

A SPECIAL CLASS OF HYPER-KÄHLER MANIFOLDS

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ABSTRACT. We consider hyper-Kähler manifolds of complex dimension 4 which are fibrations. It is known that the fibers are abelian varieties and the base is \mathbb{P}^2 . We assume that the general fiber is isomorphic to a product of two elliptic curves. We prove that such a hyper-Kähler manifold is deformation equivalent to a Hilbert scheme of two points on a K3 surface.

1. PRELIMINARIES

First we define our main objects of study, *irreducible symplectic manifolds* or *hyper-Kähler manifolds*.

Definition 1.1. *A compact complex Kähler manifold X is called irreducible symplectic if it is simply connected and if $H^0(X, \Omega_X^2)$ is spanned by an everywhere non-degenerate 2-form ω .*

Any holomorphic two-form σ induces a homomorphism $\mathcal{T}_X \rightarrow \Omega_X$. The two-form is everywhere non-degenerate if and only if $\mathcal{T}_X \rightarrow \Omega_X$ is bijective. The last condition in the definition implies that $h^{2,0}(X) = h^{0,2}(X) = 1$ and $K_X \cong \mathcal{O}_X$, i.e., $c_1(X) = 0$.

Definition 1.2. *A compact connected $4n$ -dimensional Riemannian manifold (M, g) is called irreducible hyper-Kähler if its holonomy is $\mathrm{Sp}(n)$.*

As Huybrechts notes [7], irreducible symplectic manifolds with a fixed Kähler class and compact irreducible hyper-Kähler manifolds are the same object. In the rest of the paper, we are going to refer to irreducible hyper-Kähler manifolds just as hyper-Kähler manifolds for simplicity.

Definition 1.3. *An abelian fibration on a $2n$ -dimensional hyper-Kähler manifold X is the structure of a fibration over \mathbb{P}^n whose generic fibre is a smooth abelian variety of dimension n .*

This is a higher dimensional analogue of elliptic fibrations on K3 surfaces. Any fibration structure of a hyper-Kähler manifold looks like an abelian fibration due to the following theorem by Matsushita [8]:

Theorem 1.1. *For projective symplectic manifold X , let $f : X \rightarrow B$ be a proper surjective morphism such that the generic fibre F is connected. Assume that B is smooth and $0 < \dim B < \dim X$. Then*

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- (1) F is an abelian variety up to a finite unramified cover,
- (2) B is n -dimensional and has the same Hodge numbers as \mathbb{P}^n ,
- (3) the fibration is Lagrangian with respect to the holomorphic symplectic form.

In particular, if X is 4-dimensional, we can use the Castelnuovo-Enriques classification of surfaces to deduce that the generic fibre is an abelian surface and the base is \mathbb{P}^2 .

Later on we would need the following description of a Hilbert scheme of a complex surface.

Theorem 1.2. (Fogarty, [3]) *Suppose X is non-singular and $\dim X = 2$, then the following hold:*

- (1) $\text{Hilb}^n(X)$ is non-singular of dimension $2n$;
- (2) $\pi : \text{Hilb}^n(X) \rightarrow S^n X$ is a resolution of singularities, where $S^n X$ is the n -th symmetric product of X .

2. MAIN THEOREM

In this section we prove our main theorem which is a step towards classifying hyper-Kähler fibrations.

Theorem 2.1. *Let $p : X \rightarrow \mathbb{P}^2$ be a hyper-Kähler fibration with general fiber a product of two elliptic curves. Assume that the fibration admits a section τ and that the general singular fiber is semi-stable. Then X is birational to $\text{Hilb}^2(S)$ for a K3 surface S .*

Proof. Take the open subset $U \subset \mathbb{P}^2$ over which the fibers of p are smooth. U is algebraic. The fibers over U are of the form $E_t^1 \times E_t^2$. We can form the fibration $\tilde{Y} \rightarrow U$ with fibers of the form $E_t^1 \cup E_t^2$, where the two elliptic curves are glued along the section τ . Since \tilde{Y} might not be normal, we can take the normalization $\tilde{Y}^{nor} \rightarrow \tilde{Y} \rightarrow U$. Since U is algebraic, we can use Stein factorization. The morphism $\tilde{Y}^{nor} \rightarrow U$ factors through a smooth proper morphism with connected fibers and an étale morphism of degree 2: $\tilde{Y}^{nor} \rightarrow V \rightarrow U$.

According to [4], Section 6.3., there is a unique normal variety \bar{V} , a finite degree-2 morphism $f : \bar{V} \rightarrow \mathbb{P}^2$ and a fiber diagram:

$$\begin{array}{ccc} V & \xrightarrow{2:1} & U \\ \curvearrowright & & \curvearrowright \\ \bar{V} & \xrightarrow{2:1} & \mathbb{P}^2 \end{array}$$

Since the general fiber of X is semi-stable, the fibration $\tilde{Y}^{nor} \rightarrow V$ extends to a minimal family of elliptic curves $\pi : \mathcal{E} \rightarrow \bar{V}$ with a general fiber being semi-stable. The singular fibers are from Kodaira's list of degenerations of elliptic curves. In codimension one there are no multiple fibers. However, in codimension two there might be multiple fibers, the fiber dimension can jump or \mathcal{E} might not be even defined. However, we are interested in codimension one. Notice that \mathcal{E} is a Néron model [2] since

the fibers are abelian varieties. There is an induced section σ of the fibration.

Since the map $f : \bar{V} \rightarrow \mathbb{P}^2$ is 2:1, there is an involution i acting on \bar{V} which interchanges the sheets of the fibers. The involution is well defined on V and from the fiber diagram above it is well defined on \bar{V} as well, because with $i^{-1}V$ we can construct the same fiber diagram since the maps to \mathbb{P}^2 are the same. Therefore there will be an involution on \bar{V} compatible with the involution on V . Denote the branched locus of f by D and $f^{-1}(D) = \tilde{D}$. Let G be the discriminant locus of $\pi : \mathcal{E} \rightarrow \bar{V}$. Note that the intersection $G \cap \tilde{D}$ consists of finitely many points. Indeed, if it wasn't true, then G and \tilde{D} would have a whole component in common. The fibers above this component would be very degenerate (they will be products of two degenerate elliptic curves). However, we assumed that in codimension one the fibers of the original fibration have at worst simple normal crossing singularities, so this cannot happen.

The section σ induces a section $(\sigma, \sigma \circ i)$ of the map:

$$pr_{\bar{V}} : \mathcal{E} \times_{\bar{V}} i^* \mathcal{E} \rightarrow \bar{V}$$

and the involution $i : \bar{V} \circlearrowleft$ induces an involution $i_{\mathcal{E}}$ on $\mathcal{E} \times_{\bar{V}} i^* \mathcal{E}$.

Consider the non-branched locus $\mathbb{P}^2 - D$ and its pre-image X_0 in X . Denote the induced fibration by $p_0 : X_0 \rightarrow \mathbb{P}^2 - D$. By construction,

$$(1) \quad X_0 = \mathcal{E} \times_{\bar{V}} i^* \mathcal{E} |_{V/i_{\mathcal{E}}}$$

Let $L \subset \mathbb{P}^2$ be a line intersecting D transversally at general points of D . Denote the pre-image of L in \bar{V} by L_V . The map $f_V : L_V \rightarrow L$ is 2:1. We choose L general so that L_V doesn't intersect $\tilde{D} \cap G$, so every fiber of $\mathcal{E}_{L_V} \times_{L_V} i^* \mathcal{E}_{L_V} \rightarrow L_V$ over a point of $L_V \cap \tilde{D}$ is smooth. Now we pull back our construction to L_V and we have a rational morphism:

$$L_V \times_{\mathbb{P}^2} X \dashrightarrow \mathcal{E}_{L_V} \times_{L_V} i^* \mathcal{E}_{L_V}$$

which by Weil's extension theorem [2] is regular on the smooth locus of $L_V \times_{\mathbb{P}^2} X \rightarrow L_V$ since the fibers are abelian varieties.

Every section of $p : X \rightarrow \mathbb{P}^2$ is contained in the smooth locus of $X \rightarrow \mathbb{P}^2$, so it pulls-back to a curve in the smooth locus of $L_V \times_{\mathbb{P}^2} X \rightarrow L_V$ via the section τf_V .

Since $\mathcal{K}_X = \mathcal{O}_X$, the relative sheaf of p is $\omega_{X/\mathbb{P}^2} = p^* \mathcal{O}_{\mathbb{P}^2}(3)$ and therefore $\tau^* \omega_{X/\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(3)$.

Consider the relative tangent bundle with the natural isomorphism:

$$(\tau f_V)^* N_{\tau f_V(L_V)/L_V \times_{\mathbb{P}^2} X} = f_V^*(\tau^* T_{X/\mathbb{P}^2}) \xrightarrow{\sim} (\sigma, \sigma \circ i)^* T_{\mathcal{E} \times i^* \mathcal{E}/\bar{V}} |_{L_V}$$

For every point $p \in L_V \cap \tilde{D}$, the map is $L_1 \oplus L_2 \rightarrow L_1 \oplus L_2(p)$, where $L_1 \in T_{\Delta_\mathcal{E}}|_p$ and $L_2 \in N_{\Delta_\mathcal{E}/\mathcal{E} \times \mathcal{E}} \cong T_\mathcal{E}$, and $\Delta_\mathcal{E}$ is the diagonal. Outside the branched locus, the map is the identity.

From the above we get:

$$f_V^*(\tau^* \Lambda^2 T_{X/\mathbb{P}^2}) \xrightarrow{\sim} (\sigma, \sigma \circ i)^* \Lambda^2 T_{\mathcal{E} \times i^* \mathcal{E}/\bar{V}}|_{L_V},$$

or equivalently,

$$f_V^* \omega_{X/\mathbb{P}^2} \xrightarrow{\sim} (\sigma, \sigma \circ i)^* \omega_{\mathcal{E} \times i^* \mathcal{E}/\bar{V}}|_{L_V}$$

Also, by the description of the map above, after twisting with the branched locus, we have:

$$(2) \quad \omega_{\mathcal{E} \times i^* \mathcal{E}/\bar{V}}|_{L_V} \cong f_V^* \omega_{X/\mathbb{P}^2}(-L_V \cap \tilde{D}) \cong f^*[\mathcal{O}_{\mathbb{P}^2}(3)(-\frac{1}{2}D)]|_{L_V},$$

and the last isomorphism holds because $\tilde{D} \rightarrow D$ is 2:1.

However, $\omega_{\mathcal{E} \times i^* \mathcal{E}/\bar{V}} = \omega_{\mathcal{E}/\bar{V}} \otimes \omega_{i^* \mathcal{E}/\bar{V}}$. Denote by $\bar{\mathcal{M}}_{1,1}$ the coarse moduli space of marked elliptic curves. The singular locus of a family of elliptic curves maps to the boundary of $\bar{\mathcal{M}}_{1,1}$. If we denote the pull-back of the boundary of $\bar{\mathcal{M}}_{1,1}$ by δ , then it is well known that $\omega_{\mathcal{E}/\bar{V}} = \frac{\delta}{12}$ (see for example [5]).

Therefore,

$$(3) \quad (\sigma, \sigma \circ i)^* \omega_{\mathcal{E} \times i^* \mathcal{E}/\bar{V}} \cong \mathcal{O}_{\bar{V}}\left(\frac{G + i^{-1}G}{12}\right) = \mathcal{O}_{\bar{V}}\left(\frac{f^{-1}(f(G))}{12}\right) = f^* \mathcal{O}_{\mathbb{P}^2}\left(\frac{f(G)}{12}\right)$$

When we compare the isomorphisms (2) and (3), we get:

$$f^* \mathcal{O}_{\mathbb{P}^2}\left(\frac{f(G)}{12}\right)|_{L_V} \cong f^*[\mathcal{O}_{\mathbb{P}^2}(3)(-\frac{1}{2}D)]|_{L_V},$$

or equivalently,

$$f^* \mathcal{O}_{\mathbb{P}^2}\left(\frac{D}{2} + \frac{f(G)}{12}\right)|_{L_V} \cong f^* \mathcal{O}_{\mathbb{P}^2}(3)|_{L_V}$$

Comparing the degrees, we obtain the relation:

$$\frac{1}{2} \deg(D) + \frac{1}{12} \deg(G) = 3$$

The degrees of D and G are positive even integers (otherwise we would have trivial fibrations), hence there are two possibilities: $(\deg(D), \deg(G)) = (2, 24)$ or $(4, 12)$.

Case 1: $(\deg(D), \deg(G)) = (4, 12)$

First we consider the case when D is smooth. Since $\deg(D) = 4$, \bar{V} is a del Pezzo surface ($K_{\bar{V}} < 0$). We want to show that \mathcal{E} is rationally connected. Take two general points $p, q \in \mathcal{E}$. Then $f(\pi(p)), f(\pi(q))$ are two general points in \mathbb{P}^2 . Then $f(\pi(p)) \in L_p$, where L_p is a tangent line to D at $f(\pi(p))$ and $f(\pi(q)) \in L_q$, where L_q is also a tangent line. Let $L_p \cap L_q = \{r\}$.

Take a tangent line L to D and pull it back to \bar{V} : $L_{\bar{V}} \doteq L \times_{\mathbb{P}^2} \bar{V}$. Its normalization is $\tilde{L}_{\bar{V}}$ and $\mathcal{E}|_{\tilde{L}_{\bar{V}}}$ is an elliptic fibration over \mathbb{P}^1 with 12 nodal fibers. The surface is rational, because it is deformation equivalent to \mathbb{P}^2 blown-up at 9 points in the base locus of a pencil of plane cubics ([1], section 5.12).

We can lift the lines L_p and L_q to \mathcal{E} and get $\mathcal{E}|_{\tilde{L}_{p,\bar{V}}}$ and $\mathcal{E}|_{\tilde{L}_{q,\bar{V}}}$ which are rational surfaces. We can connect any two points on a rational surface with a rational curve. Connect p to \tilde{r} and q to \tilde{r} , where $\tilde{r} \in (f\pi)^{-1}(r)$, so p is rationally chain connected to q .

The point \tilde{r} is a smooth point of \mathcal{E} . In characteristic 0, we can smooth the nodal rational curve if we have a general pair p, q and if \tilde{r} is smooth. Therefore, \mathcal{E} is rationally connected.

Fix a section s of $i^*\mathcal{E} \rightarrow \bar{V}$. Then we get a section \tilde{s} of $\mathcal{E} \times_{\bar{V}} i^*\mathcal{E} \rightarrow \mathcal{E}$. Since we have a finite morphism $\mathcal{E} \times_{\bar{V}} i^*\mathcal{E} \rightarrow X$, we get a finite morphism $\mathcal{E} \rightarrow X$.

The image of a finite morphism from a rationally connected variety to a hyper-Kähler manifold is of dimension at most $\frac{1}{2}\dim(X)$, because we have a $(2, 0)$ form on X and there are no holomorphic $(2, 0)$ or $(1, 0)$ forms on a rational variety. However, $\dim(\mathcal{E}) = 3$ and it is bigger than $\frac{1}{2}\dim(X) = 2$ - a contradiction.

Now assume D is singular. Consider the linear system of lines L containing a fixed singular point r of D . Consider the associated linear system $f^{-1}(L)$. These divisors will typically be all singular because \bar{V} is singular at $f^{-1}(r)$. However, the linear system of strict transforms $\widetilde{f^{-1}(L)}$ on $Bl_{f^{-1}(r)}\bar{V}$ is a basepoint free pencil of divisors on the normal surface $Bl_{f^{-1}(r)}\bar{V}$. Thus, by Bertini's theorem, a general member of this pencil is smooth and intersects G transversally. In particular, the surface $\mathcal{E}|_{\widetilde{f^{-1}(L)}}$ is smooth for such a member.

Let L be any line passing through a singular point $r \in D$. Every component of $f^{-1}(L)$ is a rational curve. Without loss of generality we shall consider the irreducible case. The normalization $\widetilde{f^{-1}(L)}$ is isomorphic to \mathbb{P}^1 . Therefore, $\mathcal{E}|_{\widetilde{f^{-1}(L)}}$ is an elliptic surface over \mathbb{P}^1 . Since there is the following relation of intersection numbers: $f^{-1}(L) \cdot G = L \cdot f(G) = 12$, it follows that $\mathcal{E}|_{\widetilde{f^{-1}(L)}}$ is a rational surface.

Consider the 1-parameter family \mathcal{F} of such rational surfaces parametrized by \mathbb{P}^1 (since the line L varies in \mathbb{P}^1). The 3-fold \mathcal{F} is rationally connected inside a 4-dimensional hyper-Kähler manifold which is impossible.

We ruled out the first case completely and the only remaining case is:

Case 2: $(\deg(D), \deg(G)) = (2, 24)$

Consider the case when D is smooth. Then $\bar{V} \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Take a tangent line L to the conic and pull it back to \bar{V} : $L_{\bar{V}} \doteq L \times_{\mathbb{P}^2} \bar{V}$. Its normalization $\tilde{L}_{\bar{V}}$ is reducible and consists of two copies of \mathbb{P}^1 , say $\tilde{L}_{\bar{V},1}$ and $\tilde{L}_{\bar{V},2}$.

Case 2.1:

$\mathcal{E}|_{\tilde{L}_{\bar{V},k}}$ is an elliptic fibration over \mathbb{P}^1 with 12 nodal fibers, $k = 1, 2$. Then we repeat the same argument as in Case 1 in order to exclude this case.

Case 2.2:

$\mathcal{E}|_{\tilde{L}_{\bar{V},1}}$ is an elliptic fibration over \mathbb{P}^1 with 24 nodal fibers and $\mathcal{E}|_{\tilde{L}_{\bar{V},2}}$ is an elliptic fibration over \mathbb{P}^1 with no singular fibers. Then $\mathcal{E}|_{\tilde{L}_{\bar{V},2}}$ is the trivial fibration and therefore, \mathcal{E} is the pull back of an elliptic fibration on \mathbb{P}^1 through the projection on this factor. And, since the elliptic fibration on \mathbb{P}^1 has 24 nodal fibers, it is an elliptic K3 surface $S \rightarrow \mathbb{P}^1$.

After considering all the cases, we see that:

$$\mathcal{E} \times_{(\mathbb{P}^1 \times \mathbb{P}^1)} i^* \mathcal{E} = (S \times \mathbb{P}^1) \times_{(\mathbb{P}^1 \times \mathbb{P}^1)} (\mathbb{P}^1 \times S).$$

We want to prove that $(S \times \mathbb{P}^1) \times_{(\mathbb{P}^1 \times \mathbb{P}^1)} (\mathbb{P}^1 \times S) \cong S \times S$. Indeed, we have the following commutative fiber diagram:

$$\begin{array}{ccc} S \times S & \longrightarrow & \mathbb{P}^1 \times S \\ \downarrow & \searrow & \downarrow \\ S \times \mathbb{P}^1 & \longrightarrow & \mathbb{P}^1 \times \mathbb{P}^1 \end{array}$$

Therefore,

$$\mathcal{E} \times_{(\mathbb{P}^1 \times \mathbb{P}^1)} i^* \mathcal{E} = (S \times \mathbb{P}^1) \times_{(\mathbb{P}^1 \times \mathbb{P}^1)} (\mathbb{P}^1 \times S) \cong S \times S.$$

But X is birational to the desingularization of $\mathcal{E} \times_{(\mathbb{P}^1 \times \mathbb{P}^1)} i^* \mathcal{E} / \tilde{i}$ (by (1)) which is birational to $S \times S / \mathbb{Z}_2$. Therefore, X is birational to the desingularization of $S \times S / \mathbb{Z}_2$ which is $\text{Hilb}^2(S)$ by Fogarty's theorem (Theorem 1.2.)

Now consider the case when D is singular. Since \bar{V} is normal, D cannot be a double line. Therefore, D is the union of two lines and \bar{V} is a singular quadric cone.

Let L be a line in \mathbb{P}^2 passing through the node $r \in D$. Then $f^{-1}(L)$ is the union of two lines L_1 and L_2 each one of which is a line in the cone \bar{V} passing through its vertex. On a quadric cone all lines are algebraically equivalent, and in particular L_1 and L_2 are algebraically equivalent. Since $\deg(G) = 24$, $L_1 \cdot G = L_2 \cdot G = 12$. Therefore, the surface $\mathcal{E}|_{L_i}$ is rational (for $i = 1, 2$). We constructed a rationally parametrized 1-parameter family of rational surfaces which is impossible to exist in a 4-dimensional hyper-Kähler manifold. With this we finish the proof of our main theorem. \square

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