MAT 305, Spring 2006

MIDTERM #2

due Mon. Apr.23, 2006.

This midterm contains 8 questions worth 20 points each for a total of 160 points, out of which 60 are bonus points.

1. Consider the nonlinear DE

$$\frac{dy}{dx} = (y-x)^2 + 1$$

(a) (5 pts) Show that the change of variables $u = x, v = y - x$, transforms this DE into the separable DE

$$\frac{dv}{du} = v^2$$

(b) (10 pts) Solve this DE in u, v and hence write down the general solution of the original DE in x, y.

(c) (5 pts) Show that the largest interval on which the solution y satisfying the initial condition $y(0) = y_0$ (where $y_0 > 0$ is a positive constant) is defined is the interval $-\infty < x < 1/y_0$.

2. (a) (5 points) Show that for the DE

$$\frac{dy}{dx} = y - x + \sin x$$

the transformation $T : (x, y) \mapsto (u = x + 2\pi, v = y + 2\pi)$ is a symmetry of the DE.

(b) (5 points) Given the DE

$$\frac{dy}{dx} = \sin(x/2) + \cos(y/3)$$

find two positive integers $m, n > 0$ such that the transformation $T : (x, y) \mapsto (u = x + 2m\pi, v = y + 2n\pi)$ is a symmetry of the DE.

(c) (10 points) Given the DE

$$\frac{dy}{dx} = \sin(x/2) + \cos(y/3) + \sin((x + y)/7)$$

find two positive integers $m, n > 0$ such that the transformation $T : (x, y) \mapsto (u = x + 2m\pi, v = y + 2n\pi)$ is a symmetry of the DE.

3. Consider a DE

$$\frac{dy}{dx} = f(x, y)$$

(a) (10 points) Suppose that some scaling transformation $T_t : (x, y) \mapsto (u = tx, v = ty)$ is a symmetry of the DE. Show that the function f satisfies

$$f(tx, ty) = f(x, y)$$

for all x, y.

(b) (10 points) Now suppose that *all* the scaling transformations $T_t, t \neq 0$ are symmetries of the DE. Show that the value of $f(x, y)$ only depends on the ratio y/x, by showing that

$$f(x, y) = f(1, y/x)$$

4. Consider the family of transformations $\{T_t\}_{-\infty<t<+\infty}$ given by

$$T_t : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u = x + t \\ v = e^t y \end{pmatrix}$$

(a) (10 points) Show that the family $\{T_t\}_{-\infty<t<+\infty}$ is a one-parameter group of transformations by showing the following:

$$T_0 = id \ ,$$
$$T_{t_1} \circ T_{t_2} = T_{t_2} \circ T_{t_1} = T_{t_1+t_2} \ , \text{ and}$$
$$(T_t)^{-1} = T_{-t} \ .$$

(b) (5 points) Find the equation of the flow line passing through a given point (x_0, y_0).

(c) (5 points) Find the coefficients ξ, η of the infinitesimal transformation of the group.

5. Let $\{T_t\}_{-\infty<t<+\infty}$ be the one-parameter group considered in problem 4 above. Consider the DE

$$\frac{dy}{dx} = \frac{1}{2}(y^2 e^{-x} + e^x) \ .$$

(a) (5 pts) Show that the transformations T_t are symmetries of this DE.

(b) (15 pts) By rewriting the equation in the form

$$(y^2 + e^{2x})\, dx + (-2e^x)\, dy = 0$$

use Lie's integrating factor to solve the equation.

6. Suppose a ball of mass m is falling through a thick liquid. Let $x = x(t)$ be the displacement of the ball as a function of time t. It's velocity v and acceleration are given by $v = \frac{dx}{dt}$ and $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$. There are two forces acting on in opposing directions on the ball, it's weight $W = mg$ and a frictional force which is assumed to be proportional to the square of the velocity, so $F_{friction} = kv^2$, for some constant k. By Newton's 2nd Law,

$$ma = \text{total force } F = W - F_{friction}$$

which leads to the nonlinear 2nd order DE for x,

$$m\frac{d^2x}{dt^2} = mg - k\left(\frac{dx}{dt}\right)^2$$

Suppose that at time $t = 0$ the initial velocity is $v(0) = 0$.

(a) (5 pts) Solve for the velocity v as a function of t by rewriting the above DE as a separable first-order DE in v and t.

(b) (5 pts) Show that

$$v \to \sqrt{\frac{mg}{k}} \text{ when } t \to +\infty \ ,$$

so eventually the ball falls with an almost constant velocity.

(c) (5 pts) Use part (b) to show that

$$x \to +\infty \text{ when } t \to +\infty \ ,$$

so the ball keeps falling forever.

(d) (5 pts) Use parts (b), (c) and l'Hopital's rule to show that

$$\frac{x}{\sqrt{\frac{mg}{k}}t} \to 1 \text{ when } t \to +\infty \ ,$$

so eventually $x(t) \approx \sqrt{\frac{mg}{k}}t$ for large times t.

7. Consider the 2nd order linear DE

$$\frac{d^2y}{dx^2} + 9\frac{dy}{dx} + 14y = 14x + 23$$

which be written in terms of the differential operator $D = \frac{d}{dx}$ as

$$(D^2 + 9D + 14)y = 14x + 23$$

(a) (5 pts) Find a factorization of the 2nd order linear differential operator $L = D^2 + 9D + 14$ into a composition of two 1st order linear differential operators, of the form

$$L = D^2 + 9D + 14 = (D - a)(D - b)$$

where a, b are constants.

(b) (10 pts) Hence find the general solution of the 2nd order linear DE by successively solving two 1st order linear DEs.

(c) (5 pts) Show that all solutions of the 2nd order DE are asymptotic to the function $f(x) = x + 1$ when $x \to +\infty$, ie for any solution y, $y(x) - (x + 1) \to 0$ as $x \to +\infty$.

8. Consider the 2nd order linear DE

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 32$$

which be written in terms of the differential operator $D = \frac{d}{dx}$ as

$$(D^2 - 8D + 16)y = 32$$

(a) (5 pts) Find a factorization of the 2nd order linear differential operator $L = D^2 - 8D + 16$ into a composition of two 1st order linear differential operators, of the form

$$L = D^2 - 8D + 16 = (D - a)(D - b)$$

where a, b are constants.

(b) (10 pts) Hence find the general solution of the 2nd order linear DE by successively solving two 1st order linear DEs.

(c) (5 pts) Find the solution of the 2nd order DE which satisfies the initial conditions $y(0) = 2, y'(0) = 1$.