MULTIVARIATE CALCULUS PROBLEMS

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1. Analytic Geometry

1.

Let f and g be two continuous functions on [0,1]. Their dot product is defined by, $f \cdot g = \int_0^1 f(x)g(x)dx$.

[A]

Find the length of the functions, $v_1(x) = 1$ and $v_2(x) = x$.

[B]

Find the angle between v_1 and v_2.

[C]

Show that the functions, $2x - 1$ and $6x^2 - 6x + 1$ are orthogonal to each other.

[A: $1, 1/\sqrt{2}; 30^o;$]

2.

Find the distance between the points, $(\frac{2^n}{3^{n+1}})_{n=0}^{\infty}$ and $(\frac{2^{n+1}}{3^{n+1}})_{n=0}^{\infty}$ in $\mathbb{R}^{\infty}$.

[A: $1/\sqrt{5}$]

3.

Find and sketch the surface, obtained by the union of the tangent lines of the surface,

$$S : x^2 + y^2 + z^2 - 10z + 16 = 0$$

passing through the origin.

[A: $9z^2 = 16x^2 + 16y^2$]

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4.

Find the equations of the tangent planes to the surface,

$S : x^2 + y^2 - 6x - 9 = 0$

passing through the point [-2 , 0 , 1923].

[A: $3x \pm 4y + 6 = 0$]

5.

Find the equation of the line passing through the points [2,0,0,0] and [0,1,5,6] in $\mathbb{R}^4$.

[A: $\frac{x}{2} = \frac{y-1}{-1} = \frac{z-5}{-5} = \frac{z-6}{-6}$]

6.

Find the parametric equation of the '2 dimensional' plane, passing through the points [1 1 1 1] , [1 2 3 4] , [0 0 3 9] in $\mathbb{R}^4$.

[A: [x y z t]=[1 2 3 4]+u[1 1 -2 -8]+v[0 1 2 3] , (u,v) $\in \mathbb{R}^2$]

2. Differentiation

1.

Given a paraboloid $t = x^2 + y^2 + z^2$, "t" denotes the dimension "time".

[A]

Find the equation of the normal line at the point [1,2,3,14].

[B]

Find the equation of the tangent space at this point.

[C]

Obtain the same equation via the parametric(vector) one by elimination, without any use of the gradient.

[D]

Find the equation of a plane, tangent to the paraboloid at this point.

[A: $\frac{x-1}{2} = \frac{y-2}{4} = \frac{z-3}{6} = \frac{t-14}{-1}$; $2x + 4y + 6z - t = 14$;
;[$xyzt$]=[1 2 3 14]+u[1 0 0 2]+v[0 1 0 4]]

2.

[A]

Calculate the value of, $f(x,y) = \sqrt{1+xy^2}$ at [1·9,-1·9] approximately.

[B]

Find the equation of the tangent plane to the graph of f at [2,-2], and calculate the z-coordinate of the point with x=1·9 and y=1·9 on this plane.

[C]

Comment on the numbers you have obtained.

[A: 2·8, $2x - 4y - 3z = 3$,2·8, The tg-plane approximation]

3. Implicit Functions

1.

Find the critical points of the function $z = g(x,y)$ defined by the equation,

$$e^{2zx-x^2} - 3e^{2zy+y^2} = 2$$

$[P_{0,1} = \pm\sqrt{\ln 3}, \mp\sqrt{\ln 3}]$

4. Extreme Values

1.

Find and classify the critical points of

$$f(x,y) = (1-x^2-y^2)(xy)$$

[Saddles: $(0,0), (\pm 1, 0), (0, \pm 1)$,
L.Max: $(\pm 1/2, \pm 1/2)$,
L.Min: $(\pm 1/2, \mp 1/2)$]

2.

Let P_0 be a point sliding over the ellipsoid, $x^2 + \frac{y^2}{4} + \frac{z^2}{4} = 1$. Find the location of P_0=(a,b,c) a,b,c>0 such that the volume in the first octant, bounded by the tangent plane to ellipsoid at P_0 and the coordinate planes is minimum.

[A: $[1,2,3]/\sqrt{3}$]

3.

Choose a point $P_0 = (x_0, y_0, z_0)$ on the surface $\frac{x^2}{3} + \frac{y^2}{49} + \frac{z^2}{16} = 1$, and calculate $C = x_0 y_0 z_0 + \frac{2}{3}$. This question worths C points.

[C≤10]

4.

[A]

Find the distance from the point [1 1 1 0] to the hyperplane,
$2x - y + z + \sqrt{10}t = 14$.

[B]

Obtain the same result by using the extension of the formula 'distance of a point to plane in $\mathbb{R}^3$'.

[A: 3]

5. Double Integrals

1.

Find the volume of the region in the first octant, below

$$z = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

[A:$\frac{\pi}{8}ab$]

2.

Find the average value of $f(x, y) = xy$ over the region

$$0 \leq x, 0 \leq y \leq \frac{1}{1 + x^2}$$

[A:$1/\pi$]

6. Triple Integrals

1.

Write the volume of the solid in the first octant bounded by the planes,

$$z = 2, 6x + 2y + 3z = 12$$

as a single iterated triple integral, and evaluate it in an efficient way.

[A: $\int_0^2 \int_0^{\frac{12-3z}{2}} \int_0^{\frac{12-2y-3z}{6}} dxdydz, 7$]

2.

Find the volume of a solid in $\mathbb{R}^3$, whose image in the u-v-w space lies in the first octant and bounded by

$$w = 0, w = 4, 6u + 2v + 3w = 18.$$

Where, u-v-w is related to $\mathbb{R}^3$ by the following equations,

$$u = x \sin 23 - y \cos 23 + \cosh z$$

$$v = x \cos 23 + y \sin 23 + z^2 e^{\sec z}$$

$$6w = z.$$

[A: 156]

3.

Calculate the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

4.

Evaluate $\int\int\int_E \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} dV$ where $E : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$.

[A: $\frac{4}{5}\pi abc$]

7. Line Integrals

1.

Describe the path of an ant, which is located at the center of a revolving disc about the origin, and started moving towards the $+x$ direction with a constant speed a. The disc has the angular velocity ω.

[A: $c(t) = at[\cos \omega t, \sin \omega t]$]

2.

The velocity field on the surface of a river is given by

$V(x, y) = [e^x, e^{-x}]$.

Find the path of the ball released at the point [0,1].

3.

Describe the electric field generated by the object of charge Q, placed at the origin, and show that this electric field is conservative.

F=k$\frac{q_1 q_2}{r^2}$, k=$\frac{1}{4\pi\epsilon_0}$

[A: $E[x,y,z] = kQ\frac{[x,y,z]}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \phi = \frac{kQ}{\sqrt{x^2+y^2+z^2}}$]

4.

$$B[x,y] = \frac{\mu_0 I}{2\pi}[\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}]$$

is a vector field on $\mathbb{R}^2 - \{\vec{0}\}$ where $\mu_0, I \in \mathbb{R}^+$ and $C : x^2 + y^2 = a^2$.

[A]

Evaluate $\oint_C B \cdot dr$, in the counterclockwise direction.

[B]

Determine whether B is conservative in this region. If so, find its potential.

[A: $\mu_o I$, NO]

5.

Let P,Q be two scalar fields having continuous partials with $P_y = Q_x$ except at the points $(\pm 4, 0)$ and $(0,0)$ in the plane. Let

$$C_1 : x^2 + y^2 = 1$$

$$C_2 : (x+2)^2 + y^2 = 9$$

$$C_3 : (x-2)^2 + y^2 = 9$$

$$C_4 : x^2 + y^2 = 25$$

And the integrals $\oint_C Pdx + Qdy$ over C_2, C_3, C_4 in the counterclockwise direction are $I_2 = 5, I_3 = 3, I_4 = 7$, respectively.

[A]

Find the value of I_1.

[B]

Draw any three curves, having the integrals 2 , 6 , 8.

[A: 1]

6.

Evaluate $\oint_C (y-x)dx + (x-z)dy + (x-y)dz$

where C is the boundary of the plane $x + 2y + z = 2$, in the first octant.

[A: -2]

8. Surface Integrals

1.

Evaluate $\int\int_S \frac{d\sigma}{(x^2+y^2+z^2)^{3/2}}$, where S is the plane $ax + by + cz = d \neq 0$.
[Hint : Use the Gauss Law]

[A: $2\pi\sqrt{a^2+b^2+c^2}/|d|$]

2.

Calculate $\int\int_S CurlF \cdot \vec{n} d\sigma$ where S : $x^2 + y^2 + z^2 = 4a^2$,
$x^2 + y^2 \geq a^2$, $a \neq 0$,
$\vec{n}$: unit outward normal and $F = z^2 e^x \vec{i} + x^2 e^y \vec{j} + y^2 e^z \vec{k}$

[A: 0]

Suggested Reading

- Calculus of Several Variables - R. C. Buck and A. Willcox
- Calculus II - Tom Apostol
- Multivariable Mathematics - Williamson and Trotter
- Advanced Calculus - Harold M. Edwards
- Intuitive Topology - Prasolov
- The Shape of Space - Jeffrey R. Weeks

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