\tau}+\|u_{yy}\|_{\Omega_\tau}+\|u_{yyy}\|_{\Omega_\tau} &\leq M_1\|v_y-A_yu\|_{\Omega_\tau}\\ &\leq M_6(\|v\|+\|v_y\|).\end{aligned}$$

Furthermore, $\|u_{xyyy}\|_{\Omega_\tau}$ and $\|u_{yyyy}\|_{\Omega_\tau}$ can be estimated by differentiating (2.9) with respect to x and y, respectively.

The combination of all the above estimates produces,

$$\sum_{\alpha=0}^{4}\|\sqrt{-\zeta}\partial_x^\alpha u\|_{\Omega_\tau}+\sum_{\alpha+\beta\leq 4,\ \beta\neq 0}\|\partial_x^\alpha\partial_y^\beta u\|_{\Omega_\tau}\leq M_7\|v\|_4+M_8,$$

where M_7 and M_8 are independent of τ. Then letting $\tau\to 0$ we find that $\partial_x^\alpha\partial_y^\beta u\in L^2(\Omega)$, $\alpha+\beta\leq 4$, $\beta\neq 0$, and that $\sqrt{-\zeta}\partial_x^\alpha u\in L^2(\Omega)$, $\alpha=0,\dots,4$. It follows that $u\in H^4(K)$ for every compact $K\subset\Omega$, so that $u\in C^2(\overline{\Omega})$.

We now show that $\partial_x^\alpha u\in L^2(\Omega)$, $\alpha=0,\dots,4$. Let $\varrho_1,\varrho_2\in C^\infty(\mathbb{R})$ be given by

$$\varrho_1(x)=\begin{cases}-B+\theta & \text{if } -x_0\leq x\leq \frac{-x_0}{2},\\ 0 & \text{if } 0\leq x\leq x_0.\end{cases}\quad \varrho_2(y)=\begin{cases}-y & \text{if } |y|\leq y_0,\\ 0 & \text{if } |y|\geq T,\end{cases}$$

such that $\varrho_2(y)\leq 0$ if $y>0$ and $\varrho_2(y)\geq 0$ if $y<0$, where $T>0$ is large enough so that $-1\leq\varrho_2'\leq\varepsilon$. Then define $\overline{B}=B+\varrho_1$ and $\overline{C}=C+\varrho_2-\varepsilon\mu\partial_yb_{11}=-2\mu y+\varrho_2$, and set

$$\overline{P}=\overline{B}\partial_x+\overline{C}\partial_y+A.$$

If $w\in C_c^\infty(\overline{\Omega})$, then integrating by parts yields

$$(w,\overline{P}^*w)=\iint_\Omega[-\frac{1}{2}\overline{B}_x-\frac{1}{2}\overline{C}_y+A]w^2+\int_{\partial\Omega}[-\frac{1}{2}\overline{B}n_1]w^2.$$

The boundary integral is nonnegative since $\overline{B}(-x_0,y)=\theta$ and $\overline{B}(x_0,y)=0$. Furthermore

$$-\frac{1}{2}\overline{B}_x-\frac{1}{2}\overline{C}_y+A=-\varrho_2'-a_{11}+O(\varepsilon+\theta)\geq M_9,$$

for some constant $M_9>0$. Thus

$$\|w\|\leq M_{10}\|\overline{P}^*w\|. \tag{2.10}$$

Since $v-Du_{yy}+\varrho_1u_x+(\varrho_2-\varepsilon\mu\partial_yb_{11})u_y\in L^2(\Omega)$, (2.10) implies (see the proof of Theorem 2.6 below) the existence of a weak solution $\widetilde{u}\in L^2(\Omega)$ of

$$\overline{P}\widetilde{u}=v-Du_{yy}+\varrho_1u_x+(\varrho_2-\varepsilon\mu\partial_yb_{11})u_y,\quad \widetilde{u}(-x_0,y)=0.$$

We shall now show that $u \equiv \widetilde{u}$. Since $\overline{P}$ is a first order differential operator, we may apply Peyser's extension [21] of Friedrichs' result [2] on the identity of weak and strong solutions to obtain a sequence $\{\widetilde{u}^k\}_{k=1}^{\infty}$, such that $\widetilde{u}^k \in C^{\infty}(\overline{\Omega})$ has bounded support, satisfies $\widetilde{u}^k(-x_0, y) = 0$, and

$$\|\widetilde{u} - \widetilde{u}^k\| + \|\overline{P}\widetilde{u}^k - (v - Du_{yy} + \varrho_1 u_x + (\varrho_2 - \varepsilon\mu\partial_y b_{11})u_y)\| \to 0 \text{ as } k \to \infty.$$

Set $v^k = u - \widetilde{u}^k$. Using the fact that $|y|^{-1/4}v^k \to |y|^{-1/4}(u - \widetilde{u}) \in L^2(\Omega)$ and recalling the definition of $\overline{P}$, we have

$$\begin{aligned} |(-|y|^{-1/4}v^k, \overline{P}v^k)| &\le \||y|^{-1/4}v^k\|\|\overline{P}v^k\| \\ &\le M_{11}\|v - Du_{yy} + \varrho_1 u_x + (\varrho_2 - \varepsilon\mu\partial_y b_{11})u_y - \overline{P}\widetilde{u}^k\| \to 0. \end{aligned}$$

Then the following calculation shows that $\|u - \widetilde{u}^k\|_{L^2(K)} \to 0$ for every compact $K \subset \Omega$:

$$\begin{aligned} (-|y|^{-1/4}v^k, \overline{P}v^k) &= \lim_{t\to\infty} \iint_{\Omega^{(0,t)}} [\frac{1}{2}|y|^{-1/4}\overline{B}_x + \frac{1}{2}(|y|^{-1/4}\overline{C})_y - |y|^{-1/4}A](v^k)^2 \\ &\quad + \int_{\partial\Omega^{(0,t)}} [-\frac{1}{2}|y|^{-1/4}\overline{C}n_2 - \frac{1}{2}|y|^{-1/4}\overline{B}n_1](v^k)^2 \\ &\ge \lim_{t\to\infty} \iint_{\Omega^{(0,t)}} [|y|^{-1/4}(\frac{1}{4}\mu + a_{11} - \frac{1}{2} + O(\varepsilon + \theta))](v^k)^2 \\ &\ge M_{12}\||y|^{-1/8}v^k\|_K^2. \end{aligned}$$

Therefore, $u \equiv \widetilde{u}$ in $L^2(\Omega)$.

Differentiating the equation $Pu = v$ with respect to ∂_x^{α}, $\alpha = 1, \dots, 4$, and applying the above procedure shows that $\partial_x^{\alpha}u \in L^2(\Omega)$, $\alpha = 1, \dots, 4$. We now have that $u \in H^4(\Omega)$.

To complete the case when $v \in \widetilde{C}^{\infty}(\overline{\Omega})$, we must show that $u(x_0, y) = 0$. Since $B(x_0, y) = 0$, from the equation $Pu = v$ we find that

$$(Du_{yy} + Cu_y + Au)|_{(x_0,y)} = v(x_0, y) = 0.$$

Furthermore since $u \in H^4(\Omega)$, $u \to 0$ as $|y| \to \infty$. Therefore by applying the maximum principle to the above equation, we have $u(x_0, y) = 0$.

We now consider the case when $v \in \widetilde{C}_x^{\infty}(\overline{\Omega})$. Let $h(y) \in H^{\infty}(\mathbb{R})$ be the unique solution of the ODE:

$$D(-x_0, y)h'' + C(-x_0, y)h' + A(-x_0, y)h = v(-x_0, y).$$

Then as before, the parabolicity of P guarantees the existence of a unique solution to the Cauchy problem

$$Pu = v \quad \text{in } \Omega, \quad u(-x_0, y) = h(y),$$

such that $u \in H^{\infty}(\Omega_{\tau})$ for every τ. Furthermore, $u_x(-x_0, y) = 0$ since

$$Bu_x|_{(-x_0,y)} = v(-x_0, y) - (Du_{yy} + Cu_y + Au)|_{(-x_0,y)} = 0.$$

Moreover, the same methods used above can be used here to show that $u \in H^4(\Omega)$. Lastly to show that $u_x(x_0, y) = 0$, differentiate $Pu = v$ with respect to x and use that $B(x_0, y) = 0$ to obtain

$$(D(u_x)_{yy} + C(u_x)_y + (A + B_x)u_x)|_{(x_0,y)} = v_x(x_0, y) - (C_x u_y + A_x u)|_{(x_0,y)} = 0.$$

Since $u_x \to 0$ as $|y| \to \infty$, by the maximum principle $u_x(x_0, y) = 0$. □

With Lemma 2.4 we are now in a position to establish the dual inequalities.

Proposition 2.5. *There exist constants M_1, M_2 such that:*

$$\|v\| \leq M_1\|L_\theta^* v\|_{(-1,-2)} \text{ for all } v \in \widetilde{C}^\infty(\overline{\Omega}),$$
$$\|v\| \leq M_2\|L_\theta^* v\|_{(-1,-2)} \text{ for all } v \in \widetilde{C}_x^\infty(\overline{\Omega}).$$

Proof. We first consider the case when $v \in \widetilde{C}^\infty(\overline{\Omega})$. Let $u \in C^\infty(\Omega) \cap H^4(\Omega)$ be the unique solution of

$$Pu = v \quad \text{in } \Omega, \quad u(-x_0, y) = u_x(-x_0, y) = 0, \quad u(x_0, y) = 0,$$

given by Lemma 2.4. We now show that

$$\begin{aligned}&(Au + Bu_x + Cu_y + Du_{yy}, L_\theta u)\\ &\geq C_1[\|u\|^2 + \|Eu_y\|^2 + \theta(\|u_x\|^2 + \|u_{xy}\|^2 + \|u_{yy}\|^2 + \theta\|u_{xyy}\|^2)],\end{aligned}$$

where A, B, C, D, E, and C_1 were given in Lemma 2.3. Let ν_k be given by (2.7) and define the sequence $\{u^k\}_{k=1}^\infty$, where $u^k = \nu_k u$. Then as in the proof of Lemma 2.4 we have:

(i) $u^k \in C^\infty(\Omega) \cap H^4(\Omega)$,
(ii) u^k has bounded support and $u_x^k(-x_0, y) = 0$, $u^k(x_0, y) = 0$,
(iii) $\|u - u^k\|_4 \to 0$ as $k \to \infty$,
(iv) $\|Eu_y - Eu_y^k\| \to 0$ as $k \to \infty$.

Let $\{u_k\}_{k=1}^\infty$ a C^∞ approximation of $\{u^k\}_{k=1}^\infty$ such that:

(i) $u_k \in C^\infty(\overline{\Omega})$,
(ii) u_k has bounded support and $(u_k)_x(-x_0, y) = 0$, $u_k(x_0, y) = 0$,
(iii) $\|u^k - u_k\|_4 \to 0$ as $k \to \infty$,
(iv) $\|Eu_y^k - E(u_k)_y\| \to 0$ as $k \to \infty$.

Then applying Lemma 2.3 we have

$$\begin{aligned}&(Au + Bu_x + Cu_y + Du_{yy}, L_\theta u)\\ &= \lim_{k\to\infty}(Au_k + B(u_k)_x + C(u_k)_y + D(u_k)_{yy}, L_\theta u_k)\\ &\geq \lim_{k\to\infty} C_1[\|u_k\|^2 + \|E(u_k)_y\|^2 + \theta(\|(u_k)_x\|^2\\ &\quad + \|(u_k)_{xy}\|^2 + \|(u_k)_{yy}\|^2 + \theta\|(u_k)_{xyy}\|^2)]\\ &= C_1[\|u\|^2 + \|Eu_y\|^2 + \theta(\|u_x\|^2 + \|u_{xy}\|^2 + \|u_{yy}\|^2 + \theta\|u_{xyy}\|^2)].\end{aligned}$$

By the above estimate and definition of the negative norms, it follows that

$$\begin{aligned}\|L_\theta^* v\|_{(-1,-2)}\|u\|_{(1,2)} &\geq (L_\theta^* v, u)\\ &= (v, L_\theta u)\\ &= (Au + Bu_x + Cu_y + Du_{yy}, L_\theta u)\\ &\geq C_1[\|u\|^2 + \|Eu_y\|^2 + \theta(\|u_x\|^2 + \|u_{xy}\|^2 + \|u_{yy}\|^2\\ &\quad + \theta\|u_{xyy}\|^2)].\end{aligned}$$

Furthermore using Cauchy's inequality and the equation $Pu = v$, we obtain

$$\begin{aligned}\|L_\theta^* v\|_{(-1,-2)} &\geq C_1'[\|u\| + \|Eu_y\| + \sqrt{\theta}(\|u_x\| + \|u_{xy}\| + \|u_{yy}\| + \sqrt{\theta}\|u_{xyy}\|)]\\ &\geq M_1^{-1}\|v\|,\end{aligned}$$

for some constants $C_1', M_1 > 0$. Moreover, similar arguments may be used to treat the case when $v \in \widetilde{C}_x^\infty(\overline{\Omega})$. □

The existence of weak solutions to problems (2.3) and (2.4) immediately follows from Proposition 2.5 by a standard functional analytic argument. We include the proof here for convenience.

Theorem 2.6. *For each $f \in L^2(\Omega)$ there exists a weak solution $u \in \widetilde{H}^{(1,2)}(\Omega)$, $\widetilde{H}_x^{(1,2)}(\Omega)$ of (2.3), (2.4) respectively.*

Proof. We shall first treat problem (2.3). Let $W = L_\theta^*(\widetilde{C}^\infty(\overline{\Omega}))$ and define the linear functional $F : W \to \mathbb{R}$ by

$$F(L_\theta^* v) = (f, v).$$

Using Proposition 2.5, the following calculation will show that F is bounded as a linear functional on the subspace W of $\widetilde{H}^{(-1,-2)}(\Omega)$,

$$|F(L_\theta^* v)| = |(f, v)| \le \|f\|\|v\| \le M_1 \|f\| \|L_\theta^* v\|_{(-1,-2)}.$$

Use the Hahn-Banach theorem to extend F from W to the whole space $\widetilde{H}^{(-1,-2)}(\Omega)$. It follows from the Riesz representation theorem that there exists $u \in \widetilde{H}^{(1,2)}(\Omega)$ such that

$$F(w) = (u, w) \text{ for all } w \in \widetilde{H}^{(-1,-2)}(\Omega).$$

Thus, restricting w to W we have

$$(u, L_\theta^* v) = F(L_\theta^* v) = (f, v) \text{ for all } v \in \widetilde{C}^\infty(\overline{\Omega}).$$

The case of problem (2.4) may be treated in a similar manner. □

We now prove the existence of weak solutions for the adjoint problems (2.5) and (2.6). The existence of solutions for these problems will be needed in the next section, where they will aid in proving higher regularity for solutions of (2.3).

The formal adjoint of L_θ is given by

$$\begin{aligned} L_\theta^* &= -\theta \partial_{xxyy} + a_{11}\partial_{xx} + a_{22}\partial_{yy} + (2\partial_x a_{11} - a_1)\partial_x \\ &\quad + (2\partial_y a_{22} - a_2)\partial_y + (\partial_{xx} a_{11} + \partial_{yy} a_{22} - \partial_x a_1 - \partial_y a_2 + a). \end{aligned}$$

All the coefficients of L_θ^*, denoted a_{ij}^*, a_i^*, a^*, have the same properties as the coefficients of L_θ, except $a_2^* = 2\partial_y a_{22} - a_2$. This difference will not allow us to directly apply the above procedure to obtain weak solutions for (2.5) and (2.6). However if

$$h(x, y) = e^{2\int_0^y \frac{a_2(x,t)}{a_{22}(x,t)}\,dt},$$

then by setting $v = hw$, the equation $L_\theta^* v = g$ becomes $\overline{L}_\theta^* w = g/h$, where

$$\begin{aligned} \overline{L}_\theta^* &= -\theta\partial_{xxyy} - 2\theta \frac{h_y}{h}\partial_{xxy} - 2\theta\frac{h_x}{h}\partial_{xyy} \\ &\quad + (a_{11}^* - \theta\frac{h_{yy}}{h})\partial_{xx} - 4\theta\frac{h_{xy}}{h}\partial_{xy} + (a_{22}^* - \theta\frac{h_{xx}}{h})\partial_{yy} \\ &\quad + (a_2^* + 2a_{22}^*\frac{h_y}{h} - 2\theta\frac{h_{xxy}}{h})\partial_y + (a_1^* + 2a_{11}^*\frac{h_x}{h} - 2\theta\frac{h_{xyy}}{h})\partial_x \\ &\quad + (a_{11}^*\frac{h_{xx}}{h} + a_{22}^*\frac{h_{yy}}{h} + a_1^*\frac{h_x}{h} + a_2^*\frac{h_y}{h} + a^* - \theta\frac{h_{xxyy}}{h}). \end{aligned}$$

The special choice of h guarantees that the coefficient of ∂_y in $\overline{L}_\theta^*$ is $3a_2 + O(\varepsilon + \theta)$, so that all the coefficients of $\overline{L}_\theta^*$ have the same properties as the coefficients of $\overline{L}_\theta$, where $\overline{L}_\theta w = f/h$ is the equation obtained from $L_\theta u = f$ by setting $u = hw$. Therefore if $g \in L^2(\Omega)$, the problems

$$\overline{L}_\theta^* w = g/h \quad \text{in } \Omega, \quad w|_{\partial\Omega} = 0,$$
$$\overline{L}_\theta^* w = g/h \quad \text{in } \Omega, \quad w_x|_{\partial\Omega} = 0,$$

have weak solutions of the form $w = v/h$, where $v \in \widetilde{H}^{(1,2)}(\Omega)$, $\widetilde{H}_x^{(1,2)}(\Omega)$ respectively. We then obtain the following result.

Corollary 2.7. *For each $g \in L^2(\Omega)$ there exists a weak solution $v \in \widetilde{H}^{(1,2)}(\Omega)$, $\widetilde{H}_x^{(1,2)}(\Omega)$ of (2.5), (2.6) respectively.*

3. Linear Regularity

The purpose of this section is to establish the regularity in X, of weak solutions to problem (2.3) for a particular choice of the right-hand side, f. This shall be accomplished by establishing the uniqueness of weak solutions to problems (2.3) and (2.4) in $L^2(\Omega)$, and then applying a boot-strap argument.

To obtain the uniqueness of weak solutions, we will utilize the notion of a strong solution, in particular, for first order systems. The definition of a strong solution will be given below. We first introduce the notation and terminology that will be used for first order systems. Consider a boundary value problem

$$SU = A_1 U_x + A_2 U_y + A_3 U = F \quad \text{in } \Omega, \quad U|_{\partial\Omega} \in N, \tag{3.1}$$

where A_1, A_2, A_3 are $n \times n$ matrices, U and F are n-vectors, and N is a linear subspace of the space of n-vector valued functions restricted to $\partial\Omega$. The corresponding adjoint problem is given by

$$S^* V = -A_1^* V_x - A_2^* V_y + (A_3^* - \partial_x A_1^* - \partial_y A_2^*) V = G \quad \text{in } \Omega, \quad V|_{\partial\Omega} \in N^*,$$

where A_i^* denotes the transpose of A_i, and N^* is the orthogonal complement of $\triangle N$, where $\triangle$ is the matrix defined on $\partial\Omega$ by $A_1 n_1 + A_2 n_2$, and (n_1, n_2) is the unit outward normal to $\partial\Omega$.

Let $F \in L^2(\Omega)$. The notion of a weak solution to problem (3.1) is similar to the definition given in section §2 for single equations. That is, $U \in L^2(\Omega)$ is said to be a weak solution of (3.1) whenever

$$(S^* V, U) = (V, F),$$

for every $V \in C^\infty(\overline{\Omega})$ with bounded support and such that $V|_{\partial\Omega} \in N^*$. We now give the definition of a strong solution.

Definition 3.1. *$U \in L^2(\Omega)$ is a strong solution of (3.1) if there exists a sequence $\{U_k\}_{k=1}^\infty$, such that $U_k \in C^\infty(\overline{\Omega})$ with bounded support, $U_k|_{\partial\Omega} \in N$, and*

$$\|U_k - U\| \to 0, \quad \|SU_k - F\| \to 0, \quad \text{as } k \to \infty.$$

Clearly a strong solution is a weak solution. Moreover, using techniques developed by Friedrichs [2] and Lax/Phillips [14], Peyser [21] has obtained the following converse statement.

Theorem 3.2 (Identity of Weak and Strong Solutions)**.** *Let the following conditions on the operator S and the boundary space N be satisfied:*

(i) *The matrix $\triangle$ is of constant rank in a neighborhood of the boundary,*
(ii) *N is of constant dimension at each point of the boundary,*
(iii) *N contains the nullspace of $\triangle$.*

Then a weak solution $U \in L^2(\Omega)$ of (3.1) *is also a strong solution.*

Note that for our particular domain $\triangle = A_1 n_1$, so that condition (i) is equivalent to A_1 having constant rank in a neighborhood of $\partial\Omega$.

With the aim of applying Theorem 3.2, we shall transform problems (2.3), (2.4), (2.5), and (2.6) into the setting of first order systems. Let $f, g \in L^2(\Omega)$ be the right-hand sides of (2.3), (2.4) and (2.5), (2.6) respectively, and define A_1, $\widetilde{A}_1$, A_2, $\widetilde{A}_2$, A_3, $\widetilde{A}_3$, F, and G by

$$A_1 = \widetilde{A}_1 = \begin{pmatrix} -\theta & 0 & a_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \widetilde{A}_2 = \begin{pmatrix} 0 & 0 & 0 & a_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 0 & a_1 & a_2 & a \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \widetilde{A}_3 = \begin{pmatrix} 0 & 0 & a_1^* & a_2^* & a^* \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$F = \begin{pmatrix} f \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad G = \begin{pmatrix} g \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Define boundary spaces

$$N_1 = \{(u_1, \dots, u_5)|_{\partial\Omega} : u_5|_{\partial\Omega} = 0\},$$
$$N_2 = \{(u_1, \dots, u_5)|_{\partial\Omega} : (-\theta u_1 + a_{11} u_3)|_{\partial\Omega} = 0\}.$$

Furthermore define boundary value problems

$$S_\theta U = A_1 U_x + A_2 U_y + A_3 U = F \quad \text{in } \Omega, \quad U|_{\partial\Omega} \in N_1, \tag{3.2}$$
$$S_\theta U = F \quad \text{in } \Omega, \quad U|_{\partial\Omega} \in N_2, \tag{3.3}$$
$$\widetilde{S}_\theta V = \widetilde{A}_1 V_x + \widetilde{A}_2 V_y + \widetilde{A}_3 V = G \quad \text{in } \Omega, \quad V|_{\partial\Omega} \in N_1, \tag{3.4}$$
$$\widetilde{S}_\theta V = G \quad \text{in } \Omega, \quad V|_{\partial\Omega} \in N_2. \tag{3.5}$$

We now show that the weak solutions of (2.3), (2.4), (2.5), and (2.6) given by Theorem 2.6 and Corollary 2.7 are also weak solutions of (3.2), (3.3), (3.4), and (3.5) respectively.

Lemma 3.3. *Let $u \in \widetilde{H}^{(1,2)}(\Omega)$, $\widetilde{H}_x^{(1,2)}(\Omega)$ be a weak solution of* (2.3), (2.4) *respectively, then $U = (u_{xyy}, u_{yy}, u_x, u_y, u) \in L^2(\Omega)$ is a weak solution of* (3.2), (3.3) *respectively. Similarly if $v \in \widetilde{H}^{(1,2)}(\Omega)$, $\widetilde{H}_x^{(1,2)}(\Omega)$ is a weak solution of* (2.5), (2.6) *respectively, then $V = (v_{xyy}, v_{yy}, v_x, v_y, v) \in L^2(\Omega)$ is a weak solution of* (3.4), (3.5) *respectively.*

Proof. Let $u \in \widetilde{H}^{(1,2)}(\Omega)$ be a weak solution of problem (2.3). We will show that

$$\iint_\Omega U^* S_\theta^* V = \iint_\Omega F^* V \tag{3.6}$$

for all $V \in C^\infty(\overline{\Omega})$ with bounded support such that $V|_{\partial\Omega} \in N_1^*$, where

$$N_1^* = \{(v_1, \dots, v_5)|_{\partial\Omega} : v_1|_{\partial\Omega} = v_5|_{\partial\Omega} = 0\}.$$

A calculation shows that

$$\begin{aligned}\iint_\Omega U^* S_\theta^* V = \iint_\Omega &(\theta u_{xyy} - a_{11}u_x)\partial_x v_1 - a_{22}u_y\partial_y v_1 - (u\partial_x v_2 + u_x v_2)\\ &+ [(a_1 - \partial_x a_{11})u_x + (a_2 - \partial_y a_{22})u_y + au]v_1\\ &- (u\partial_y v_3 + u_y v_3) - (u_y\partial_y v_4 + u_{yy}v_4) - (u_{yy}\partial_x v_5 + u_{xyy}v_5).\end{aligned} \tag{3.7}$$

Since $V|_{\partial\Omega} \in N_1^*$ and $u \in \widetilde{H}^{(1,2)}(\Omega)$ is a weak solution of (2.3), we can integrate by parts to obtain

$$\iint_\Omega U^* S_\theta^* V = \iint_\Omega u L_\theta^* v_1 = \iint_\Omega f v_1 = \iint_\Omega F^* V,$$

showing that U is a weak solution of (3.2).

Let $u \in \widetilde{H}_x^{(1,2)}(\Omega)$ be a weak solution of (2.4). We now show that (3.6) holds for all $V \in C^\infty(\overline{\Omega})$ with bounded support such that $V|_{\partial\Omega} \in N_2^*$, where

$$N_2^* = \{(v_1, \dots, v_5)|_{\partial\Omega} : v_2|_{\partial\Omega} = v_5|_{\partial\Omega} = 0\}.$$

From (3.7) it follows that

$$\begin{aligned}\iint_\Omega U^* S_\theta^* V = \iint_\Omega &(\theta u_{xyy} - a_{11}u_x)\partial_x v_1 - a_{22}u_y\partial_y v_1\\ &+ [(a_1 - \partial_x a_{11})u_x + (a_2 - \partial_y a_{22})u_y + au]v_1.\end{aligned} \tag{3.8}$$

To integrate by parts we construct an approximating sequence $\{v_1^k\}_{k=1}^\infty$ for v_1, such that $v_1^k \in \widetilde{C}_x^\infty(\overline{\Omega})$ and

$$\|v_1^k - v_1\| + \|\partial_x v_1^k - v_1\| \to 0 \quad \text{as } k \to \infty.$$

Take a sequence $\{v_k\}_{k=1}^\infty \subset \widetilde{C}^\infty(\overline{\Omega})$ with the property that $\|v_k - \partial_x v_1\| \to 0$ as $k \to \infty$, and define

$$v_1^k = \int_{-x_0}^x v_k(t, y)dt + v_1(-x_0, y).$$

Then since

$$\begin{aligned}(v_1^k - v_1)^2 &= \Big(\int_{-x_0}^x \partial_t(v_1^k(t, y) - v_1(t, y))dt\Big)^2\\ &\le 2x_0 \int_{-x_0}^x (\partial_t v_1^k(t, y) - \partial_t v_1(t, y))^2 dt,\end{aligned}$$

we have

$$\iint_\Omega (v_1^k - v_1)^2 \le 4x_0^2 \iint_\Omega (\partial_x v_1^k - \partial_x v_1)^2 = 4x_0^2 \iint_\Omega (v_k - \partial_x v_1)^2 \to 0,$$

so that v_1^k satisfies the desired properties. Recalling that $a_1|_{\partial\Omega} = \partial_x a_{11}|_{\partial\Omega} = 0$ by (ii) of Lemma 2.2, and using the fact that u is a weak solution of (2.4), we can integrate by parts in (3.8) to obtain

$$\iint_\Omega U^* S_\theta^* V = \lim_{k\to\infty} \iint_\Omega u L_\theta^* v_1^k = \lim_{k\to\infty} \iint_\Omega f v_1^k = \iint_\Omega f v_1 = \iint_\Omega F^* V,$$

showing that U is a weak solution of (3.3). Similar arguments show that if $v \in \widetilde{H}^{(1,2)}(\Omega)$, $\widetilde{H}_x^{(1,2)}(\Omega)$ is a weak solution of (2.5), (2.6) respectively, then $V = (v_{xyy}, v_{yy}, v_x, v_y, v) \in L^2(\Omega)$ is a weak solution of (3.4), (3.5) respectively. □

Now that the weak solutions of the previous section have been translated into the setting of first order systems, Theorem 3.2 is applicable. As a result, we obtain

Proposition 3.4. *The weak solutions of problems* (2.3) *and* (2.4)*, given by Theorem 2.6, are unique in* $L^2(\Omega)$.

Proof. Let $u \in \widetilde{H}^{(1,2)}(\Omega)$ be a weak solution of problem (2.3) with $f = 0$, then

$$(L_\theta^* w, u) = 0 \quad \text{for all } w \in \widetilde{C}^\infty(\overline{\Omega}). \tag{3.9}$$

We will show that $u = 0$ in $L^2(\Omega)$.

Let $v \in \widetilde{H}^{(1,2)}(\Omega)$ be the weak solution of (2.5) with $g = u$. Then by Lemma 3.3 $V = (v_{xyy}, v_{yy}, v_x, v_y, v)$ is a weak solution of (3.4). We now show that the conditions of Theorem 3.2 are satisfied for problem (3.4). Condition (ii) is immediately satisfied, and since $a_{11}^* \le -\theta$ in a neighborhood of $\partial\Omega$, condition (i) is satisfied with $\triangle = \pm\widetilde{A}_1$ having the constant rank of 3. Furthermore the nullspace of $\triangle$ is given by

$$\{(v_1, \dots, v_5)|_{\partial\Omega} \mid (-\theta v_1 + a_{11} v_3)|_{\partial\Omega} = v_2|_{\partial\Omega} = v_5|_{\partial\Omega} = 0\},$$

which is contained in N_1 so that condition (iii) is satisfied. Therefore we can apply Theorem 3.2 to obtain an approximating sequence $\{V_k\}_{k=1}^\infty$ for V, such that $V_k \in C^\infty(\overline{\Omega})$ with bounded support, $V_k|_{\partial\Omega} \in N_1$, and

$$\|V_k - V\| \to 0, \quad \|\widetilde{S}_\theta V_k - G\| \to 0 \quad \text{as } k \to \infty. \tag{3.10}$$

From (3.10) it follows that

$$\begin{gathered}
\|v_k^1 - v_{xyy}\| \to 0, \quad \|v_k^2 - v_{yy}\| \to 0, \quad \|v_k^3 - v_x\| \to 0, \\
\|v_k^4 - v_y\| \to 0, \quad \|v_k^5 - v\| \to 0, \\
\|(-\theta\partial_x v_k^1 + a_{11}^*\partial_x v_k^3 + a_{22}^*\partial_y v_k^4 + a_1^* v_k^3 + a_2^* v_k^4 + a^* v_k^5) - u\| \to 0.
\end{gathered}$$

Hence

$$\begin{aligned}
(u,u) &= \lim_{k\to\infty} \iint_\Omega [-\theta\partial_x v_k^1 + a_{11}^*\partial_x v_k^3 + a_{22}^*\partial_y v_k^4 + a_1^* v_k^3 + a_2^* v_k^4 + a^* v_k^5]u \\
&= \lim_{k\to\infty} \iint_\Omega (\theta v_k^1 - a_{11}^* v_k^3)u_x - a_{22}^* v_k^4 u_y + [(a_1^* - \partial_x a_{11}^*)v_k^3 \\
&\quad + (a_2^* - \partial_y a_{22}^*)v_k^4 + a^* v_k^5]u \\
&= \iint_\Omega (\theta v_{xyy} - a_{11}^* v_x)u_x - a_{22}^* v_y u_y + [(a_1^* - \partial_x a_{11}^*)v_x \\
&\quad + (a_2^* - \partial_y a_{22}^*)v_y + a^* v]u.
\end{aligned}$$

Let $\{v_n\}_{n=1}^{\infty} \subset \widetilde{C}^{\infty}(\overline{\Omega})$ be an approximating sequence for v in $\widetilde{H}^{(1,2)}(\Omega)$. Then integrating by parts and using (3.9), we obtain

$$(u,u) = \lim_{n\to\infty} \iint_{\Omega} (L_\theta^* v_n)u = 0.$$

Similar arguments hold for problem (2.4) □

Having established the uniqueness of weak solutions, we are now ready to apply a boot-strap procedure to obtain higher regularity for problem (2.3) in the x-direction.

Theorem 3.5. *Let u and f be as in problem (2.3). Let $s \le r-4$ and $f \in H^s(\Omega)$ be such that $\partial_x^\alpha f|_{\partial\Omega} = 0$ for $\alpha \le s-1$. If $\varepsilon = \varepsilon(s)$ is sufficiently small, then for all $\alpha \le s$, $\partial_x^\alpha u \in \widetilde{H}^{(1,2)}(\Omega)$ when α is even, and $\partial_x^\alpha u \in \widetilde{H}_x^{(1,2)}(\Omega)$ when α is odd.*

Proof. The case $s=0$ is given by Theorem 2.6. Consider the case $s=1$. Let $w = u_x$ and formally differentiate the equation $L_\theta u = f$ with respect to x:

$$\begin{aligned} L_1 w &:= -\theta w_{xxyy} + a_{11}w_{xx} + a_{22}w_{yy} + (a_1 + \partial_x a_{11})w_x + a_2 w_y + (a + \partial_x a_1)w \\ &= f_x - u_{yy}\partial_x a_{22} - u_y \partial_x a_2 - u\partial_x a := f_1. \end{aligned}$$

Observe that since $\partial_x a_{11}, \partial_x a_1 = O(\varepsilon)$ and both vanish outside X, the operator L_1 has the same existence and uniqueness properties as L_θ. Furthermore, by restricting $L_\theta u = f$ to the boundary of Ω and using $u|_{\partial\Omega} = a_1|_{\partial\Omega} = 0$, we obtain the following ODE

$$(-\theta u_{xxyy} + a_{11}u_{xx})|_{\partial\Omega} = 0, \tag{3.11}$$

for which the only solution in $L^2(\partial\Omega)$ is $u_{xx}|_{\partial\Omega} = 0$. Therefore, in the regular case $w = u_x$ satisfies problem (2.4) with L_θ and f replaced by L_1 and f_1.

Let $u \in \widetilde{H}^{(1,2)}(\Omega)$ be the weak solution of problem (2.3). We now show that $u_x \in L^2(\Omega)$ is a weak solution of (2.4) with L_θ and f replaced by L_1 and $f_1 \in L^2(\Omega)$; we denote this problem by $(2.4)_1$. Let $v \in \widetilde{C}_x^{\infty}(\overline{\Omega})$, then

$$\begin{aligned} &(u_x, L_1^* v) \\ &= -(u, (L_1^* v)_x) = -(u, L^*(v_x)) + (u, L^*(v_x) - (L_1^* v)_x) \\ &= -(f, v_x) + (u, -v_{yy}\partial_x a_{22} + v_y[\partial_x a_2 - 2\partial_{xy}a_{22}] + v[-\partial_x a - \partial_{xyy}a_{22} + \partial_{xy}a_2]) \\ &= (f_x, v) + (-u_{yy}\partial_x a_{22} - u_y\partial_x a_2 - u\partial_x a, v) = (f_1, v). \end{aligned}$$

Therefore u_x is a weak solution of $(2.4)_1$, and by the uniqueness result Proposition 3.4, u_x must coincide with the solution in $\widetilde{H}_x^{(1,2)}(\Omega)$ given by Theorem 2.6. Hence $u_x \in \widetilde{H}_x^{(1,2)}(\Omega)$.

We now consider the case $s=2$. Let $w = u_{xx}$ and formally differentiate the equation $L_1 u_x = f_1$ with respect to x:

$$\begin{aligned} L_2 w &:= -\theta w_{xxyy} + a_{11}w_{xx} + a_{22}w_{yy} \\ &\quad + (a_1 + 2\partial_x a_{11})w_x + a_2 w_y + (a + 2\partial_x a_1 + \partial_{xx}a_{11})w \\ &= \partial_x f_1 - u_{xyy}\partial_x a_{22} - u_{xy}\partial_x a_2 - u_x(\partial_x a + \partial_{xx}a_1) := f_2. \end{aligned}$$

Again since $\partial_x a_{11}, \partial_{xx}a_{11}, \partial_x a_1 = O(\varepsilon)$ and all three vanish outside X, the operator L_2 has the same existence and uniqueness properties as L_θ, provided that ε is sufficiently small. Also, when u is regular $u_{xx}|_{\partial\Omega} = 0$ from (3.11). Thus in the regular case $w = u_{xx}$ satisfies (2.3) with L_θ and f replaced by L_2 and $f_2 \in L^2(\Omega)$; we denote this problem by $(2.3)_2$.

Let $u \in \widetilde{H}^{(1,2)}(\Omega)$ be the weak solution of (2.3), then we know that $u_x \in \widetilde{H}_x^{(1,2)}(\Omega)$. We now show that $u_{xx} \in L^2(\Omega)$ is a weak solution of $(2.3)_2$. Note that $L_\theta u = f$ in $L^2(\Omega)$ and let $v \in \widetilde{C}^\infty(\overline{\Omega})$, then a calculation produces

$$\begin{aligned}(u_{xx}, L_2^* v) &= (u_{xxyy}, -\theta v_{xx}) + (u_{xx}, (a_{11}v)_{xx}) + (u_{yy}, (a_{22}v)_{xx}) + (u_y, (a_2 v)_{xx}) \\ &\quad + (u_x, [(a_1 + 2\partial_x a_{11})v]_{xx}) + (u, [(a + 2\partial_x a_1 + \partial_{xx} a_{11})v]_{xx}) \\ &= (L_\theta u, v_{xx}) + (f_2 - f_{xx}, v) = (f, v_{xx}) + (f_2 - f_{xx}, v) = (f_2, v).\end{aligned}$$

By the uniqueness of weak solutions for problem $(2.3)_2$, u_{xx} must coincide with the solution in $\widetilde{H}^{(1,2)}(\Omega)$. Thus $u_{xx} \in \widetilde{H}^{(1,2)}(\Omega)$.

To obtain the regularity of higher order derivatives, we observe that the above procedure applied to $L_\theta u = f$ holds for $L_2 u_{xx} = f_2$, since for $\alpha \geq 1$

$$\partial_x^\alpha a_{11}|_{\partial\Omega} = \partial_x^\alpha a_{22}|_{\partial\Omega} = \partial_x^\alpha a_i|_{\partial\Omega} = \partial_x^\alpha a|_{\partial\Omega} = 0,$$

so that $f_2|_{\partial\Omega} = 0$. Therefore $u_{xxx} \in \widetilde{H}_x^{(1,2)}(\Omega)$ and $u_{xxxx} \in \widetilde{H}^{(1,2)}(\Omega)$. Furthermore, we can continue this process until f and the coefficients of L_θ run out of derivatives, as long as ε is chosen sufficiently small depending on the size of s. □

We now prove regularity in the y-direction for the weak solution of problem (2.3). The following standard lemma concerning difference quotients will be needed.

Lemma 3.6. *Let $w \in L^2(\Omega)$ have bounded support, and define*

$$w^h = \frac{1}{h}(w(x, y+h) - w(x, y)).$$

If $\|w^h\| \leq M$ where M is independent of h, then $w \in H^{(0,1)}(\Gamma)$ for any compact $\Gamma \subset \Omega$. Furthermore, if $w \in H^{(0,1)}(\Omega)$ then $\|w^h\| \leq M\|w_y\|$.

Theorem 3.7. *Let the hypotheses of Theorem 3.5 hold, then $u \in H^s(X)$.*

Proof. From Theorem 3.5 we know that $\partial_x^\alpha u \in H^{(1,2)}(\Omega)$ for $0 \leq \alpha \leq s$. Therefore the following equality holds in $L^2(\Omega)$,

$$\widetilde{L} u_{yy} := -\theta u_{xxyy} + a_{22} u_{yy} = f - a_{11} u_{xx} - a_1 u_x - a_2 u_y - au := \widetilde{f}. \tag{3.12}$$

Since $|a_2| = O(|y|)$ as $|y| \to \infty$, we do not necessarily know that $\widetilde{f} \in H^{(0,1)}(\Omega)$; however, we do have $\widetilde{f} \in H^{(0,1)}(\Gamma)$ for any compact $\Gamma \subset \Omega$. Fix a constant $k > y_0$ and set $w = \nu_k u_{yy}$, where ν_k is given by (2.7). Then

$$\widetilde{L} w^h = (\nu_k \widetilde{f})^h - \nu_k(y+h) u_{yy}(x, y+h) a_{22}^h. \tag{3.13}$$

Since $u \in \widetilde{H}^{(1,2)}(\Omega)$, by multiplying (3.13) on both sides by w^h and integrating by parts we obtain

$$\|w^h\| + \|w_x^h\| \leq M_1(\|(\nu_k \widetilde{f})^h\| + 1),$$

for some M_1 independent of h. By Lemma 3.6

$$\|w^h\| + \|w_x^h\| \leq M_2(\|\nu_k \widetilde{f}\|_{(0,1)} + 1),$$

independent of h. Therefore w_y, $w_{xy} \in L^2(X)$, which implies that $\partial_y^3 u$, $\partial_x \partial_y^3 u \in L^2(X)$. Furthermore by differentiating $L_\theta u = f$ with respect to x, $\alpha = 1, \dots, s-3$ times, the same procedure yields $\partial_x^\alpha \partial_y^3 u \in L^2(X)$.

Proceeding by induction on l, assume that $\partial_x^\alpha \partial_y^\beta u \in L^2(X)$, $\alpha \leq s-\beta$, $\beta \leq l$, and $3 \leq l < s$. Differentiate (3.12) with respect to y, $l-2$ times:

$$\widetilde{L}\partial_y^l u = \partial_y^{l-2}\widetilde{f} - \sum_{i=0}^{l-3} \partial_y^i(\partial_y a_{22}\partial_y^{l-3-i}u_{yy}). \tag{3.14}$$

Note that this equation holds in $L^2(\Omega)$, and that the right-hand side is in $H^{(0,1)}(\Gamma)$ for any compact $\Gamma \subset \Omega$. Applying the method above yields $\partial_y^{l+1}u, \partial_x\partial_y^{l+1}u \in L^2(X)$. Moreover differentiating (3.14) with respect to x, $\alpha = 1, \dots, s-(l+1)$ times, and applying the same procedure, yields $\partial_x^\alpha \partial_y^{l+1} u \in L^2(X)$. The desired conclusion now follows by induction. □

4. The Moser Estimate

Having established the existence of regular solutions to a small perturbation of the linearized equation for (1.5), we intend to apply a Nash-Moser type iteration procedure in the following section, to obtain a smooth solution of (1.5) in a subdomain of X which contains the origin. In the current section, we shall make preparations for the Nash-Moser procedure by establishing a certain a priori estimate. This estimate, referred to as the Moser estimate, will establish the dependence of the solution u of (2.3), on the coefficients of L_θ as well as on the right-hand side, f. The Moser estimate that we seek has the form

$$\|u\|_{H^s(X)} \leq C_s(\|f\|_{H^s(X)} + \Lambda_{s+s_0}\|f\|_{H^2(X)}), \tag{4.1}$$

where

$$\Lambda_{s+s_0} = \sum \|a_{ij}\|_{H^{s+s_0}(X)} + \|a_i\|_{H^{s+s_0}(X)} + \|a\|_{H^{s+s_0}(X)}$$

for some $s_0 > 0$, and C_s is a constant independent of ε and θ.

Estimate (4.1) will first be established in the coordinates (ξ, η), which we have been denoting by (x, y) for convenience, and later converted into the original coordinates (x, y) of the introduction. We will need the following preliminary lemmas. The first is a modification of Lemma 2.3, and the second contains standard consequences of the interpolation inequalities for Sobolev spaces.

Lemma 4.1. *Let $w \in \widetilde{H}^{(2,2)}(\Omega)$ (or $\widetilde{H}_x^{(2,2)}(\Omega)$) be such that $yw \in L^2(\Omega)$, and let $p_1 = \varepsilon\theta\widetilde{p_1}$, $p_2 = \varepsilon\widetilde{p_2}$, $p_3 = \varepsilon\widetilde{p_3}$, where $\widetilde{p_i} \in C_c^\infty(X)$, $i = 1, 2, 3$. Then for ε and θ sufficiently small, there exists a constant M independent of ε and θ, such that*

$$\|w\| + \|w_y\| \leq M\|p_1 w_{xyy} + p_2 w_x + p_3 w + L_\theta w\|.$$

Proof. Assume temporarily that $w \in \widetilde{C}^\infty(\overline{\Omega})$ (or $\widetilde{C}_x^\infty(\overline{\Omega})$). The properties of p_2 and p_3 guarantee that Lemma 2.3 holds for the operator $p_2\partial_x + p_3 + L_\theta$. Therefore

$$\begin{gathered}(Aw + Cw_y + Dw_{yy}, p_2 w_x + p_3 w + L_\theta w) \geq \\ C_1[\|w\|^2 + \|w_y\|^2 + \theta(\|w_x\|^2 + \|w_{xy}\|^2 + \|w_{yy}\|^2)]\end{gathered} \tag{4.2}$$

where A, C, D, and C_1 were given in Lemma 2.3. Furthermore integrating by parts yields

$$\begin{aligned}&(Aw + Cw_y + Dw_{yy}, p_1 w_{xyy}) \\ &= \iint_\Omega [-\frac{1}{2}(Dp_1)_x]w_{yy}^2 + [-Cp_1]w_{xy}w_{yy} + [\frac{1}{2}(Cp_1)_{xy} + \frac{1}{2}(Ap_1)_x]w_y^2 \\ &\quad + [(Ap_1)_y]w_x w_y + [-\frac{1}{2}(Ap_1)_{xyy}]w^2.\end{aligned} \tag{4.3}$$

All the boundary integrals vanish since $p_1 \in C_c^\infty(X)$. Moreover the properties of p_1 guarantee that by choosing ε and θ sufficiently small, we obtain the following by adding (4.2) and (4.3),

$$\begin{aligned}&(Aw + Cw_y + Dw_{yy}, p_1 w_{xyy} + p_2 w_x + p_3 w + L_\theta w)\\ &\geq C_1[\|w\|^2 + \|w_y\|^2 + \theta(\|w_x\|^2 + \|w_{xy}\|^2 + \|w_{yy}\|^2)].\end{aligned}$$

Then an application of Cauchy's inequality, and the use of an approximating sequence $\{w_k\}_{k=1}^\infty$, as was constructed in Proposition 2.5, removes the assumption that $w \in \widetilde{C}^\infty(\overline{\Omega})$ (or $\widetilde{C}_x^\infty(\overline{\Omega})$) and completes the proof. □

Lemma 4.2 ([24])**.** *Let $u, v \in H^s(X)$.*

(i) *If $0 \leq i \leq j \leq s$, then there exists a constant $\mathcal{M}_{i,j,s}$ such that*

$$\|u\|_{H^j(X)} \leq \mathcal{M}_{i,j,s} \|u\|_{H^i(X)}^{\frac{s-j}{s-i}} \|u\|_{H^s(X)}^{\frac{j-i}{s-i}}.$$

(ii) *If α and β are multi-indices such that $|\alpha| + |\beta| = s$, then there exists a constant $\mathcal{M}_s$ such that*

$$\|\partial^\alpha u \partial^\beta v\|_{L^2(X)} \leq \mathcal{M}_s(|u|_{L^\infty(X)} \|v\|_{H^s(X)} + \|u\|_{H^s(X)} |v|_{L^\infty(X)}).$$

(iii) *Let $\Gamma \subset \mathbb{R}^N$ be compact and contain the origin, and let $G \in C^\infty(\Gamma)$. If $u \in H^{s+2}(X, \Gamma)$ and $\|u\|_{H^2(X)} \leq \mathcal{C}$ for some fixed $\mathcal{C}$, then there exists a constant $\mathcal{M}_s$ such that*

$$\|G \circ u\|_{H^s(X)} \leq \mathrm{Vol}(X)|G(0)| + \mathcal{M}_s \|u\|_{H^{s+2}(X)}.$$

Estimate (4.1) will be established by induction on s, and we begin by estimating the x-derivatives. Let $\|\cdot\|_{s,X}$ denote $\|\cdot\|_{H^s(X)}$, and $|\cdot|_\infty$ denote $|\cdot|_{L^\infty(X)}$.

Proposition 4.3. *Let u and f be as in Theorem 3.5. If $\varepsilon = \varepsilon(s)$ is sufficiently small then*

$$\|\partial_x^s u\| + \|\partial_x^s u_y\| \leq C_s(\|f\|_s + \|u\|_{s-1,X} + \Lambda_{s+2}\|f\|_{2,X})$$

for $s \leq r - 6$, where C_s is independent of ε and θ, and

$$\Lambda_{s+2} = \sum \|a_{ij}\|_{s+2,X} + \|a_i\|_{s+2,X} + \|a\|_{s+2,X}.$$

Proof. We proceed by induction on s. The case $s = 0$ follows from Lemma 2.3. Differentiate $L_\theta u = f$ s-times with respect to x and put $w = \partial_x^s u$, then

$$\begin{aligned}&-\theta w_{xxyy} + a_{11} w_{xx} + a_{22} w_{yy} + (a_1 + s\partial_x a_{11}) w_x + a_2 w_y + a_s w\\ &= \partial_x^s f - \sum_{i=0}^{s-1} \partial_x^i(\partial_x a_{22} \partial_x^{s-1-i} u_{yy}) - \sum_{i=0}^{s-1} \partial_x^i(\partial_x a_2 \partial_x^{s-1-i} u_y)\\ &\quad - \sum_{i=0}^{s-1} \partial_x^i(\partial_x a_{s-1-i} \partial_x^{s-1-i} u) := f_s\end{aligned} \tag{4.4}$$

where $a_s = a + s\partial_x a_1 + \frac{s(s-1)}{2}\partial_x^2 a_{11}$. A calculation shows that

$$\begin{aligned}&\sum_{i=0}^{s-1} \partial_x^i(\partial_x a_{22} \partial_x^{s-1-i} u_{yy})\\ &= s\partial_x a_{22} \partial_x^{s-1} u_{yy} + \frac{s(s-1)}{2} \partial_x^2 a_{22} \partial_x^{s-2} u_{yy} + \sum_{i=2}^{s-1} \sum_{j=2}^{i} \binom{i}{j} \partial_x^{j+1} a_{22} \partial_x^{s-1-j} u_{yy}.\end{aligned}$$

Note that the term $\partial_x^{s-1}u_{yy}$ contains too many derivatives. However since $a_{22} = 1 + O(\varepsilon)$, we can solve for $\partial_x^{s-1}u_{yy}$ in (4.4) with s replaced by $s-1$ to obtain a more manageable expression:

$$\partial_x^{s-1}u_{yy} = \frac{1}{a_{22}}[\theta w_{xyy} - a_{11}w_x - (a_1 + s\partial_x a_{11})w - a_2\partial_x^{s-1}u_y - a_{s-1}\partial_x^{s-1}u + f_{s-1}].$$

Substituting back into (4.4), we have

$$\begin{aligned}
&\frac{s\theta\partial_x a_{22}}{a_{22}} w_{xyy} + (s\partial_x a_{11} - \frac{sa_{11}\partial_x a_{22}}{a_{22}})w_x \\
&+ (a_s - a - \frac{s\partial_x a_{22}}{a_{22}}(a_1 - s\partial_x a_{11}))w + L_\theta w \\
&= \partial_x^s f - \frac{s(s-1)}{2}\partial_x^2 a_{22}\partial_x^{s-2}u_{yy} - \sum_{i=2}^{s-1}\sum_{j=2}^{i}\binom{i}{j}\partial_x^{j+1}a_{22}\partial_x^{s-1-j}u_{yy} \\
&\quad - \sum_{i=0}^{s-1}\partial_x^i(\partial_x a_2\partial_x^{s-1-i}u_y) - \sum_{i=0}^{s-1}\partial_x^i(\partial_x a_{s-1-i}\partial_x^{s-1-i}u) \\
&\quad + \frac{s\partial_x a_{22}}{a_{22}}[a_2\partial_x^{s-1}u_y + a_{s-1}\partial_x^{s-1}u - f_{s-1}] := \widetilde{f}_s.
\end{aligned}$$

If $\varepsilon = \varepsilon(s)$ and θ are sufficiently small, we can apply Lemma 4.1 to obtain

$$\|\partial_x^s u\| + \|\partial_x^s u_y\| \leq M\|\widetilde{f}_s\|. \tag{4.5}$$

We now estimate each term of $\widetilde{f}_s$. Using Lemma 4.2 (ii), Lemma 2.2 (iii), and the fact that $\partial_x a_{22}$ vanishes outside of X, produces

$$\begin{aligned}
\|\sum_{i=2}^{s-1}\sum_{j=2}^{i}\binom{i}{j}\partial_x^{j+1}a_{22}\partial_x^{s-1-j}u_{yy}\| &= \|\sum_{i=2}^{s-1}\sum_{j=2}^{i}\binom{i}{j}\partial_x^{j+1}a_{22}\partial_x^{s-1-j}u_{yy}\|_{0,X} \\
&\leq M_1(|\partial_x^3 a_{22}|_\infty\|u\|_{s-1,X} + \|\partial_x^3 a_{22}\|_{s-1,X}|u|_\infty) \\
&\leq M_1'(\|u\|_{s-1,X} + \|a_{22}\|_{s+2,X}|u|_\infty).
\end{aligned}$$

A calculation shows that

$$\sum_{i=0}^{s-1}\partial_x^i(\partial_x a_2\partial_x^{s-1-i}u_y) = s\partial_x a_2\partial_x^{s-1}u_y + \sum_{i=1}^{s-1}\sum_{j=1}^{i}\binom{i}{j}\partial_x^{j+1}a_2\partial_x^{s-1-j}u_y.$$

Then using the same procedure as above, we have

$$\|\sum_{i=0}^{s-1}\partial_x^i(\partial_x a_2\partial_x^{s-1-i}u_y)\| \leq M_2\|\partial_x^{s-1}u_y\| + M_2'(\|u\|_{s-1,X} + \|a_2\|_{s+2,X}|u|_\infty).$$

Furthermore the following estimates are obtained in the same way:

$$\begin{aligned}
&\|\sum_{i=0}^{s-1}\partial_x^i(\partial_x a_{s-1-i}\partial_x^{s-1-i}u)\| + \|\frac{s\partial_x a_{22}}{a_{22}}\sum_{i=0}^{s-2}\partial_x^i(\partial_x a_{s-2-i}\partial_x^{s-2-i}u)\| \\
&\leq M_3(\|u\|_{s-1,X} + (\|a\|_{s+2,X} + \|a_1\|_{s+2,X} + \|a_{11}\|_{s+2,X})|u|_\infty)
\end{aligned}$$

and

$$\|\frac{s\partial_x a_{22}}{a_{22}}\sum_{i=0}^{s-2}\partial_x^i(\partial_x a_2\partial_x^{s-2-i}u_y)\| \leq M_4(\|u\|_{s-1,X} + \|a_2\|_{s+2,X}|u|_\infty).$$

Also since

$$\sum_{i=0}^{s-2}\partial_x^i(\partial_x a_{22}\partial_x^{s-2-i}u_{yy}) = (s-1)\partial_x a_{22}\partial_x^{s-2}u_{yy} + \sum_{i=1}^{s-2}\sum_{j=1}^{i}\binom{i}{j}\partial_x^{j+1}a_{22}\partial_x^{s-2-j}u_{yy}$$

and $\partial_x a_{22} = O(\varepsilon)$, we find that

$$\begin{aligned}&\|\frac{s\partial_x a_{22}}{a_{22}}\sum_{i=0}^{s-2}\partial_x^i(\partial_x a_{22}\partial_x^{s-2-i}u_{yy})\|\\ &\le \varepsilon s^2 M_5\|\partial_x^{s-2}u_{yy}\|_{0,X} + M_5'(\|u\|_{s-1,X} + \|a_{22}\|_{s+2,X}|u|_\infty),\end{aligned}$$

where M_5 is independent of ε and s. Summing the above estimates produces

$$\|\widetilde{f}_s\| \le M_6(\|f\|_s + \|u\|_{s-1,X} + \|\partial_x^{s-1}u_y\| + \varepsilon s^2\|\partial_x^{s-2}u_{yy}\|_{0,X} + \Lambda_{s+2}|u|_\infty). \tag{4.6}$$

Therefore if we estimate $\|\partial_x^{s-2}u_{yy}\|_{0,X}$ appropriately and show that

$$|u|_\infty \le M_7\|f\|_{2,X},$$

the proof will be complete by induction.

We now estimate $\|\partial_x^{s-2}u_{yy}\|_{0,X}$. Differentiate the equation

$$\widetilde{L}u_{yy} := -\theta u_{xxyy} + a_{22}u_{yy} = f - a_{11}u_{xx} - a_1u_x - a_2u_y - au := \widetilde{g}$$

with respect to x $(s-2)$-times, then

$$\widetilde{L}\partial_x^{s-2}u_{yy} = \partial_x^{s-2}\widetilde{g} - \sum_{i=0}^{s-3}\partial_x^i(\partial_x a_{22}\partial_x^{s-3-i}u_{yy}) := \widetilde{g}_{s-2}.$$

Multiply the above equation by $\partial_x^{s-2}u_{yy}$ and integrate by parts in X to obtain,

$$\|\partial_x^{s-2}u_{yy}\|_{0,X} \le M_8\|\widetilde{g}_{s-2}\|_{0,X}.$$

We now estimate $\|\widetilde{g}_{s-2}\|_{0,X}$. Using the same methods as above, we have

$$\|\partial_x^{s-2}(a_1u_x+a_2u_y+au)+\sum_{i=0}^{s-3}\partial_x^i(\partial_x a_{22}\partial_x^{s-3-i}u_{yy})\|_{0,X} \le M_9(\|u\|_{s-1,X}+\Lambda_{s+2}|u|_\infty).$$

Furthermore

$$\partial_x^{s-2}(a_{11}u_{xx}) = a_{11}\partial_x^s u + \sum_{i=1}^{s-2}\binom{s-2}{i}\partial_x^i a_{11}\partial_x^{s-2-i}u_{xx};$$

thus

$$\|\partial_x^{s-2}(a_{11}u_{xx})\|_{0,X} \le M_{10}(\|\partial_x^s u\|_{0,X} + \|u\|_{s-1,X} + \Lambda_{s+2}|u|_\infty).$$

It follows that

$$\|\partial_x^{s-2}u_{yy}\|_{0,X} \le M_{11}(\|\partial_x^s u\|_{0,X} + \|u\|_{s-1,X} + \Lambda_{s+2}|u|_\infty). \tag{4.7}$$

The coefficient of $\|\partial_x^{s-2}u_{yy}\|_{0,X}$ in (4.6) is $\varepsilon s^2 M_6$. If $\varepsilon = \varepsilon(s)$ is chosen sufficiently small so that $\varepsilon s^2 M M_6 M_{11} < \frac{1}{2}$, we can then bring $\varepsilon s^2 M M_6 M_{11}\|\partial_x^s u\|_{0,X}$ from (4.7) to the left-hand side of (4.5), so that by induction on s

$$\|\partial_x^s u\| + \|\partial_x^s u_y\| \le M_6'(\|f\|_s + \|u\|_{s-1,X} + \Lambda_{s+2}(|u|_\infty + \|f\|_{2,X})).$$

We now estimate $|u|_\infty$ to complete the proof. The above methods can be used to show that

$$\|u\|_{2,X} \le M_{12}\|f\|_{2,X}.$$

Then by the Sobolev lemma,

$$|u|_\infty \le M_{13}\|u\|_{2,X} \le M'_{13}\|f\|_{2,X}.$$

□

We now estimate the remaining derivatives.

Proposition 4.4. *Let u, f, s, and ε be as in Proposition 4.3. Then*

$$\|\partial_x^\alpha \partial_y^\beta u\|_{0,X} \le C_s(\|f\|_{s,X} + \|u\|_{s-1,X} + \Lambda_{s+2}\|f\|_{2,X})$$

for $\alpha+\beta \le s$, where C_s is independent of ε and θ.

Proof. The cases $\beta = 0,1,2$ follow from (4.7) and Proposition 4.3. We proceed by induction on β. Assume that the desired estimate holds for $0 \le \alpha \le s-\beta$, and $0 \le \beta \le k-1$, for some $k \le s$.

Differentiate the equation

$$\widetilde{L}u_{yy} := -\theta u_{xxyy} + a_{22}u_{yy} = f - a_{11}u_{xx} - a_1 u_x - a_2 u_y - au := \widetilde{g}$$

with respect to $\partial_x^\alpha \partial_y^{k-2}$ where $0 \le \alpha \le s-k$, then

$$\begin{aligned}\widetilde{L}\partial_x^\alpha \partial_y^k u &= \partial_x^\alpha \partial_y^{k-2}\widetilde{g} - \sum_{i=0}^{\alpha-1}\partial_y^{k-2}\partial_x^i(\partial_x a_{22}\partial_x^{\alpha-1-i}u_{yy}) - \sum_{i=0}^{k-3}\partial_y^i(\partial_y a_{22}\partial_y^{k-3-i}\partial_x^\alpha u_{yy})\\ &:= \widetilde{g}_{\alpha,k-2}.\end{aligned}$$

Multiply the above equation by $\partial_x^\alpha \partial_y^k u$, and integrate by parts in X to obtain

$$\|\partial_x^\alpha \partial_y^k u\|_{0,X} \le M\|\widetilde{g}_{\alpha,k-2}\|_{0,X}.$$

We now estimate $\|\widetilde{g}_{\alpha,k-2}\|_{0,X}$. Using Lemma 4.2 (ii), we have

$$\begin{aligned}&\|\partial_x^\alpha \partial_y^{k-2}(a_{11}u_{xx})\|_{0,X}\\ &\le M_1(\|\partial_x^{\alpha+2}\partial_y^{k-2}u\|_{0,X} + \sum_{p\le\alpha, q\le k-2,\ (p,q)\ne(0,0)}\|\partial_x^p\partial_y^q a_{11}\partial_x^{\alpha-p}\partial_y^{k-2-q}u_{xx}\|_{0,X})\\ &\le M'_1(\|\partial_x^{\alpha+2}\partial_y^{k-2}u\|_{0,X} + |a_{11}|_{C^1(\overline{X})}\|u\|_{s-1,X} + \|a_{11}\|_{s,X}|u|_\infty)\\ &\le M''_1(\|\partial_x^{\alpha+2}\partial_y^{k-2}u\|_{0,X} + \|u\|_{s-1,X} + \Lambda_{s+2}\|f\|_{2,X}).\end{aligned}$$

Furthermore if $\alpha < s-k$ then $\|\partial_x^{\alpha+2}\partial_y^{k-2}u\|_{0,X} \le \|u\|_{s-1,X}$, and if $\alpha = s-k$ the induction assumption implies that

$$\|\partial_x^{\alpha+2}\partial_y^{k-2}u\|_{0,X} \le M_2(\|f\|_{s,X} + \|u\|_{s-1,X} + \Lambda_{s+2}\|f\|_{2,X}).$$

Thus

$$\|\partial_x^\alpha \partial_y^{k-2}(a_{11}u_{xx})\|_{0,X} \le M_3(\|f\|_{s,X} + \|u\|_{s-1,X} + \Lambda_{s+2}\|f\|_{2,X}).$$

Moreover, the methods of Proposition 4.3 may be used to estimate the remaining terms of $\|\widetilde{g}_{\alpha,k-2}\|_{0,X}$ by

$$M_4(\|u\|_{s-1,X} + \Lambda_{s+2}\|f\|_{2,X}).$$

The desired conclusion now follows by combining the above estimates. □

From Proposition 4.4 we obtain the following Moser estimate by induction on s.

Theorem 4.5. *Let u and f be as in Theorem 3.5. If $\varepsilon = \varepsilon(s)$ is sufficiently small then*
$$\|u\|_{s,X} \leq C_s(\|f\|_{s,X} + \Lambda_{s+2}\|f\|_{2,X})$$
for $s \leq r-6$, where C_s is independent of ε and θ.

The estimate of Theorem 4.5 is in terms of the variables (ξ, η) of Lemma 2.2, which we have been denoting by (x, y) for convenience. We now swap notation and denote the original variables of (2.1) by (x, y), and the change of variables by (ξ, η). Furthermore, let $\|\cdot\|_s$ and $\|\cdot\|'_s$ denote the $H^s(X)$ norm with respect to the variables (x, y) and (ξ, η) respectively. Similarly for Λ_s and Λ'_s. We now obtain the analogue of Theorem 4.5 with respect to the variables (x, y). We will need the following lemma.

Lemma 4.6. *If $\varepsilon = \varepsilon(s)$ is sufficiently small then*
$$\|\xi_x\|_s \leq C_s(\|a_{12}\|_{s+3} + \|a_{22}\|_{s+5})$$
for $s \leq r-7$, where C_s is independent of ε and θ.

Proof. We prove the estimate by induction on s. The case $s = 0$ follows from the estimate
$$0 < M_1 \leq |\xi_x| \leq M_2,$$
obtained in the proof of Lemma 2.2. Now assume that the estimate holds for $s-1$. We first estimate the x-derivatives. Differentiate the equation
$$(\frac{a_{12}}{a_{22}})(\xi_x)_x + (\xi_x)_y = -(\frac{a_{12}}{a_{22}})_x\xi_x \tag{4.8}$$
with respect to x s-times to obtain
$$(\frac{a_{12}}{a_{22}})(\partial_x^s\xi_x)_x + (\partial_x^s\xi_x)_y = -\partial_x^s[(\frac{a_{12}}{a_{22}})_x\xi_x] - \sum_{i=0}^{s-1}\partial_x^i((\frac{a_{12}}{a_{22}})_x\partial_x^{s-i}\xi_x) := h_s.$$
Then estimating $\partial_x^s\xi_x$ along the characteristics of (4.8) as in the proof of Lemma 2.2, we have
$$|\partial_x^s\xi_x|_{C^0(\overline{X})} \leq M_3|h_s|_{C^0(\overline{X})}.$$
Recalling that $a_{12} = O(\varepsilon)$, and using the analogue of Lemma 4.2 (ii) for $C^s(\overline{X})$-norms in the same way that the Sobolev version was used in Proposition 4.3, produces
$$\begin{aligned}|h_s|_{C^0(\overline{X})} &\leq \varepsilon(s+1)M_4|\partial_x^s\xi_x|_{C^0(\overline{X})} + M_4'(|(\frac{a_{12}}{a_{22}})_{xx}|_{C^0(\overline{X})}|\xi_x|_{C^{s-1}(\overline{X})}\\ &\quad + |(\frac{a_{12}}{a_{22}})_{xx}|_{C^{s-1}(\overline{X})}|\xi_x|_{C^0(\overline{X})}).\end{aligned}$$
Therefore if ε is small enough to guarantee that $\varepsilon(s+1)M_3M_4 < \frac{1}{2}$, we can bring $\varepsilon(s+1)M_3M_4|\partial_x^s\xi_x|_{C^0(\overline{X})}$ to the left-hand side:
$$|\partial_x^s\xi_x|_{C^0(\overline{X})} \leq M_5(|\xi_x|_{C^{s-1}(\overline{X})} + |\frac{a_{12}}{a_{22}}|_{C^{s+1}(\overline{X})}). \tag{4.9}$$

We now estimate the remaining derivatives. Assume that
$$|\partial_x^\alpha\partial_y^\beta\xi_x|_{C^0(\overline{X})} \leq M_6(|\xi_x|_{C^{s-1}(\overline{X})} + |\frac{a_{12}}{a_{22}}|_{C^{s+1}(\overline{X})}) \tag{4.10}$$

for all $0 \leq \alpha \leq s-\beta$, $0 \leq \beta \leq s-1$. The case $\beta = 0$ is given by (4.9). Differentiate (4.8) with respect to $\partial_x^{\alpha-1}\partial_y^{\beta}$ to obtain

$$\begin{aligned}
&\partial_x^{\alpha-1}\partial_y^{\beta+1}\xi_x \\
&= -\partial_y^{\beta}[(\frac{a_{12}}{a_{22}})(\partial_x^{\alpha-1}\xi_x)_x] - \partial_y^{\beta}\partial_x^{\alpha-1}[(\frac{a_{12}}{a_{22}})_x\xi_x] - \partial_y^{\beta}\sum_{i=0}^{s-1}\partial_x^{i}((\frac{a_{12}}{a_{22}})_x\partial_x^{\alpha-1-i}\xi_x).
\end{aligned}$$

Using assumption (4.10) on the first term on the right-hand side, and applying Lemma 4.2 (ii) to the remaining terms, we find

$$|\partial_x^{\alpha-1}\partial_y^{\beta+1}\xi_x|_{C^0(\overline{X})} \leq M_7(|\xi_x|_{C^{s-1}(\overline{X})} + |\frac{a_{12}}{a_{22}}|_{C^{s+1}(\overline{X})}).$$

Thus by induction on β estimate (4.10) holds for all $0 \leq \alpha \leq s-\beta$, $0 \leq \beta \leq s$. By induction on s, (4.10) implies that

$$|\xi_x|_{C^s(\overline{X})} \leq M_8|\frac{a_{12}}{a_{22}}|_{C^{s+1}(\overline{X})}.$$

Then the Sobolev lemma gives

$$\|\xi_x\|_s \leq M_9\|\frac{a_{12}}{a_{22}}\|_{s+3}.$$

Moreover, by Lemma 4.2 (ii) and (iii) we have

$$\begin{aligned}
\|\frac{a_{12}}{a_{22}}\|_{s+3} &\leq M_{10}(|a_{12}|_\infty\|\frac{1}{a_{22}}\|_{s+3} + \|a_{12}\|_{s+3}|\frac{1}{a_{22}}|_\infty) \\
&\leq M_{11}(\|a_{22}\|_{s+5} + \|a_{12}\|_{s+3}).
\end{aligned}$$

□

Theorem 4.7. *Let u and f be as in Theorem 3.5. If $\varepsilon = \varepsilon(s)$ is sufficiently small then*

$$\|u\|_s \leq C_s(\|f\|_s + \Lambda_{s+11}\|f\|_2)$$

for $s \leq r-13$, where C_s is independent of ε and θ.

Proof. Let σ be a multi-index with $|\sigma| \leq s$. A calculation shows that

$$\|\partial_{x,y}^{\sigma}u\| \leq M_1\|\sum_{|\gamma|\leq s}G_\gamma\partial_{\xi,\eta}^{\gamma}u\|,$$

where G_γ are polynomials in the variables $x_\xi^{-1} = \xi_x$, $\partial_{\xi,\eta}^{\gamma_1}x_\xi$, and $\partial_{\xi,\eta}^{\gamma_2}x_\eta$, such that $\sum_i|\gamma_i| \leq s-|\gamma|$ for each term of G_γ. Then using Lemma 4.2 (ii) and (iii), we find that

$$\|\partial_{x,y}^{\sigma}u\| \leq M_2(\|u\|_s' + (\|x_\xi\|_{s+2}' + \|x_\eta\|_{s+2}')|u|_\infty).$$

Similarly

$$\|\partial_{\xi,\eta}^{\sigma}u\| \leq M_3(\|u\|_s + (\|\xi_x\|_{s+2} + \|\xi_y\|_{s+2})|u|_\infty). \tag{4.11}$$

Then by Theorem 4.5 and the Sobolev lemma, we have

$$\|\partial_{x,y}^{\sigma}u\| \leq M_4(\|f\|_s' + \Lambda_{s+2}'\|f\|_2') + M_4'(\|x_\xi\|_{s+2}' + \|x_\eta\|_{s+2}')\|f\|_2. \tag{4.12}$$

We now estimate the terms on the right-hand side of (4.12). Use Lemma 4.2 (ii), (iii), and (4.11) to obtain

$$\begin{aligned}\|x_\xi\|'_{s+2} = \|\frac{1}{\xi_x}\|'_{s+2} &\le M_5\|\xi_x\|'_{s+4}\\ &\le M_6(\|\xi_x\|_{s+4} + (\|\xi_x\|_{s+6} + \|\xi_y\|_{s+6})|\xi_x|_\infty)\\ &\le M_7(\|\xi_x\|_{s+6} + \|\frac{a_{12}}{a_{22}}\xi_x\|_{s+6})\\ &\le M_8(\|a_{12}\|_{s+9} + \|a_{22}\|_{s+11}).\end{aligned}$$

Similarly

$$\|x_\eta\|'_{s+2} = \|\frac{\xi_y}{\xi_x}\|'_{s+2} \le M_9(\|a_{12}\|_{s+7} + \|a_{22}\|_{s+9}).$$

Furthermore

$$\begin{aligned}\|f\|'_s &\le M_{10}(\|f\|_s + (\|\xi_x\|_{s+2} + \|\xi_y\|_{s+2})|f|_\infty)\\ &\le M_{11}(\|f\|_s + (\|a_{12}\|_{s+5} + \|a_{22}\|_{s+7})\|f\|_2),\end{aligned}$$

and hence

$$\|f\|'_2 \le M_{12}(\|a_{12}\|_7 + \|a_{22}\|_9)\|f\|_2 \le M_{13}\|f\|_2.$$

Also

$$\begin{aligned}\|a_{ij}\|'_{s+2} &\le M_{14}(\|a_{ij}\|_{s+2} + (\|\xi_x\|_{s+4} + \|\xi_y\|_{s+4})|a_{ij}|_\infty)\\ &\le M_{15}(\|a_{ij}\|_{s+2} + \|a_{12}\|_{s+7} + \|a_{22}\|_{s+9}),\end{aligned}$$

so that

$$\Lambda'_{s+2} \le M_{16}\Lambda_{s+9}.$$

Therefore by using the above estimates and summing over all $|\sigma| \le s$, (4.12) produces

$$\|u\|_s \le M_{17}(\|f\|_s + \Lambda_{s+11}\|f\|_2).$$

□

5. The Nash-Moser Procedure

In this section we will modify the Nash-Moser iteration procedure to obtain a solution of

$$\Phi(w) = 0 \quad \text{in } X_\infty, \tag{5.1}$$

where $X_\infty \subset X$ is a sufficiently small neighborhood of the origin that will be defined below. In order to accommodate the requirement (Theorem 3.5) that $\partial_x^\alpha f|_{\partial\Omega} = 0$, $\alpha \le s-1$, we will cut off the right-hand side of the modified linearized equation

$$L_\theta u = f,$$

near ∂X at each iteration, and then estimate the error in a smaller domain at the next step. Furthermore the constant θ will be chosen sufficiently small at each iteration, to guarantee that the procedure converges.

Let $\mu > 5$. Define a sequence of domains $\{X_n\}_{n=1}^\infty$ by

$$X_1 = X, \quad X_n = (1 - \sum_{i=1}^{n-1}\mu^{-i})X,$$

where $\lambda X = \{\lambda x : x \in X\}$. Then $X_\infty = (1 - \frac{1}{\mu-1})X$. In addition, let $\frac{3}{2} < \tau < 2$ and define $\mu_n = \mu^{\tau^{n+n_0}}$, where $n_0 > 0$ will be chosen sufficiently large.

We now construct smoothing operators on $L^2(X_n)$. Fix $\widehat{\psi} \in C_c^\infty(\mathbb{R}^2)$ such that $\widehat{\psi} \equiv 1$ in some neighborhood of the origin. Let $\psi(x) = \iint_{\mathbb{R}^2} \widehat{\psi}(\eta) e^{2\pi i \eta \bullet x} d\eta$ be the inverse Fourier transform of $\widehat{\psi}$. Then ψ is a Schwartz function and satisfies $\iint_{\mathbb{R}^2} \psi(x)dx = 1$, and $\iint_{\mathbb{R}^2} x^\alpha \psi(x)dx = 0$ for any multi-index $\alpha \neq 0$. If $g \in L^2(\mathbb{R}^2)$ and $\gamma \geq 1$, we define the smoothing operators $S'_\gamma : L^2(\mathbb{R}^2) \to H^\infty(\mathbb{R}^2)$ by

$$(S'_\gamma g)(x) = \gamma^2 \iint_{\mathbb{R}^2} \psi(\gamma(x-y))g(y)dy.$$

Then we have the following result (see [22]).

Lemma 5.1. *Let $a, b \in \mathbb{Z}_{\geq 0}$ and $g \in H^a(\mathbb{R}^2)$, then*

(i) $\|S'_\gamma g\|_{H^b(\mathbb{R}^2)} \leq C_{a,b} \|g\|_{H^a(\mathbb{R}^2)}$, $b \leq a$,
(ii) $\|S'_\gamma g\|_{H^b(\mathbb{R}^2)} \leq C_{a,b} \gamma^{b-a} \|g\|_{H^a(\mathbb{R}^2)}$, $a \leq b$,
(iii) $\|g - S'_\gamma g\|_{H^b(\mathbb{R}^2)} \leq C_{a,b} \gamma^{b-a} \|g\|_{H^a(\mathbb{R}^2)}$, $b \leq a$.

To complete the construction, we also need the following extension theorem.

Theorem 5.2 ([23])**.** *Let D be a bounded convex domain in $\mathbb{R}^2$ with Lipschitz smooth boundary. Then there exists a linear operator $T_D : L^2(D) \to L^2(\mathbb{R}^2)$ such that:*

(i) $T_D(g)|_D = g$,
(ii) $T_D : H^a(D) \to H^a(\mathbb{R}^2)$ *continuously for each $a \in \mathbb{Z}_{\geq 0}$.*

To obtain smoothing operators on X_n, $S_n : L^2(X_n) \to H^\infty(X_n)$, we set $S_n g = (S'_{\mu_n} T_{X_n} g)|_{X_n}$. Furthermore, it is clear that the corresponding results of Lemma 5.1 hold for each S_n.

We now set up the iteration procedure. A sequence of functions $\{w_n\}_{n=1}^\infty$ will be shown to converge to a solution of (5.1), and shall be defined inductively as follows. Set $w_1 = 0$ and suppose that w_j, $j \leq n$, are already defined in X_j, then set $w_{n+1} = w_n + S_n u_n$ in X_{n+1}, where u_n is defined in X_n and will be specified below. Set $f_n = -\Phi(w_n)$ in X_n, and let ϕ_n be a C^∞ cut-off function

$$\phi_n(x) = \begin{cases} 1 & \text{if } x \in X_{n+1}, \\ 0 & \text{if } x \in X - X_n, \end{cases}$$

such that

$$|\phi_n|_{C^s(X_n)} \leq M_s \mu^{sn}.$$

Let

$$L(w_n) = \sum_{i,j} a_{ij}(w_n)\partial_{ij} + \sum_i a_i(w_n)\partial_i + a(w_n)$$

denote the linearization of $\Phi(w)$ evaluated at w_n, and let $\{\theta_n\}_{n=1}^\infty$ be a sequence of positive numbers tending towards zero that will be specified later. Then define u_n in X_n by $u_n = v_n|_{X_n}$, where v_n is the solution of

$$L_{\theta_n}(w_n)v_n = \phi_n f_n \quad \text{in } X,$$

given by Theorem 2.6. Since $\mu > 5$ we have $\frac{3}{4}X \subset X_\infty$. Therefore, it follows from the definition of $\Phi(w)$ in (1.5) that the coefficients of $L_{\theta_n}(w_n)$ are well-defined in all of X, even though w_n is only defined in X_n.

For simplicity we denote the Sobolev norms $\|\cdot\|_{H^s(X_n)}$ by $\|\cdot\|_s^n$, and the $C^s(\overline{X}_n)$ norms by $|\cdot|_s^n$. Let $s_* \in \mathbb{Z}_{\geq 0}$ be fixed such that $\Phi(0) \in H^{s_*}(X)$, and define

$$\sigma = n(n+1)\tau^{-(n+1+n_0)}, \quad \delta = \frac{16}{\tau - 1}.$$

The convergence of the sequence $\{w_n\}_{n=1}^{\infty}$ to a solution of (5.1) will follow from the following four statements. Each will be proven by induction on j, for some constants C_1, C_2, and C_3 independent of j and dependent on μ and s_*. We shall require that $s \leq s_* - 18 - 2\delta - \frac{6\tau}{2-\tau}$ and $s_* \geq 22 + 2\delta + \frac{6\tau}{2-\tau}$.

(I_j) $\|w_j\|_{s+15}^j \leq \mu_j^{\sigma s+\delta}\|f_1\|_{s_*-15}^1$

(II_j) $\|u_{j-1}\|_s^{j-1} \leq C_1\mu_{j-1}^{\tau^{-1}(s-s_*+18+2\delta)}\|f_1\|_{s_*-15}^1$

(III_j) $\|f_j\|_s^j \leq C_2\mu_j^{\tau^{-1}(s-s_*+18+2\delta)}\|f_1\|_{s_*-15}^1$

(IV_j) $\|w_j\|_{14}^j \leq C_3$

To start the induction process observe that I_1, II_1, and IV_1 are trivial, and that III_1 holds if we set $C_2 = \mu_1$. Now assume that $\mathrm{I}_j,\dots,\mathrm{IV}_j$ hold for $1 \leq j \leq n$. The next four propositions will prove the induction step. Note that the coefficients of $L(w_j)$ satisfy the conditions placed on (2.1) with $r = s_* - 2$. Therefore the results of the previous sections apply to $L_{\theta_j}(w_j)$, $1 \leq j \leq n$, as long as $\varepsilon(s_*)$ and θ_j are sufficiently small and $s \leq s_* - 15$.

Proposition 5.3. *If $s \leq s_* - 15$ and $\mu(s_*)$ is sufficiently large, then*

$$\|w_{n+1}\|_{s+15}^{n+1} \leq \mu_{n+1}^{\sigma s+\delta}\|f_1\|_{s_*-15}^1.$$

Proof. We have

$$\|w_{n+1}\|_{s+15}^{n+1} \leq \|w_n\|_{s+15}^n + \|S_n u_n\|_{s+15}^n.$$

Furthermore by Theorem 4.7 and Lemma 4.2 (iii),

$$\begin{aligned}\|S_n u_n\|_{s+15}^n &\leq M_1\mu_n^{15}\|u_n\|_s^n \\ &\leq M_2\mu_n^{15}(\|\phi_n f_n\|_s^n + \|w_n\|_{s+15}^n\|\phi_n f_n\|_2^n).\end{aligned}$$

Using Lemma 4.2 (ii), we obtain

$$\begin{aligned}\|\phi_n f_n\|_s^n &\leq M_3(\|f_n\|_s^n + \|\phi_n\|_s^n|f_n|_0^n) \\ &\leq M_4(\|f_n\|_s^n + \|\phi_n\|_s^n\|f_n\|_2^n) \\ &\leq M_5\mu^{sn}\|f_n\|_s^n.\end{aligned}$$

Moreover by the definition of f_n and Lemma 4.2 (iii)

$$\|f_n\|_s^n \leq M_6(\|f_1\|_{s_*-15}^n + \|w_n\|_{s+4}^n), \tag{5.2}$$

so that

$$\|\phi_n f_n\|_s^n \leq M_7\mu^{sn}(\|f_1\|_{s_*-15}^n + \|w_n\|_{s+4}^n).$$

Similarly using IV_n,

$$\|\phi_n f_n\|_2^n \leq M_7\mu^{2n}(\|f_1\|_{s_*-15}^n + \|w_n\|_6^n) \leq M_8\mu^{2n}.$$

We now have

$$\|S_n u_n\|_{s+15}^n \leq M_9\mu_n^{16}\mu^{sn}(\|f_1\|_{s_*-15}^1 + \|w_n\|_{s+15}^n).$$

Therefore

$$\begin{aligned}\|w_{n+1}\|_{s+15}^{n+1} &\leq 2M_9\mu_n^{16}\mu^{sn}(\|f_1\|_{s_*-15}^1 + \|w_n\|_{s+15}^n)\\ &\leq \mu_n^{16}\mu^{2sn}(\|f_1\|_{s_*-15}^1 + \|w_n\|_{s+15}^n),\end{aligned}$$

where the last inequality holds if μ is chosen so large that $2M_9\mu^{-1} \leq 1$. It follows that

$$\|w_{n+1}\|_{s+15}^{n+1} \leq (\prod_{i=1}^{n}\mu_i^{16}\mu^{2si})M_{10}\|f_1\|_{s_*-15}^1,$$

where

$$M_{10} = 1 + \mu_1^{-16}\mu^{-2s} + \dots + \prod_{i=1}^{n-1}\mu_i^{-16}\mu^{-2si} \leq 2,$$

if μ is large. Hence

$$\|w_{n+1}\|_{s+15}^{n+1} \leq 2\mu^{sn(n+1)+\frac{16}{\tau-1}(\tau^{n+1+n_0}-\tau^{1+n_0})}\|f_1\|_{s_*-15}^1 \leq \mu_{n+1}^{\sigma s+\delta}\|f_1\|_{s_*-15}^1,$$

where $\sigma = n(n+1)\tau^{-(n+1+n_0)}$ and $\delta = \frac{16}{\tau-1}$. □

Proposition 5.4. *If $s \leq s_* - 20 - 2\delta$ and $n_0(s_*)$ is sufficiently large then*

$$\|u_n\|_s^n \leq C_1\mu_n^{\tau^{-1}(s-s_*+18+2\delta)}\|f_1\|_{s_*-15}^1,$$

where C_1 depends on μ and s_*.

Proof. By Theorem 4.7

$$\|u_n\|_{s_*-15}^n \leq M_1(\|\phi_n f_n\|_{s_*-15}^n + \|w_n\|_{s_*}^n\|\phi_n f_n\|_2^n),$$

where M_1 depends only on s_*. By Lemma 4.2 (ii), (5.2), and I_n

$$\begin{aligned}\|\phi_n f_n\|_{s_*-15}^n &\leq M_2(\|f_n\|_{s_*-15}^n + \|\phi_n\|_{s_*-15}^n\|f_n\|_2^n)\\ &\leq M_3(1+\mu^{(s_*-15)n})\mu_n^{\sigma(s_*-26)+\delta}\|f_1\|_{s_*-15}^1\\ &\leq M_4\mu_n^{2s_*\sigma+\delta}\|f_1\|_{s_*-15}^1,\end{aligned}$$

where M_3 depends only on s_*. Similarly III_n yields

$$\|\phi_n f_n\|_2^n \leq M_5C_2\mu_n^{2\sigma+\tau^{-1}(20-s_*+2\delta)}\|f_1\|_{s_*-15}^1.$$

Therefore for some constant M_6 depending on μ and s_*, we have

$$\begin{aligned}\|u_n\|_{s_*-15}^n &\leq M_6(\mu_n^{2s_*\sigma+\delta} + \mu_n^{\sigma(s_*-15)+\delta}\mu_n^{2\sigma+\tau^{-1}(20-s_*+2\delta)})\|f_1\|_{s_*-15}^1\\ &\leq 2M_6\mu_n^{2s_*\sigma+\delta}\|f_1\|_{s_*-15}^1,\end{aligned} \tag{5.3}$$

since $s_* \geq 20 + 2\delta$. Furthermore Lemma 2.3 and III_n produce

$$\|u_n\|_0^n \leq M_7\|f_n\|_0^n \leq M_7C_2\mu_n^{\tau^{-1}(18-s_*+2\delta)}\|f_1\|_{s_*-15}^1.$$

Then applying Lemma 4.2 (i), we find

$$\begin{aligned}\|u_n\|_s^n &\leq M_8(\|u_n\|_0^n)^{1-\frac{s}{s_*-15}}(\|u_n\|_{s_*-15}^n)^{\frac{s}{s_*-15}}\\ &\leq M_9\mu_n^{\tau^{-1}(18-s_*+2\delta)(1-\frac{s}{s_*-15})+(2s_*\sigma+\delta)(\frac{s}{s_*-15})}\|f_1\|_{s_*-15}^1\\ &\leq M_9\mu_n^{\tau^{-1}(s-s_*+18+2\delta)}\|f_1\|_{s_*-15}^1\end{aligned}$$

if σ is sufficiently small. Note that σ may be made arbitrarily small by choosing n_0 sufficiently large. We then set $C_2 = M_9$ to obtain the desired result. □

Proposition 5.5. *If $s \le s_* - 18 - 2\delta - \frac{6\tau}{2-\tau}$, $s_* \ge 22 + 2\delta + \frac{6\tau}{2-\tau}$, $n_0(s_*)$ and $\mu(s_*)$ are sufficiently large, and $\varepsilon(s_*)$ is sufficiently small, then*

$$\|f_{n+1}\|_s^{n+1} \le C_2 \mu_{n+1}^{\tau^{-1}(s-s_*+18+2\delta)} \|f_1\|_{s_*-15}^1.$$

Proof. Expanding $\Phi(w_{n+1})$ in a Taylor series yields

$$f_{n+1} = f_n - L(w_n)S_n u_n + Q_n = f_n - \theta_n (S_n u_n)_{\eta\eta\xi\xi} - L_{\theta_n}(w_n)S_n u_n + Q_n,$$

where (ξ, η) are the change of variables given in section §2 by

$$a_{12}(w_n)\xi_x + a_{22}(w_n)\xi_y = 0 \quad \text{in } X, \quad \xi(x,0) = x, \quad \xi(\pm x_0, y) = \pm x_0, \quad \eta = y,$$

and where Q_n is the quadratic error term given by

$$Q_n = \int_0^1 (t-1)\partial_t^2 \Phi(w_n + tS_n u_n)dt.$$

Since $L_{\theta_n}(w_n)u_n = f_n$ in X_{n+1} we have

$$f_{n+1} = L_{\theta_n}(w_n)(u_n - S_n u_n) - \theta_n (S_n u_n)_{\eta\eta\xi\xi} + Q_n, \tag{5.4}$$

in X_{n+1}. Each term of (5.4) shall be estimated separately. First note that θ_n may be chosen sufficiently small to guarantee that

$$\|\theta_n (S_n u_n)_{\eta\eta\xi\xi}\|_s^{n+1} \le \frac{1}{3} C_2 \mu_{n+1}^{\tau^{-1}(s-s_*+18+2\delta)} \|f_1\|_{s_*-15}^1.$$

We now estimate $L_{\theta_n}(w_n)(u_n - S_n u_n)$. By Lemma 4.2 and IV_n,

$$\begin{aligned}
&\|L_{\theta_n}(w_n)(u_n - S_n u_n)\|_s^{n+1} \\
&\le \|L_{\theta_n}(w_n)(u_n - S_n u_n)\|_s^{n} \\
&\le M_1(\|u_n - S_n u_n\|_{s+2}^n + \|w_n\|_{s+4}^n |u_n - S_n u_n|_0^n) + O(\theta_n) \\
&\le M_2(\|u_n - S_n u_n\|_{s+2}^n + \|w_n\|_{s+4}^n \|u_n - S_n u_n\|_2^n) + O(\theta_n) \\
&\le M_3(\mu_n^{s+17-s_*} \|u_n\|_{s_*-15}^n + \mu_n^{17-s_*} \|w_n\|_{s+4}^n \|u_n\|_{s_*-15}^n) + O(\theta_n).
\end{aligned}$$

Furthermore by (5.3),

$$\|u_n\|_{s_*-15}^n \le M_4 \mu_n^{2s_*\sigma+\delta} \|f_1\|_{s_*-15}^1.$$

If θ_n and σ are sufficiently small and μ is sufficiently large, it follows that

$$\begin{aligned}
\|L_{\theta_n}(w_n)(u_n - S_n u_n)\|_s^{n+1} &\le M_5 \mu^{-1}(\mu_n^{3s_*\sigma+s-s_*+17+\delta} + \mu_n^{3s_*\sigma-s_*+17+2\delta}) \|f_1\|_{s_*-15}^1 \\
&\le \frac{1}{3} C_2 \mu_{n+1}^{\tau^{-1}(s-s_*+18+2\delta)} \|f_1\|_{s_*-15}^1.
\end{aligned}$$

We now estimate Q_n. Apply Lemma 4.2 (ii) to obtain

$$\begin{aligned}
\|Q_n\|_s^{n+1} &\le \|Q_n\|_s^n \\
&\le \int_0^1 \sum_{|\alpha|,|\beta|,|\rho|\le 2} \|\partial^\rho \Phi(w_n + tS_n u_n)\partial^\alpha (S_n u_n)\partial^\beta (S_n u_n)\|_s^n dt \\
&\le \int_0^1 \sum_{|\alpha|,|\beta|,|\rho|\le 2} M_6(|\partial^\rho \Phi(w_n + tS_n u_n)|_0^n \|\partial^\alpha (S_n u_n)\partial^\beta (S_n u_n)\|_s^n \\
&\quad + \|\partial^\gamma \Phi(w_n + tS_n u_n)\|_s^n |\partial^\alpha (S_n u_n)\partial^\beta (S_n u_n)|_0^n)dt.
\end{aligned}$$

Then the Sobolev lemma and the interpolation inequality $\|u^2\|_{L^2} \leq C\|u\|_{H^1}^2$, show that

$$\|Q_n\|_s^{n+1} \leq \int_0^1 \sum_{|\rho|\leq 2} M_7(\|\partial^\rho \Phi(w_n + tS_n u_n)\|_2^n (\|S_n u_n\|_{s+3}^n)^2 + \|\partial^\rho \Phi(w_n + tS_n u_n)\|_s^n (\|S_n u_n\|_4^n)^2) dt.$$

Furthermore by Lemma 4.2 (iii), I_n, IV_n, and Proposition 5.4

$$\begin{aligned}\|Q_n\|_s^{n+1} &\leq M_8[(\|w_n\|_6^n + \mu_n^2\|u_n\|_4^n)(\mu_n^3\|u_n\|_s^n)^2 + (\|w_n\|_{s+4}^n + \mu_n^4\|u_n\|_s^n)(\|u_n\|_4^n)^2] \\ &\leq (M_9\|f_1\|_{s_*-15}^1)[(1+\mu_n^{2+\tau^{-1}(-s_*+22+2\delta)})\mu_n^{6+2\tau^{-1}(s-s_*+18+2\delta)} \\ &\quad + (\mu_n^{\sigma(s-11)+\delta} + \mu_n^{4+\tau^{-1}(s-s_*+18+2\delta)})\mu_n^{2\tau^{-1}(-s_*+22+2\delta)}]\|f_1\|_{s_*-15}^1 \\ &\leq (M_{10}\|f_1\|_{s_*-15}^1)\mu_n^{s-s_*+18+2\delta}\|f_1\|_{s_*-15}^1,\end{aligned}$$

since $s \leq s_* - 18 - 2\delta - \frac{6\tau}{2-\tau}$ and $s_* \geq 22 + 2\delta + \frac{6\tau}{2-\tau}$. If $\varepsilon(s_*)$ is sufficiently small to guarantee that $M_{10}\|f_1\|_{s_*-15}^1 \leq \frac{1}{3}C_2$, then

$$\|Q_n\|_s^{n+1} \leq \frac{1}{3}C_2\mu_{n+1}^{\tau^{-1}(s-s_*+18+2\delta)}\|f_1\|_{s_*-15}^1.$$

By combining the estimates for each term of (5.4) we obtain the desired result. □

Proposition 5.6. *If $n_0(s_*)$ is sufficiently large then*

$$\|w_{n+1}\|_{14}^{n+1} \leq C_3,$$

where C_3 depends on μ and s_*.

Proof. Let $a = 14 + \tau^{-1}(18 + 2\delta - s_*)$ and note that since $s_* \geq 22 + 2\delta + \frac{6\tau}{2-\tau}$, $\tau \geq \frac{3}{2}$, we have $a < 0$. If n_0 is sufficiently large, we may apply Proposition 5.4 to obtain

$$\|w_{n+1}\|_{14}^{n+1} \leq \sum_{i=1}^{n} \|S_i u_i\|_{14}^i \leq \sum_{i=1}^{n} \mu_i^{14}\|u_i\|_0^i \leq \sum_{i=1}^{\infty} \mu_i^a \|f_1\|_{s_*-15}^1 := C_3.$$

□

To obtain the largest value for s and smallest lower bound for s_* which satisfy the conditions of Propositions 5.3, 5.4, 5.5, 5.6, we choose $\tau = 1.6$ so that $s_* \geq 100$ and $s \leq s_* - 96$. We now establish two corollaries which will complete the proof of Theorem 1.3.

Corollary 5.7. *$w_n \to w$ in $H^{s_*-96}(X_\infty)$.*

Proof. If $s \leq s_* - 96$ then by II_n,

$$\|w_i - w_j\|_s^\infty \leq \sum_{k=j}^{i} \|u_k\|_s^k \leq C_1 \sum_{k=j}^{i} \mu_k^{\tau^{-1}(s-s_*+18+2\delta)}\|f_1\|_{s_*-15}^1.$$

Hence $\{w_n\}$ is Cauchy in $H^s(X_\infty)$ for all $s \leq s_* - 96$, since $18 + 2\delta < 96$. □

Corollary 5.8. *$\Phi(w_n) \to 0$ in $H^{s_*-96}(X_\infty)$.*

Proof. If $s \leq s_* - 96$ then by III_n,

$$\|\Phi(w_n)\|_s^\infty \leq \|f_n\|_s^n \leq C_2\mu_n^{\tau^{-1}(s-s_*+18+2\delta)}\|f_1\|_{s_*-15}^1 \to 0.$$

□

Since $s_* \geq 100$, it follows that $w_n \to w$ in $C^2(\overline{X}_\infty)$. Therefore $\Phi(w_n) \to \Phi(w)$, showing that w is a solution of (5.1). Furthermore if l is as in Theorem 1.3, then we have $w \in C^{l-98}$, $l \geq 100$. This completes the proof of Theorem 1.3.

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