

HYPER-KÄHLER FOURFOLDS FIBERED BY ELLIPTIC PRODUCTS

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ABSTRACT. Every fibration of a projective hyper-Kähler fourfold has fibers which are Abelian surfaces. In case the Abelian surface is a Jacobian of a genus two curve, these have been classified by Markushevich. We classify those cases where the Abelian surface is a product of two elliptic curves, under some mild genericity hypotheses. Unlike the genus two case, we prove there are infinitely many deformation families for hyper-Kähler fourfolds fibered by elliptic products.

1. INTRODUCTION

Among projective complex manifolds with zero first Chern class, hyper-Kähler manifolds play a distinguished role, generalizing the class of K3 surfaces. In fact the hyper-Kähler manifolds of (complex) dimension 2 are precisely the K3 surfaces. In higher dimensions, dimension 4 and higher, few hyper-Kähler manifolds are known: two infinite classes introduced by Beauville ([2]) and two exceptional cases discovered by O’Grady ([12] and [13]). Nonetheless, even in the first case of hyper-Kähler fourfolds, we are still far from a classification or even a proof that hyper-Kähler fourfolds form a bounded family.

Just as elliptically fibered K3 surfaces are more amenable to study, so also fibered hyper-Kähler manifolds are better understood. By works of Matsushita and Hwang, ([11] and [9]), every holomorphic fibration whose total space is a $2n$ -dimensional hyper-Kähler has base manifold \mathbb{P}^n and has general fiber an n -dimensional Abelian variety, up to a finite (unramified) cover. A hyper-Kähler manifold fibered over \mathbb{P}^n with general fiber an Abelian variety is called an *Abelian fibration*. Every Abelian fibered manifold is a “Tate-Shafarevich twist” of an Abelian fibration with a section, and the set of these twists coming from a given Abelian fibration is classified by a well-studied Tate-Shafarevich group. Thus from now on we assume that the Abelian fibration admits a section.

The principal case is when the fibers are *principally polarized* Abelian varieties. For hyper-Kähler fourfolds, the fiber will be a principally polarized Abelian surface. There are two types of such surfaces, depending on whether or not the polarizing divisor is irreducible or reducible, i.e., whether the Abelian surface is the Jacobian of a genus 2 curve or a product of elliptic curves. Markushevich classified the first case, when the Abelian surface is the Jacobian of a genus 2 curve.

Theorem 1.1. [10, Markushevich] *Every irreducible holomorphic symplectic fourfold X which is fibered by Jacobians of genus two curves and such that the fibration $X \rightarrow \mathbb{P}^2$*

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admits a section is birational to $\text{Hilb}^2(S)$, for a K3 surface S which is a double cover of \mathbb{P}^2 branched over a plane sextic.

Our goal is to classify those Abelian fibrations where the generic fiber is a product of two elliptic curves, an *elliptic product*. But unlike Markushevich's theorem, there are infinitely many deformation classes of projective hyper-Kähler fourfolds together with a fibration by elliptic products. Moreover, these hyper-Kähler fourfolds can become “wild” when the fibration is allowed to become pathologically degenerate. In fact we conjecture that these pathological degenerations never occur, at least for a generic member of a deformation family. But, at the moment, we add our genericity conditions as hypotheses: the conditions are quite natural and are satisfied in every case we know of.

In order to explain the genericity conditions, we first introduce some notation.

As we shall see, every hyper-Kähler fibration by elliptic products $p : X \rightarrow \mathbb{P}^2$ arises from a degree-2 branched cover $f : \bar{V} \rightarrow \mathbb{P}^2$, where \bar{V} is a normal variety, and a family of elliptic curves $\pi : \mathcal{E} \rightarrow \bar{V}$. Let D be the branch locus of f , $\tilde{D} = f^{-1}(D)$ and let G be the discriminant locus of π . These are our *genericity conditions*:

- (1) Over every generic point of G , the fiber of π is irreducible and semistable.
- (2) The image $f(G)$ contains no singular point of D . In fact it suffices to assume that $f(G)$ contains no intersection point of two irreducible components of D which are lines.

Here is the main theorem.

Theorem 1.2. *Let $p : X \rightarrow \mathbb{P}^2$ be a projective hyper-Kähler fibration whose general fiber is a product of two elliptic curves and which satisfies the genericity conditions above. Assume that the fibration admits a section τ . Then X is birational to $\text{Hilb}^2(S)$ for a projective, elliptic K3 surface S with a section.*

Notice that for every projective, elliptic K3 surface S with a section, $\text{Hilb}^2(S)$ is a projective hyper-Kähler fourfold fibered by elliptic products and which admits a section. There are countably many families of projective, elliptic K3 surfaces S , hence countably many families of elliptic product hyper-Kähler fourfolds. And for the general member of each of these families, the genericity conditions are satisfied. As mentioned, we conjecture that every sufficiently general elliptic product hyper-Kähler fourfold satisfies the genericity conditions. All such are deformation equivalent to one of the countably many families arising from projective, elliptic K3 surfaces.

2. PRELIMINARIES

First we define our main objects of study, *irreducible holomorphic symplectic manifolds* or *irreducible hyper-Kähler manifolds*.

Definition 2.1. A compact complex Kähler manifold X is called *irreducible holomorphic symplectic* if it is simply connected and if $H^0(X, \Omega_X^2)$ is spanned by an everywhere non-degenerate 2-form ω .

Any holomorphic two-form σ induces a homomorphism $\mathcal{T}_X \rightarrow \Omega_X$. The two-form is *everywhere non-degenerate* if $\mathcal{T}_X \rightarrow \Omega_X$ is bijective. The last condition in the definition implies that $h^{2,0}(X) = h^{0,2}(X) = 1$ and $K_X \cong \mathcal{O}_X$, i.e., $c_1(X) = 0$.

Definition 2.2. A compact connected $4n$ -dimensional Riemannian manifold (M, g) is called *irreducible hyper-Kähler* if its holonomy is $\mathrm{Sp}(n)$.

As Huybrechts notes [8], irreducible holomorphic symplectic manifolds with a fixed Kähler class are equivalent to irreducible hyper-Kähler manifolds. In the rest of the paper, we refer to irreducible hyper-Kähler manifolds just as hyper-Kähler manifolds for simplicity.

Definition 2.3. An *Abelian fibration* on a $2n$ -dimensional hyper-Kähler manifold X is the structure of a fibration over \mathbb{P}^n whose generic fibre is a smooth abelian variety of dimension n .

This is a higher dimensional analogue of elliptic fibrations on K3 surfaces. Any fibration structure of a projective hyper-Kähler manifold is an abelian fibration due to the following theorems by Matsushita [11] and Hwang [9]:

Theorem 2.4. (Matsushita, [11]) *For a projective holomorphic symplectic manifold X , let $f : X \rightarrow B$ be a proper surjective morphism such that the generic fibre F is connected. Assume that B is smooth and $0 < \dim B < \dim X$. Then*

- (1) F is an abelian variety up to a finite unramified cover,
- (2) B is n -dimensional and has the same Hodge numbers as \mathbb{P}^n ,
- (3) the fibration is Lagrangian with respect to the holomorphic symplectic form.

Theorem 2.5. (Hwang, [9]) *In the setting of Matsushita's theorem, B is biholomorphic to \mathbb{P}^n .*

Fogarty gives the following description of the Hilbert schemes of complex surfaces.

Theorem 2.6. (Fogarty, [4]) *For a nonsingular surface X and $n \in \mathbb{N}$*

- (1) $\mathrm{Hilb}^n(X)$ is non-singular of dimension $2n$ and
- (2) $\pi : \mathrm{Hilb}^n(X) \rightarrow S^n X$ is a resolution of singularities, where $S^n X$ is the n -th symmetric product of X .

3. PROOF OF THE MAIN THEOREM

In this section we prove our main theorem.

Proof. Let us be more precise about the hypothesis that the general singular fiber is irreducible and semistable. Precisely, there exists a closed codimension two subscheme N of \mathbb{P}^2 such that the fiber of p over every point outside of N is either smooth or irreducible and semi-stable.

Take the open subset $U \subset \mathbb{P}^2$ over which the fibers of p are smooth. U is algebraic. The fibers over U are of the form $E_t^1 \times E_t^2$. We can form the fibration $Y \rightarrow U$ with

fibers of the form $E_t^1 \cup E_t^2$, where the two elliptic curves are glued along the section τ . Take the normalization $\tilde{Y} \rightarrow Y \rightarrow U$. Since U is algebraic, we can apply Stein factorization. The morphism $\tilde{Y} \rightarrow U$ factors through a smooth proper morphism with connected fibers and an étale morphism of degree 2: $\tilde{Y} \rightarrow V \rightarrow U$.

According to Grothendieck's lemma ([5], Section 6.3.), there is a unique normal variety \bar{V} , a finite degree-2 morphism $f : \bar{V} \rightarrow \mathbb{P}^2$ and a fiber diagram:

$$\begin{array}{ccc} V & \xrightarrow{2:1} & U \\ \curvearrowright & & \curvearrowright \\ \bar{V} & \xrightarrow{2:1} & \mathbb{P}^2 \end{array}$$

Since the general fiber of X is semi-stable, the fibration $\tilde{Y} \rightarrow V$ extends to a minimal family of elliptic curves $\pi : \mathcal{E} \rightarrow \bar{V}$ with a general fiber being semi-stable. The singular fibers are from Kodaira's list of degenerations of elliptic curves. In codimension one there are no multiple fibers. In codimension two there might be multiple fibers, the fiber dimension can jump or \mathcal{E} might not be even defined. However, we are interested in codimension one. Notice that \mathcal{E} is a Néron model [3] since the fibers are abelian varieties. There is an induced section σ of the fibration.

Since the map $f : \bar{V} \rightarrow \mathbb{P}^2$ is 2:1, there is an involution i acting on \bar{V} which interchanges the sheets of the fibers. The involution is well defined on V and from the fiber diagram above it is well defined on \bar{V} as well, because with $i^{-1}V$ we can construct a similar fiber diagram since the maps to \mathbb{P}^2 are the same. Therefore there will be an involution on \bar{V} compatible with the involution on V . We recall the data from the Introduction: denote the branched locus of f by D and $f^{-1}(D) = \tilde{D}$. Let G be the discriminant locus of $\pi : \mathcal{E} \rightarrow \bar{V}$. Note that the intersection $G \cap \tilde{D}$ consists of finitely many points. Indeed, if it wasn't true, then $f(G)$ and D would have a whole component in common. The fibers of p above this component would be very degenerate (they will be products of two degenerate elliptic curves). According to the genericity conditions, in codimension one the fibers of the original fibration have at worst simple normal crossing singularities, so this cannot happen.

The section σ induces a section $(\sigma, \sigma \circ i)$ of the map:

$$pr_{\bar{V}} : \mathcal{E} \times_{\bar{V}} i^* \mathcal{E} \rightarrow \bar{V}$$

and the involution $i : \bar{V} \circlearrowleft$ induces an involution $i_{\mathcal{E}}$ on $\mathcal{E} \times_{\bar{V}} i^* \mathcal{E}$.

Consider the non-branched locus $\mathbb{P}^2 - D$ and its pre-image X_0 in X . Denote the induced fibration by $p_0 : X_0 \rightarrow \mathbb{P}^2 - D$. By construction,

$$(1) \quad X_0 \cong \mathcal{E} \times_{\bar{V}} i^* \mathcal{E} |_{V/i_{\mathcal{E}}}$$

Let $L \subset \mathbb{P}^2$ be a line intersecting D transversally at general points of D . Denote the pre-image of L in \bar{V} by L_V . The restricted map $f_V : L_V \rightarrow L$ is 2:1. We choose L general so that L_V doesn't intersect $\tilde{D} \cap G$, so every fiber of $\mathcal{E}_{L_V} \times_{L_V} i^* \mathcal{E}_{L_V} \rightarrow L_V$ over a point of $L_V \cap \tilde{D}$ is smooth. After pulling back our construction to L_V , there a rational morphism:

$$L_V \times_{\mathbb{P}^2} X \dashrightarrow \mathcal{E}_{L_V} \times_{L_V} i^* \mathcal{E}_{L_V}$$

which by Weil's extension theorem [3] is regular on the smooth locus of $L_V \times_{\mathbb{P}^2} X \rightarrow L_V$ since the fibers are abelian varieties.

Every section of $p : X \rightarrow \mathbb{P}^2$ is contained in the smooth locus of $X \rightarrow \mathbb{P}^2$, so it pulls-back to a curve in the smooth locus of $L_V \times_{\mathbb{P}^2} X \rightarrow L_V$ via the section τf_V .

Since $\mathcal{K}_X = \mathcal{O}_X$, the relative sheaf of $p : X \rightarrow \mathbb{P}^2$ is $\omega_{X/\mathbb{P}^2} = p^* \mathcal{O}_{\mathbb{P}^2}(3)$ and therefore $\tau^* \omega_{X/\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(3)$.

Consider the relative tangent bundle with the natural isomorphism:

$$(\tau f_V)^* N_{\tau f_V(L_V)/L_V \times_{\mathbb{P}^2} X} = f_V^*(\tau^* T_{X/\mathbb{P}^2}) \xrightarrow{\sim} (\sigma, \sigma \circ i)^* T_{\mathcal{E} \times i^* \mathcal{E}/\bar{V}}|_{L_V}$$

For every point $p \in L_V \cap \tilde{D}$, the map is $L_1 \oplus L_2 \rightarrow L_1 \oplus L_2(p)$, where $L_1 \in T_{\Delta_{\mathcal{E}}}|_p$ and $L_2 \in N_{\Delta_{\mathcal{E}}/\mathcal{E} \times \mathcal{E}} \cong T_{\mathcal{E}}$, and $\Delta_{\mathcal{E}}$ is the diagonal. Outside the branched locus, the map is the identity.

From the above we get:

$$f_V^*(\tau^* \Lambda^2 T_{X/\mathbb{P}^2}) \xrightarrow{\sim} (\sigma, \sigma \circ i)^* \Lambda^2 T_{\mathcal{E} \times i^* \mathcal{E}/\bar{V}}|_{L_V},$$

or equivalently,

$$f_V^* \omega_{X/\mathbb{P}^2} \xrightarrow{\sim} (\sigma, \sigma \circ i)^* \omega_{\mathcal{E} \times i^* \mathcal{E}/\bar{V}}|_{L_V}$$

Also, by the description of the map above, after twisting with the branched locus, we have:

$$(2) \quad \omega_{\mathcal{E} \times i^* \mathcal{E}/\bar{V}}|_{L_V} \cong f_V^* \omega_{X/\mathbb{P}^2}(-L_V \cap \tilde{D}) \cong f^*[\mathcal{O}_{\mathbb{P}^2}(3)(-\frac{1}{2}D)]|_{L_V},$$

and the last isomorphism holds because $\tilde{D} \rightarrow D$ is 2:1.

However, $\omega_{\mathcal{E} \times i^* \mathcal{E}/\bar{V}} = \omega_{\mathcal{E}/\bar{V}} \otimes \omega_{i^* \mathcal{E}/\bar{V}}$. Denote by $\bar{\mathcal{M}}_{1,1}$ the coarse moduli space of marked elliptic curves. The singular locus of a family of elliptic curves maps to the boundary of $\bar{\mathcal{M}}_{1,1}$. If we denote the pull-back of the boundary of $\bar{\mathcal{M}}_{1,1}$ by δ , then it is well known that $\omega_{\mathcal{E}/\bar{V}} = \frac{\delta}{12}$ (see for example [6]).

Therefore,

$$(3) \quad (\sigma, \sigma \circ i)^* \omega_{\mathcal{E} \times i^* \mathcal{E}/\bar{V}} \cong \mathcal{O}_{\bar{V}}\left(\frac{G + i^{-1}G}{12}\right) = \mathcal{O}_{\bar{V}}\left(\frac{f^{-1}(f(G))}{12}\right) = f^* \mathcal{O}_{\mathbb{P}^2}\left(\frac{f(G)}{12}\right)$$

When we compare the isomorphisms (2) and (3), we get:

$$f^* \mathcal{O}_{\mathbb{P}^2}\left(\frac{f(G)}{12}\right)|_{L_V} \cong f^*[\mathcal{O}_{\mathbb{P}^2}(3)(-\frac{1}{2}D)]|_{L_V},$$

or equivalently,

$$f^* \mathcal{O}_{\mathbb{P}^2} \left(\frac{D}{2} + \frac{f(G)}{12} \right) |_{L_V} \cong f^* \mathcal{O}_{\mathbb{P}^2}(3) |_{L_V}$$

Comparing the degrees, we obtain the relation:

$$\frac{1}{2} \deg(D) + \frac{1}{12} \deg f(G) = 3$$

The degrees of D and $f(G)$ are positive even integers (otherwise we would have trivial fibrations), hence there are two possibilities: $(\deg(D), \deg(G)) = (2, 24)$ or $(4, 12)$.

Case 1: $(\deg(D), \deg(f(G))) = (4, 12)$

First we consider the case when D is smooth. Since $\deg(D) = 4$, \bar{V} is a del Pezzo surface ($K_{\bar{V}} < 0$). We want to show that \mathcal{E} is rationally connected. Take two general points $p, q \in \mathcal{E}$. Then $f(\pi(p)), f(\pi(q))$ are two general points in \mathbb{P}^2 . Then $f(\pi(p)) \in L_p$, where L_p is a tangent line to D at $f(\pi(p))$ and $f(\pi(q)) \in L_q$, where L_q is also a tangent line. Let $L_p \cap L_q = \{r\}$.

Take a tangent line L to D and pull it back to \bar{V} : $L_{\bar{V}} \doteq L \times_{\mathbb{P}^2} \bar{V}$. Its normalization is $\tilde{L}_{\bar{V}}$ and $\mathcal{E}|_{\tilde{L}_{\bar{V}}}$ is an elliptic fibration over \mathbb{P}^1 with 12 nodal fibers. The surface is rational, because it is deformation equivalent to \mathbb{P}^2 blown-up at 9 points in the base locus of a pencil of plane cubics ([1], section 5.12).

We can lift the lines L_p and L_q to \mathcal{E} and get $\mathcal{E}|_{\tilde{L}_{p, \bar{V}}}$ and $\mathcal{E}|_{\tilde{L}_{q, \bar{V}}}$ which are rational surfaces. We can connect any two points on a rational surface with a rational curve. Connect p to \tilde{r} and q to \tilde{r} , where $\tilde{r} \in (f\pi)^{-1}(r)$, so p is rationally chain connected to q .

The point \tilde{r} is a smooth point of \mathcal{E} . In characteristic 0, we can smooth the nodal rational curve if we have a general pair p, q and if \tilde{r} is smooth. Therefore, \mathcal{E} is rationally connected.

Fix a section s of $i^* \mathcal{E} \rightarrow \bar{V}$. Then we get a section \tilde{s} of $\mathcal{E} \times_{\bar{V}} i^* \mathcal{E} \rightarrow \mathcal{E}$. Since we have a finite morphism $\mathcal{E} \times_{\bar{V}} i^* \mathcal{E} \rightarrow X$, we get a finite morphism $\mathcal{E} \rightarrow X$.

The image of a finite morphism from a rationally connected variety to a hyper-Kähler manifold is of dimension at most $\frac{1}{2} \dim(X)$, because we have a $(2, 0)$ form on X and there are no holomorphic $(2, 0)$ or $(1, 0)$ forms on a rational variety. However, $\dim(\mathcal{E}) = 3$ and it is bigger than $\frac{1}{2} \dim(X) = 2$ - a contradiction.

Now assume D is singular. Consider the linear system of lines l containing a fixed singular point r of D . Consider the associated linear system $f^{-1}(l)$. These divisors will typically be all singular because \bar{V} is singular at $f^{-1}(r)$. However, the linear system of strict transforms $\widetilde{f^{-1}(l)}$ on $Bl_{f^{-1}(r)} \bar{V}$ is a basepoint free pencil of divisors on the normal surface $Bl_{f^{-1}(r)} \bar{V}$. Thus, by Bertini's theorem, a general member of

this pencil is smooth and intersects G transversally. In particular, the surface $\mathcal{E}|_{\widetilde{f^{-1}(L)}}$ is smooth for such a member.

Let L be any line passing through a singular point $r \in D$. Every component of $f^{-1}(L)$ is a rational curve. Without loss of generality we shall consider the irreducible case. The normalization $\widetilde{f^{-1}(L)}$ is isomorphic to \mathbb{P}^1 . Therefore, $\mathcal{E}|_{\widetilde{f^{-1}(L)}}$ is an elliptic surface over \mathbb{P}^1 . Since there is the following relation of intersection numbers: $f^{-1}(L) \cdot G = L \cdot f(G) = 12$, it follows that $\mathcal{E}|_{\widetilde{f^{-1}(L)}}$ is a rational surface.

Consider the 1-parameter family \mathcal{F} of such rational surfaces parametrized by \mathbb{P}^1 (since the line L varies in \mathbb{P}^1). The 3-fold \mathcal{F} is rationally connected inside a 4-dimensional hyper-Kähler manifold which is impossible.

We ruled out the first case completely and the only remaining case is:

Case 2: $(\deg(D), \deg(f(G))) = (2, 24)$

First we consider the case when D is smooth. Then $\bar{V} \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Take a tangent line L to the conic and pull it back to \bar{V} : $L_{\bar{V}} \doteq L \times_{\mathbb{P}^2} \bar{V}$. Its normalization $\tilde{L}_{\bar{V}}$ is reducible and consists of two copies of \mathbb{P}^1 , say $\tilde{L}_{\bar{V},1}$ and $\tilde{L}_{\bar{V},2}$.

Case 2.1:

$\mathcal{E}|_{\tilde{L}_{\bar{V},k}}$ is an elliptic fibration over \mathbb{P}^1 with 12 nodal fibers, $k = 1, 2$. Then we repeat the same argument as in Case 1 in order to exclude this case.

Case 2.2:

$\mathcal{E}|_{\tilde{L}_{\bar{V},1}}$ is an elliptic fibration over \mathbb{P}^1 with 24 nodal fibers and $\mathcal{E}|_{\tilde{L}_{\bar{V},2}}$ is an elliptic fibration over \mathbb{P}^1 with no singular fibers. Then $\mathcal{E}|_{\tilde{L}_{\bar{V},2}}$ is the trivial fibration and therefore, \mathcal{E} is the pull back of an elliptic fibration on \mathbb{P}^1 through the projection on this factor. And, since the elliptic fibration on \mathbb{P}^1 has 24 nodal fibers, it is an elliptic K3 surface $S \rightarrow \mathbb{P}^1$.

After considering all the cases, we see that:

$$\mathcal{E} \times_{(\mathbb{P}^1 \times \mathbb{P}^1)} i^* \mathcal{E} = (S \times \mathbb{P}^1) \times_{(\mathbb{P}^1 \times \mathbb{P}^1)} (\mathbb{P}^1 \times S).$$

We want to prove that $(S \times \mathbb{P}^1) \times_{(\mathbb{P}^1 \times \mathbb{P}^1)} (\mathbb{P}^1 \times S) \cong S \times S$. Indeed, we have the following commutative fiber diagram:

$$\begin{array}{ccc} S \times S & \longrightarrow & \mathbb{P}^1 \times S \\ \downarrow & \searrow & \downarrow \\ S \times \mathbb{P}^1 & \longrightarrow & \mathbb{P}^1 \times \mathbb{P}^1 \end{array}$$

Therefore,

$$\mathcal{E} \times_{(\mathbb{P}^1 \times \mathbb{P}^1)} i^* \mathcal{E} = (S \times \mathbb{P}^1) \times_{(\mathbb{P}^1 \times \mathbb{P}^1)} (\mathbb{P}^1 \times S) \cong S \times S.$$

But X is birational to the desingularization of $\mathcal{E} \times_{(\mathbb{P}^1 \times \mathbb{P}^1)} i^* \mathcal{E}/\tilde{i}$ (by (1)) which is birational to $S \times S/\mathbb{Z}_2$. Therefore, X is birational to the desingularization of $S \times S/\mathbb{Z}_2$ which is $\text{Hilb}^2(S)$ by Fogarty's theorem (Theorem 2.6.)

Now consider the case when D is singular. Since \bar{V} is normal, D cannot be a double line. Therefore, D is the union of two lines and \bar{V} is a singular quadric cone.

Let L be a line in \mathbb{P}^2 passing through the node $r \in D$. Then $f^{-1}(L)$ is the union of two lines L_1 and L_2 each one of which is a line in the cone \bar{V} passing through its vertex. On a quadric cone all lines are algebraically equivalent, and in particular L_1 and L_2 are algebraically equivalent. Since $\deg(G) = 24$, $L_1 \cdot G = L_2 \cdot G = 12$. Therefore, the surface $\mathcal{E}|_{L_i}$ is rational (for $i = 1, 2$). We constructed a rationally parametrized 1-parameter family of rational surfaces which is impossible to exist in a 4-dimensional hyper-Kähler manifold. With this we finish the proof of our main theorem. \square

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