1 Differential forms

1.1 A word on terminology

We will begin by stating various terminology, which is frequently abbreviated and misused: A *vector* is a point in $\mathbb{R}^n$. A *k-tensor* on $\mathbb{R}^n$ is a multi-linear function $T\colon (\mathbb{R}^n)^k \to \mathbb{R}$. We denote the space of k-tensors on $\mathbb{R}^n$ by $\mathcal{T}^k(\mathbb{R}^n)$. An alternating k-tensor is sometimes called a *form*. In fact, k-tensor is sometimes called a form. A *vector field* on an open set $U \subset \mathbb{R}^n$ is a function $v\colon U \to \mathbb{R}^n$. A *$k$-tensor field* on U is a function $T\colon U \to \mathcal{T}^k(\mathbb{R}^n)$. Just as a vector field on U is a choice of vector for every point of U, a tensor field on U is a choice of tensor for every point of U. A k-tensor field is sometimes called a tensor field, or a k-tensor, or a tensor. Calling a tensor field a tensor is confusing and misleading (no one calls a vector field a vector); it is also common practice. The "strain tensor" that one learns in engineering (I have only the vaguest idea of what it is) is presumably a tensor field, in that it assigns a tensor (also called the strain tensor) to every point of the material. The curvature tensor of Riemannian geometry is likewise a tensor field; we evaluate the curvature tensor field at a given point to determine the curvature tensor at that point. An alternating k-tensor field is a field of alternating k-tensors; it is a choice of alternating k-tensor at every point of U. It is also called a form field, or a form, or a differential form. The latter terminology is probably the most common. A differential form is most precisely and properly called an alternating k-tensor field.

1.2 Tensors

We say that $T\colon (\mathbb{R}^n)^k \to \mathbb{R}$ is *multilinear* if it is linear in each coordinate; in other words,

$$T(v_0, \ldots, \lambda v_i, \ldots, v_k) = \lambda T(v_0, \ldots, v_i, \ldots, v_k),$$

and

$$T(v_0, \ldots, v_i + v_i', \ldots, v_k) = T(v_0, \ldots, v_i, \ldots v_k) + T(v_0, \ldots, v_i', \ldots, v_k).$$

We call such a multilinear form a *k-tensor* (on $\mathbb{R}^n$). We let $\mathcal{T}^k(\mathbb{R}^n)$ denote the space of k-tensors. Then $\mathcal{T}^k(\mathbb{R}^n)$ is a vector space (over $\mathbb{R}$) with pointwise addition and scalar multiplication. We let $e_1, \ldots e_n$ be the standard basis vectors for $\mathbb{R}^n$. We let $[n]$ denote the sequence $1, \ldots, n$ (or the associated set), and we let $[n]^k$ be the set of sequences $\tau \colon [k] \to [n]$. Given such a τ, we let e_τ be the sequence $e_{\tau(1)}, \ldots, e_{\tau(k)}$, so that $T(e_\tau)$ is shorthand for $T(e_{\tau(1)}, \ldots e_{\tau(k)})$. We will prove the following:

1 Theorem
For every $t \colon [n]^k \to \mathbb{R}$ there is a unique $T \in \mathcal{T}^k(\mathbb{R})$ such that

$$T(e_\tau) = t(\tau)$$

for every $\tau \in [n]^k$.

Before we prove Theorem 1, we will define $\phi^\tau \in \mathcal{T}^k(\mathbb{R}^n)$ where $\tau \in [n]^k$, by

$$\phi^\tau(v_{[k]}) = \prod_{i=1}^{k} v_i^{\tau(i)}.$$

(Recall that v^j is the j^{th} coordinate of v). Therefore, for $\tau, \sigma \in [n]^k$, $\phi^\tau(e_\sigma) = \delta_{\tau\sigma}$, where $\delta_{\tau\sigma}$ is 1 if $\tau = \sigma$, and zero otherwise.
Proof of 1:
Given $t \colon [n]^k \to \mathbb{R}$, we let $T_t = \sum_{\tau \in [n]^k} t(\tau)\phi^\tau$. Then

$$\begin{aligned} T_t(e_\tau) &= \sum_{\sigma \in [n]^k} t(\sigma)\phi^\sigma(e_\tau) \\ &= \sum_{\sigma \in [n]^k} t(\sigma)\delta_{\sigma\tau} \\ &= t(\tau). \end{aligned}$$

This shows existence. For uniqueness, we can use multilinearity to write, for a k-tensor T,

$$T(v_{[k]}) = \sum_{\sigma \in [n]^k} \mathcal{T}(e_\sigma) \prod_{i=1}^{k} v_i^{\sigma(i)}.$$

$\boxed{1}$

We can derive a corollary of the uniqueness part of this theorem:

2 Corollary
The $\{\phi^\tau\}_{\tau\in[n]^k}$ form a basis for $\mathcal{T}^k(\mathbb{R}^n)$.

Proof:
First, if $\sum \lambda_\tau \phi^\tau \equiv 0$, then $\lambda_\sigma = \sum_{\tau\in[n]^k} \lambda_\tau \phi^\tau(e_\sigma) = 0$ for all $\sigma \in [n]^k$; it follows that the ϕ^τ are linearly independent. Second, given $T \in \mathcal{T}^k(\mathbb{R}^n)$, we write $\hat{T} = \sum_{\tau\in[n]^k} T(e_\tau)\phi^\tau$. Then $\hat{T}(e_\sigma) = T(e_\sigma)$ for all $\sigma \in [n]^k$, so $\hat{T} \equiv T$ by Theorem 1. $\boxed{2}$

We can define the tensor product $\bigotimes: \mathcal{T}^k(\mathbb{R}^n) \times \mathcal{T}^l(\mathbb{R}^n) \to T^{k+l}(\mathbb{R}^n)$ by $(S \otimes T)(v_1, \ldots, v_{k+l}) = S(v_1, \ldots, v_k)T(v_{k+1}, \ldots, v_{k+l})$. The tensor product is bilinear. If we denote by (σ, τ) the sequence $(\sigma(1), \ldots, \sigma(k), \tau(1), \ldots, \tau(l))$ then $\phi^\sigma \otimes \phi^\tau = \phi^{(\sigma,\tau)}$, and we can write $\phi^\sigma = \phi^{\sigma(1)} \otimes \cdots \otimes \phi^{\sigma(k)}$, where $\phi^i(e_j) = \delta_{ij}$.

If $Q: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, and $T \in \mathcal{T}^k(\mathbb{R}^m)$, we can define $Q^*T \in \mathcal{T}^k(\mathbb{R}^n)$ by $(Q^*T)(v_1, \ldots, v_k) = T(Q(v_1), \ldots, Q(v_k))$. We observe that $Q^*(S \otimes T) = (Q^*S) \times (Q^*T)$.

1.2.1 Exercises

1. Find
$$\phi^{1,2,1}((2,3),(4,1),(5,7)).$$

2. Find tensors S and T such that $S \otimes T \not\equiv T \otimes S$.

3. For every non-zero 1-tensor R on $\mathbb{R}^n$ there exists an invertible linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ such that $T^*R = \phi^1$.

4. Find a tensor $T \in \mathcal{T}^2(\mathbb{R}^2)$ such that $T \neq R \otimes S$ for all 1-tensors R, S on $\mathbb{R}^2$. What 2-tensors T on $\mathbb{R}^n$ can we write as $R \otimes S$?

5. Show that we can write any $T \in \mathcal{T}^2(\mathbb{R}^n)$ as
$$T = \sum_{i=1}^{n} R_i \otimes S_i,$$
where $R_i, S_i \in \mathcal{T}^1(\mathbb{R}^n)$.

1.3 Alternating tensors and the wedge product

1.3.1 Generalized signs

We first define various extensions of the sign of a permutation. We define $\operatorname{sgn}\colon \mathbb{N} \to \{-1, 0, 1\}$ by

$$\operatorname{sgn} n = \begin{cases} -1 & \text{if } n < 0 \\ 0 & \text{if } n = 0 \\ 1 & \text{if } n > 0 \end{cases}$$

For $\sigma\colon [k] \to [k]$ a permutation (we will write $\sigma \in \mathcal{S}_k$), we can write

$$\operatorname{sgn} \sigma = \operatorname{sgn} \prod_{1 \le i < j \le k} \sigma(j) - \sigma(i),$$

or equivalently,

$$\operatorname{sgn} \sigma = \operatorname{sgn} \prod_{\{i,j\} \in \mathcal{I}_2^k} (\sigma(i) - \sigma(j))(i - j),$$

where $\mathcal{I}_2^k$ is the set of two-element subsets of $[k]$. We can then observe that

$$\begin{aligned} \operatorname{sgn}(\sigma \circ \tau) &= \operatorname{sgn} \prod_{\{i,j\} \in \mathcal{I}_2^k} \big(\sigma(\tau(i)) - \sigma(\tau(j))\big)(j - i) \\ &= \operatorname{sgn} \prod_{\{i,j\} \in \mathcal{I}_2^k} \big(\sigma(\tau(i)) - \sigma(\tau(j))\big)(\tau(i) - \tau(j)) \ \operatorname{sgn} \prod_{\{i,j\} \in \mathcal{I}_2^k} (\tau(i) - \tau(j))(i - j) \\ &= (\operatorname{sgn} \sigma)(\operatorname{sgn} \tau). \end{aligned}$$

If σ is a transposition, then $\operatorname{sgn} \sigma = -1$. More generally, for any $\sigma \in [n]^k$ we can define $\operatorname{sgn} \sigma$ by the same formula; then it is non-zero if and only if σ is injective, and 1 when σ is increasing. We can define even more generally, for $\sigma\colon I \to \mathbb{N}$, where $I \subset \mathbb{N}$ is finite,

$$\operatorname{sgn} \sigma = \operatorname{sgn} \prod_{\substack{i,j \in I \\ i < j}} \sigma(j) - \sigma(i).$$

Then if $\sigma(I) = J$, and $\tau\colon J \to \mathbb{N}$ has domain J, then $\operatorname{sgn}(\tau \circ \sigma) = (\operatorname{sgn} \tau)(\operatorname{sgn} \sigma)$. For any finite $I \subset \mathbb{N}$ we can let $|I|$ be the cardinality of I, and identify I with the increasing function defined on $[k]$ (where $k = |I|$) whose image is I, so we can write $I = \{I(1) < \ldots < I(k)\}$. Thus by (v_I) we mean $(v_{I(1)}, \ldots, v_{I(k)})$. For $I, J \subset \mathbb{N}$, we let

$$\operatorname{sgn}(I, J) = \operatorname{sgn} \prod_{i \in I, j \in J} j - i.$$

More generally, for $I_1, \ldots, I_k \subset \mathbb{N}$, we let

$$\operatorname{sgn}(I_1, \ldots, I_k) = \operatorname{sgn} \prod_{\substack{r \in I_i, s \in I_j \\ i < j}} s - r.$$

We show the following:

3 Lemma

1. *$\operatorname{sgn}(I_1, \ldots, I_k) \neq 0$ if and only if the I_i are pairwise disjoint.*

2. *$\operatorname{sgn}(J, I) = (-1)^{|I||J|} \operatorname{sgn}(I, J)$.*

3. *$\operatorname{sgn}(I \cup J, K) \operatorname{sgn}(I, J) = \operatorname{sgn}(I, J, K)$.*

4. *For $\sigma \colon K \to \mathbb{N}$, and $I \cup J = K$, $I \cap J = \emptyset$,*

$$\operatorname{sgn}(\sigma) = \operatorname{sgn}(I, J) \operatorname{sgn}(\sigma|_I) \operatorname{sgn}(\sigma|_J) \operatorname{sgn}(\sigma(I), \sigma(J))$$

.

Proof:

1. This is immediate.

2. Interchanging I and J in $\operatorname{sgn}(I, J)$ multiplies every term in the product by -1; there are $|I||J|$ such terms.

3. We write, for I, J, K mutually disjoint,

$$\begin{aligned} \operatorname{sgn}(I, J, K) &= \operatorname{sgn} \prod_{r \in I, s \in J \text{ or } r \in I \cup J, s \in K} (s - r) \\ &= \operatorname{sgn} \prod_{r \in I, s \in J} (s - r) \prod_{r \in I \cup J, s \in K} s - r \\ &= \operatorname{sgn}(I, J) \operatorname{sgn}(I \cup J, K). \end{aligned}$$

More generally we can verify that

$$\operatorname{sgn}(I_1, \ldots, I_{n-1}) \operatorname{sgn}\left(\bigcup_{i=1}^{n-1} I_i, I_n\right) = \operatorname{sgn}(I_1, \ldots, I_n).$$

4. We write

$$\begin{aligned}
\operatorname{sgn} \sigma &= \operatorname{sgn} \prod_{\{i,j\} \in \mathcal{I}_2^K} (\sigma(j) - \sigma(i))(j - i) \\
&= \operatorname{sgn} \prod_{i \in I, j \in J} (\sigma(j) - \sigma(i)) \prod_{i \in I, j \in J} (j - i) \\
&\quad \prod_{\{i,j\} \in \mathcal{I}_2^I} (\sigma(j) - \sigma(i))(j - i) \prod_{\{i,j\} \in \mathcal{I}_2^J} (\sigma(j) - \sigma(i))(j - i) \\
&= \operatorname{sgn}(\sigma(I), \sigma(J)) \operatorname{sgn}(I, J) \operatorname{sgn}(\sigma|_I) \operatorname{sgn}(\sigma|_J)
\end{aligned}$$

(where $\mathcal{I}_2^I$ is the set of two-element subsets of I).

3

1.3.2 The Determinant

We define $\mathrm{Det} \in \mathcal{T}^k(\mathbb{R}^k)$ by

$$\mathrm{Det} = \sum_{\sigma \in \mathcal{S}_k} \operatorname{sgn}(\sigma)\, \phi^\sigma.$$

Then, for any $\tau \in \mathcal{S}_k$,

$$\begin{aligned}
\mathrm{Det} &= \sum_{\sigma \in \mathcal{S}_n k} \phi^{\sigma \circ \tau} \\
&= (\operatorname{sgn} \tau) \sum_{\sigma \in \mathcal{S}_k} (\operatorname{sgn} \sigma) \phi^{\sigma \circ \tau},
\end{aligned}$$

and then, for any sequence $(v_{[k]}) \in (\mathbb{R}^k)^k$,

$$\begin{aligned}\mathrm{Det}(v_\tau) &= (\operatorname{sgn}\tau) \sum_{\sigma\in\mathcal{S}_k} (\operatorname{sgn}\sigma) \prod_{i=1}^{k} v_{\tau(i)}^{\sigma(\tau(i))} \\ &= (\operatorname{sgn}\tau) \sum_{\sigma\in\mathcal{S}_k} (\operatorname{sgn}\sigma) \prod_{i=1}^{k} v_j^{\sigma(j)} \\ &= (\operatorname{sgn}\tau)\,\mathrm{Det}(v_{[k]}). \end{aligned} \tag{1}$$

1.3.3 Alternating tensors

We say that a k-tensor T on $\mathbb{R}^n$ is *alternating* (or anti-symmetric) if

$$T(v_\sigma) = (\operatorname{sgn}\sigma) T(v_{[k]}) \tag{2}$$

for every sequence $(v_{[k]}) \in (\mathbb{R}^n)^k$ and permutation $\sigma\colon [k] \to [k]$. Because every permutation can be written as a product of transpositions, it is enough to check (2) for all transpositions (or even transpositions of consecutive numbers); this is the usual definition of an alternating tensor. Note that if $T(v_{[k]})$ has a repeated value, then there is a transposition that interchanges those two inputs; it follows that $T(v_{[k]}) = -T(v_{[k]})$ and therefore $T(v_{[k]}) = 0$. Thus equation 2 can be extended to non-injective σ. We can then write, for any $\sigma \in [n]^k$, $T(e_\sigma) = (\operatorname{sgn}\sigma)T(e_{\sigma([k])})$ when σ is injective, and $T(e_\sigma) = 0$ otherwise. (By $(e_{\sigma([k])})$ we of course mean $(e_{I(1)}, \ldots, e_{I(k)})$ where $I = \{I(1) < \ldots < I(k)\} = \sigma([k])$). It follows that the values of $T(e_\sigma)$ for $\sigma \in [n]^k$ can be deduced from the values of $T(e_I)$ for $I \subset [n]$, $|I| = k$. Thus an alternating k-tensor T is determined by $t_T\colon \mathcal{I}_k^n \to \mathbb{R}$ defined by $t_T(I) = T(e_I)$. We denote the space of alternating k-tensors on $\mathbb{R}^n$ by $\Omega^k(\mathbb{R}^n)$.

By (1), we observe that $\mathrm{Det} \in \Omega^k(\mathbb{R}^k)$. In fact, if $\lambda \in \Omega^k(\mathbb{R}^k)$, then $\lambda = \lambda(e_{[k]})\cdot\mathrm{Det}$ because the two alternating k-tensors agree on $[k]$, the unique element of $\mathcal{I}_k^k$. In particular, if $T\colon \mathbb{R}^k \to \mathbb{R}^k$ is a linear transformation, then then $T^*\,\mathrm{Det} = \mathrm{Det}(Te_1, \ldots, Te_k)\,\mathrm{Det}$; if we let $\det T$ denote $\mathrm{Det}(Te_1, \ldots, Te_k)$; we obtain $T^*\,\mathrm{Det} = (\det T)\,\mathrm{Det}$, and thereby $\det(S \circ T) = \det S \cdot \det T$ for all linear transformations $S, T\colon \mathbb{R}^k \to \mathbb{R}^k$.

Suppose that $I \subset [n]$, and $|I| = k$. We define $\pi^I\colon \mathbb{R}^n \to \mathbb{R}^k$ by $\pi^I(v_{[n]}) = (v_I)$. We can then define $\phi^I \in \Omega^k(\mathbb{R}^n)$ by

$$\phi(v_1, \ldots, v_k) = \mathrm{Det}(\pi^I(v_1), \ldots, \pi^I(v_k)).$$

Then $\phi^I \in \Omega^k(\mathbb{R}^n)$ because $\mathrm{Det} \in \Omega^k(\mathbb{R}^k)$. We can easily verify that $\phi^I(e_J) = \delta_{IJ}$, where δ_{IJ} is the Kroneker delta on $\mathcal{I}_k^n$. Given an arbitrary function $t\colon \mathcal{I}_k^n \to \mathbb{R}$, we can define $T \in \Omega^k(\mathbb{R}^n)$ by $T = \sum_{I \in \mathcal{I}_k^n} t(I)\phi^I$; then $T(e_I) = t(I)$. Thus we have shown:

4 Theorem
Given $t\colon \mathcal{I}_k^n \to \mathbb{R}$, there is a unique $T \in \Omega^k(\mathbb{R}^n)$ such that

$$T(e_I) = t(I)$$

for every $I \in \mathcal{I}_k^n$.

We can likewise show

5 Lemma
The $\{\phi^I\}_{I \in \mathcal{I}_k^n}$ form a basis for $\Omega^k(\mathbb{R}^n)$.

1.3.4 The wedge product

We now define the *wedge product*, which is a bilinear form from $\Omega^k(\mathbb{R}^n) \times \Omega^l(\mathbb{R}^n)$ to $\Omega^{k+l}(\mathbb{R}^n)$. The wedge product of α and β is written $\alpha \wedge \beta$. We define

$$(\alpha \wedge \beta)(v_K) = \sum_{\substack{I \cup J = K \\ |I|=k, |J|=l}} \mathrm{sgn}(I, J)\alpha(v_I)\beta(v_J),$$

where $K = [k + l]$. So far we know only that $\alpha \wedge \beta \in \mathcal{T}^n(\mathbb{R}^{k+l})$; we will verify as 1 below that $\alpha \wedge \beta \in \Omega^{k+l}(\mathbb{R}^n)$. The reader should check that the equation will also hold if we replace $[k + l]$ by some other index set $K \subset \mathbb{N}$ of size $k + l$; this will be a convenient observation. We will then show:

1. If $\alpha \in \Omega^k(\mathbb{R}^n)$, and $\beta \in \Omega^l(\mathbb{R}^n)$, then $\alpha \wedge \beta \subset \Omega^{k+l}(\mathbb{R}^n)$.
2. $\beta \wedge \alpha = (-1)^{|\alpha||\beta|}\alpha \wedge \beta$, where $|\alpha| = k$ if $\alpha \in \Omega^k(\mathbb{R}^n)$.
3. $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$.
4. $\phi^I \wedge \phi^J = \mathrm{sgn}(I, J)\phi^{I \cup J}$ when $I \cap J = \emptyset$; $\phi^I \wedge \phi^J = 0$ otherwise.

Proof:

1. We must show that if $\sigma \in \mathcal{S}_n$, then $(\alpha\wedge\beta)(v_\sigma) = (\operatorname{sgn}\sigma)(\alpha\wedge\beta)(v_{[k+l]})$. We write

$$\begin{aligned}
(\alpha\wedge\beta)(v_\sigma) &= \sum_{I\cup J=[k+l]} \operatorname{sgn}(I,J)\alpha(v_{\sigma|_I})\beta(v_{\sigma|_J}) \\
&= \sum_{I\cup J=[k+l]} \operatorname{sgn}(I,J)\operatorname{sgn}(\sigma|_I)\operatorname{sgn}(\sigma|_J)\alpha(v_{\sigma(I)})\beta(v_{\sigma(J)}) \\
&= \sum_{I\cup J=[k+l]} \operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma(I),\sigma(J))\alpha(v_{\sigma(I)})\beta(v_{\sigma(J)}) \\
&= \sum_{I'\cup J'=[k+1]} \operatorname{sgn}(\sigma)\operatorname{sgn}(I',J')\alpha(v_{I'})\beta(v_{J'}) \\
&\quad \text{(where } I'=\sigma(I), J'=\sigma(J)\text{)} \\
&= (\operatorname{sgn}\sigma)(\alpha\wedge\beta)(v_{[k+l}).
\end{aligned}$$

2. We write

$$\begin{aligned}
(\alpha\wedge\beta)(v_K) &= \sum_{I\cup J=K} \operatorname{sgn}(I,J)\alpha(v_I)\beta(v_J) \\
&= \sum_{I\cup J=K} \operatorname{sgn}(I,J)\beta(v_J)\alpha(v_I) \\
&= \sum_{J\cup I=K} \operatorname{sgn}(J,I)\beta(v_I)\alpha(v_J) \\
&= \sum_{J\cup I=K} (-1)^{|I||J|}\operatorname{sgn}(I,J)\beta(v_I)\alpha(v_J) \\
&= \sum_{J\cup I=K} (-1)^{|\alpha||\beta|}\operatorname{sgn}(I,J)\beta(v_I)\alpha(v_J) \\
&= (-1)^{|\alpha||\beta|}(\beta\wedge\alpha)(v_K).
\end{aligned}$$

3. We write

$$\begin{aligned}
\big((\alpha\wedge\beta)\wedge\gamma\big)(v_M) &= \sum_{K\cup L=M} \operatorname{sgn}(K,L)(\alpha\wedge\beta)(v_K)\gamma(v_L) \\
&= \sum_{K\cup L=M} \operatorname{sgn}(K,L)\Big(\sum_{I\cup J=K} \operatorname{sgn}(I,J)\alpha(v_I)\beta(v_J)\Big)\gamma(v_L) \\
&= \sum_{I\cup J\cup L=M} \operatorname{sgn}(I,J)\operatorname{sgn}(I\cup J,L)\alpha(v_I)\beta(v_J)\gamma(v_L) \\
&= \sum_{I\cup J\cup L=M} \operatorname{sgn}(I,J,L)\alpha(v_I)\beta(v_J)\gamma(v_L).
\end{aligned}$$

We can then likewise write

$$\big(\alpha \wedge (\beta \wedge \gamma)\big)(v_M) = \sum_{I \cup J \cup L = M} \operatorname{sgn}(I, J, L)\alpha(v_I)\beta(v_J)\gamma(v_L).$$

In fact, we can write

$$(\alpha_1 \wedge \ldots \wedge \alpha_k)(v_M) = \sum_{\cup_{i=1}^k I_i = M} \operatorname{sgn}(I_1, \ldots, I_k) \prod_{i=1}^k \alpha_i(v_{I_i}).$$

4. We write, for $|K| = |I| + |J|$,

$$\begin{aligned} (\phi^I \wedge \phi^J)(e_K) &= \sum_{R \cup S = K} \operatorname{sgn}(I, J)\phi^I(e_R)\phi^J(e_S) \\ &= \sum_{R \cup S = K} \operatorname{sgn}(I, J)\delta_{IR}\delta_{JS} \\ &= \operatorname{sgn}(I, J)\delta_{(I \cup J)K}, \end{aligned}$$

because the only possible non-zero term of the sum occurs when $R = I$ and $S = J$, and that can only occur if $K = I \cup J$. Therefore $\phi^I \wedge \phi^J = \operatorname{sgn}(I, J)\phi^{I \cup J}$ if $|I \cup J| = |I| + |J|$. (because the two sides agree on $\{e_K\}_{|K|=|I|+|J|}$) and $\phi^I \wedge \phi^J = 0$ otherwise.

5

Thus we can write $\phi^I = \phi^{I(1)} \wedge \ldots \wedge \phi^{I(|I|)}$; in particular $\operatorname{Det} = \phi[n] = \phi^1 \wedge \ldots \wedge \phi^n$.

We observe that if $Q\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, and $\alpha \in \Omega^k(\mathbb{R}^m), \beta \in \Omega^l(\mathbb{R}^m)$, then $Q^*(\alpha \wedge \beta) = Q^*\alpha \wedge Q^*\beta$. This "naturality" follows directly from the coordinate-free definition of the wedge product.

1.3.5 exercises

1. Find $\phi^{\{1,3\}}\big((1, 2, 3, 4), (5, 6, 7, 8)\big)$.

2. What is the dimension of $\Omega^k(\mathbb{R}^n)$?

3. What is $\phi^{\{1,3,5\}} \wedge \phi^{\{2,4,5\}}$?

4. Prove that

$$\operatorname{sgn}(I, J) \operatorname{sgn}(K, L) \operatorname{sgn}(I \cup J, K \cup L) = \operatorname{sgn}(I, J, K, L).$$

5. Prove that

$$\operatorname{sgn}\Big(\bigcup_{i=1}^{n-1} I_i, I_n\Big)\operatorname{sgn}(I_1,\ldots,I_{n-1}) = \operatorname{sgn}(I_1,\ldots,I_n).$$

   Use that to prove that

$$\alpha_1\wedge\ldots\wedge\alpha_k(v_M) = \sum_{I_1\cup\ldots\cup I_k=M} \operatorname{sgn}(I_1,\ldots,I_k)\alpha(v_{I_1})\cdots\alpha(v_{I_k}).$$

6. Prove that

$$\operatorname{sgn}(\sigma) = \operatorname{sgn}(I_1,\ldots,I_k)\operatorname{sgn}(\sigma(I_1),\ldots\sigma(I_k))\prod_{i=1}^{k}\operatorname{sgn}(\sigma|_{I_i}),$$

   where the domain of σ is partitioned into $I_1,\ldots,I_k$.

7. Given a linear transformation $T\colon\mathbb{R}^n\to\mathbb{R}^n$, let $T^*\colon\mathbb{R}^n\to\mathbb{R}^n$ be defined by $\langle Tx,y\rangle = \langle x,T^*y\rangle$ for all $x,y\in\mathbb{R}^n$. Prove that $\det T^* = \det T$.

8. Find $\omega\in\Omega^2(\mathbb{R}^4)$ such that $\omega\wedge\omega\neq 0$.

1.4 Form fields and the exterior derivative

Let $U\subset\mathbb{R}^n$ be open. A *vector field* v on U is a map $v\colon U\to\mathbb{R}^n$. A *k-tensor field* τ on U is a map $\tau\colon U\to\mathcal{T}^k(\mathbb{R}^n)$. If $v_1,\ldots,v_k$ are vector fields on U, and τ is a k-tensor field, then $\tau(v_1,\ldots,v_k)$ is a function from U to $\mathbb{R}$. We say that τ is C^r (for $r\geq 0$) if, for every sequence $v_1,\ldots,v_k$ of C^r vector fields on U, the function $\tau(v_1,\ldots,v_k)$ is C^k. We denote the space of C^r k-tensor fields on U by $C^r(U;\mathcal{T}^k(\mathbb{R}^n))$; likewise we denote the space of C^r vector fields on U by $C^r(U;\mathbb{R}^n)$. If $\tau\in C^r(U;\mathcal{T}^k(\mathbb{R}^n))$ then we can write $\tau\colon(C^r(U;\mathbb{R}^n))\to C^r(U,\mathbb{R})$; because $\tau(v_1,\ldots,v_k)$ is a real-valued function on U when $v_1,\ldots,v_k$ are vector fields on U. On the other hand, not every function τ from k vector fields to real-valued functions on U is a k-tensor field; we require that τ is *C^r-linear*; which means not only that τ is additive in each coordinate, but that

$$\tau(v_1,\ldots,g\cdot v_i,\ldots,v_k) = g\cdot\tau(v_1,\ldots,v_k)$$

for every C^r function $g\colon U\to\mathbb{R}$.

If $x \in U$, then $\tau(x) \in \mathcal{T}^k(\mathbb{R}^n)$. Therefore we can write, for every $x \in U$, $\tau(x) = \sum_{\sigma\in[n]^k} f_\tau(x,\sigma)\phi^\sigma$, where $f_\tau(x,\sigma) = \tau(x; e_\sigma)$. We can also write $\tau = \sum_{\sigma\in[n]^k} \tau_\sigma\phi^\sigma$, where $\tau_\sigma(x) = \tau(x; e_\sigma)$. Thus every k-tensor field on U can be written as a linear combination of the tensor *fields* ϕ^σ, with coefficients that are real-valued functions on U. If τ is C^k then the functions τ_σ are C^k; conversely, if $\tau = \sum_{\sigma\in[n]^k} \tau_\sigma\phi^\sigma$ and the τ_σ are in C^k, then $\tau \in C^k$ (the reader can verify).

We say that τ is a k-form field (or more precisely and long-windedly, an alternating k-tensor field), if one of the following three equivalent conditions hold:

1. For every $x \in U$ and vectors $v_1, \dots, v_k$ in $\mathbb{R}^n$, and every permutation $\sigma \in \mathcal{S}_k$ $\tau(x; v_\sigma) = (\operatorname{sgn}\sigma)\tau(x; v_{[k]})$.

2. For every sequence $v_{[k]}$ of vector *fields* on U, $\tau(v_\sigma) = (\operatorname{sgn}\sigma)\tau(v_{[k]})$.

3. For every $\sigma \in [n]^k$, $\tau_\sigma = (\operatorname{sgn}\sigma)\tau_{\sigma([k])}$

The third condition means that τ is an alternating k-tensor field if and only if we can write $\tau = \sum_{I\in\mathcal{I}_k^n} \tau_I\phi^I$, where the τ_I are real-valued functions on U, and ϕ^I in this context is the alternating k-tensor *field* given by $\phi^I(x) = \phi^I$; in other words, its value at every point $x \in U$ is the alternating k-tensor ϕ^I.

The form field ϕ^I is usually written dx^I.

We can take the wedge product of form fields by taking it pointwise; in particular, we can write $\mathrm{dx}^I = \mathrm{dx}^{I(1)} \wedge \ldots \wedge \mathrm{dx}^{I(|I|)}$.

A zero-tensor field is a function; any zero-tensor or one-tensor field is automatically alternating. If $f \in C^{r+1}(U)$, we define $df \in C^r(U; \Omega^k(\mathbb{R}^n))$ by $df(x; v) = Df(x)v$. We can compute

$$df(x) = \sum_{i=1}^{n} D_i f(x) dx^i.$$

We have thus defined the exterior derivative $d\colon C^\infty(U; \Omega^0(\mathbb{R}^n)) \to C^\infty(U; \Omega^1(\mathbb{R}^n))$; we will extend it to $d\colon C^\infty(U; \Omega^k(\mathbb{R}^n)) \to C^\infty(U; \Omega^{k+1}(\mathbb{R}^n))$ by writing

$$d\Big(\sum_I f_I dx^I\Big) = \sum_I df_I dx^I.$$

We will prove the following:

6 Lemma

The linear operator $d: C^\infty(U;\Omega^k(\mathbb{R}^n)) \to C^\infty(U;\Omega^{k+1}(\mathbb{R}^n))$ *is the unique such linear transformation satisfying*

1. $df(x;v) = Df(x)v$ *for every function* f *and vector* $(x;v)$,
2. $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge d\beta$ *for all form fields* α *and* β, *and*
3. $d(df) = 0$ *for every function* f.

Proof:

We first show that our standard exterior derivative defined in (1.4) satisfies 1, 2, and 3. 1 is obvious. For 2, we write

$$\begin{aligned} d(f\,dx^I \wedge g\,dx^J) &= \operatorname{sgn}(I,J) d(fg\,dx^{I\cup J}) \\ &= \operatorname{sgn}(I,J)(g\,df + f\,dg) \wedge dx^{I\cup J} \\ &= (df \wedge dx^I) \wedge (g\,dx^J) + (-1)^{|I|} f\,dx^I \wedge dg \wedge \mathrm{dx}^J . \\ &= d(f\,dx^I) \wedge (g\,dx^J) + (-1)^{|I|} f\,dx^I \wedge d(g\ \mathrm{dx}^J) \end{aligned}$$

For 3, we observe

$$\begin{aligned} d(df) &= \sum_i d(D_i f) \wedge dx^i \\ &= \sum_{i,j} D_j(D_i f) dx^j \wedge dx^i \\ &= 0 \qquad \text{by the law of mixed partials.} \end{aligned}$$

Next we show that if $\hat{d}: C^\infty(U;\Omega^k(\mathbb{R}^n)) \to C^\infty(U;\Omega^{k+1}(\mathbb{R}^n))$ is a linear transformation that satisfies 1, 2, and 3, then $\hat{d} = d$ on all form fields. By 1 $\hat{d}f = df$ for every function f. Next we show by induction that $\hat{d}(dx^I) = 0$. Let $I = \{i\} \cup I'$, with $i = I(1)$ and $|I| = |I'| + 1$. We assume by induction that $\hat{d}(dx^{I'}) = 0$. Then $dx^I = dx^i \wedge dx^{I'}$, and

$$\begin{aligned} \hat{d}(dx^I) &= \hat{d}(dx^i) \wedge dx^{I'} - dx^i \wedge (\hat{d}(dx^{I'})) \text{ (by 2)} \\ &= 0 \wedge dx^{I'} - dx^i \wedge 0 \text{ (by 3 and induction).} \end{aligned}$$

Finally, by 1 and 2,

$$\begin{aligned} \hat{d}(f dx^I) &= df \wedge dx^I + f\hat{d}(dx^I) \\ &= df \wedge dx^I . \end{aligned}$$

Thus, by linearity $\hat{d}\alpha = d\alpha$ for all form fields $\alpha = \sum_I \alpha_I dx^I$. $\boxed{6}$

Suppose that $f: U \to V$ is C^∞, where $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are open, and $\alpha \in C^\infty(V; \Omega^k(\mathbb{R}^n))$. We define the *pullback* $f^*\alpha \in C^\infty(U; \Omega^k(\mathbb{R}^n))$ by

$$(f^*\alpha)(x; v_1, \ldots, v_k) = \alpha(f(x); Df(x)v_1, \ldots, Df(x)v_k). \tag{3}$$

We observe that, for $g \in C^\infty(U)$,

$$\begin{aligned} f^*(dg)(x; v) &= dg(f(x); Df(x)v) \\ &= Dg(f(x))Df(x)v \\ &= D(g \circ f)(x)v \\ &= d(g \circ f)(x; v). \end{aligned}$$

So $f^*(dg) = d(g \circ f) = d(f^*g)$. With f as before, and $I \in \mathcal{I}_k^n$, we let

$$df^I = df^{I(1)} \wedge \ldots \wedge df^{I(k)},$$

so $d\pi^I = dx^I$. We require the following:

7 Lemma
$d(df^I) = 0$.

Proof:
We can write $I = \{i\} \cup I'$ as in the proof of Theorem 6. Then by induction, $d(df^{I'}) = 0$. Then

$$\begin{aligned} d(df^I) &= d(df^i \wedge df^{I'}) \\ &= d(df^i) \wedge df^{I'} - df^i \wedge d(df^{I'}) \\ &= 0. \end{aligned}$$

$\boxed{7}$

We can then write

$$\begin{aligned} d(f^*(g\, dx^I)) &= d((f^*g) df^I) \\ &= d(f^*g) \wedge df^I \\ &= f^*(dg \wedge dx^I) \\ &= f^*(d(g\, dx^I)). \end{aligned}$$

It follows then that $d(f^*\alpha) = f^*d\alpha$; in other words, d is natural for the pullback operation.

1.5 Two simple special cases of Stokes' theorem

If $f \in C^\infty(U)$, the *support* of f, written $\operatorname{supp} f$, is $\operatorname{Cl}\{x \in U \mid f(x) \neq 0\}$. If $\omega \in C^\infty(U; \Omega^k(\mathbb{R}^n))$, then we can write $\omega = \sum_{I \in \mathcal{I}_k^n} \omega_I dx^I$; the support of ω is the union $\bigcup_{I \in \mathcal{I}_k^n} \operatorname{supp} \omega_I$. Note that $f \in C^\infty(\mathbb{R}^n)$ has compact support if and only if it is non-zero on a bounded subset of $\mathbb{R}^n$.

If $\omega \in C^\infty(U; \Omega^n(\mathbb{R}^n))$, then we can write $\omega = f dx^1 \wedge \ldots \wedge dx^n$; we define

$$\int_U \omega = \int_U f dx^1 \wedge \ldots \wedge dx^n = \int_U f.$$

We now prove

8 Proposition
Suppose that $\omega \in C^\infty(\mathbb{R}^n, \Omega^{n-1}(\mathbb{R}^n))$ has compact support. Then

$$\int_{\mathbb{R}^n} d\omega = 0.$$

Proof:
We let $\overline{dx^i}$ be $dx^1 \wedge \ldots \widehat{dx^i} \ldots \wedge \mathrm{dx}^n$, where the $\widehat{dx^i}$ indicates that dx^i has been ommitted. Then

$$dx^j \wedge \overline{dx^i} = \begin{cases} 0 & \text{if } i \neq j \\ (-1)^i \, dx^1 \wedge \ldots \wedge dx^n & \text{if } i = j \end{cases} \tag{4}$$

We can write $\omega = \sum_{i=1}^n \omega_i \, \overline{dx^i}$, where $\omega \in C^\infty(\mathbb{R}^n)$ has compact support. Then

$$\begin{aligned} d\omega &= \sum_{i=1}^n \sum_{j=1}^n D_j \omega_i \, dx^j \wedge \overline{dx^i} \\ &= \sum_{i=1}^n (-1)^{i-1} D_i \omega_i \, dx^1 \wedge \ldots \wedge dx^n \end{aligned}$$

We can write

$$\begin{aligned} \int_{\mathbb{R}^n} d\omega &= \sum_{i=1}^n (-1)^i \int_{\mathbb{R}^n} D_i \omega_i \\ &= \sum_{i=1}^n (-1)^i \int_{(x^1, \ldots \widehat{x^i} \ldots, x^n) \in \mathbb{R}^{n-1}} \int_{x_i = -\infty}^{\infty} D_i \omega_i (x_1, \ldots, x_i, \ldots, x_n) \\ &= 0. \end{aligned}$$

8

Now define $i_n\colon \mathbb{R}^{n-1} \to \mathbb{R}^n$ by $i_n(x_1, \ldots, x_{n-1}) = (0, x_1, \ldots, x_{n-1})$, or, equivalently, $i_n(x_2, \ldots, x_n) = (0, x_2, \ldots, x_n)$. We define the lower half-space H_n by $H_n = \{x \in R^n \mid x^1 \le 0\}$. We will prove:

9 Proposition

Suppose that $\omega \in C^\infty(H_n, \mathcal{T}^{k-1}(\mathbb{R}^n))$ has compact support. Then

$$\int_{H_n} d\omega = \int_{R^{n-1}} i_n^* \omega. \tag{5}$$

Proof:

We let $\omega = \sum_{i=1}^n \omega_i \, \overline{dx^i}$ as in the proof of Proposition 8. Then $d\omega = \sum_{i=1}^n (-1)^{i-1} D_i \omega_i dx^1 \wedge \ldots \wedge dx^n$ as before. Then for $i > 1$,

$$\begin{aligned}\int_{H_n} D_i \omega_i &= \int_{(x_1, \ldots, \widehat{x_i}, \ldots, x_n) \in H_{n-1}} \int_{x_i = -\infty}^{\infty} D_i \omega_i(x_1, \ldots, x_n) \\ &= 0\end{aligned}$$

by the fundamental theorem of calculus. For $i = 1$ we can write

$$\begin{aligned}\int_{H_n} D_1 \omega_1 &= \int_{(x_2, \ldots, x_n) \in \mathbb{R}^{n-1}} \int_{x_1 = -\infty}^{0} D_1 \omega_1(x_1, \ldots, x_n) \\ &= \int_{(x_2, \ldots, x_n) \in \mathbb{R}^{n-1}} \omega_1(0, x_2, \ldots, x_n).\end{aligned}$$

Finally

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} i_n^* \omega &= \sum_{i=1}^n \int_{\mathbb{R}^{n-1}} (\omega_i \circ i_n) \, i_n^* \overline{dx^i} \\ &= \int_{\mathbb{R}^{n-1}} (\omega_1 \circ i_n) \, dx^1 \wedge \ldots \wedge dx^{n-1} \\ &= \int_{(x_2, \ldots, x_n) \in \mathbb{R}^{n-1}} \omega_1(0, x_2, \ldots, x_n),\end{aligned}$$

which completes the proof of the proposition. 9

2 Integration and partions of unity

2.1 Iterated integration

10 Lemma

Suppose that $f\colon [a,b]\times R \to \mathbb{R}$ *is continuous, where* $R \subset \mathbb{R}^n$*. Then* $\forall y \in R$, $\epsilon > 0$, $\exists \delta > 0$ *such that* $\forall x \in [a,b]$, $y' \in R$,

$$|y-y'|<\delta \implies |f(x,y)-f(x,y')|<\epsilon.$$

Proof:

Given $y \in R$, $\epsilon > 0$: For all $x \in [a,b]$ we can find an open interval $I_x \subset \mathbb{R}$ (with $x \in I_x$) and $\delta_x > 0$ such that if $(x',y') \in [a,b]\times\mathbb{R}$, and $x' \in I_x$, and $|y'-y| < \delta_x$, then

$$|f(x',y')-f(x,y)|<\frac{\epsilon}{2},$$

and

$$|f(x',y)-f(x,y)|<\frac{\epsilon}{2},$$

and then

$$|f(x',y)-f(x',y')|<\epsilon.$$

We can find finitely many x_i such that $[a,b] \subset \bigcup_i I_{x_i}$; we then let $\delta = \min_i \delta_{x_i}$. Then any $x \in [a,b]$ lies in I_{x_i} for some i, so if $|y'-y| < \delta < \delta_{x_i}$, then $|f(x,y')-f(x,y)| < \epsilon$. $\boxed{10}$

We will use the usual notation $\int_a^b f(x)\,dx$ (where f may depend on other variables as well) as shorthand for the notation $\int_{x=a}^b f(x)$.

11 Theorem

Suppose that $f\colon [a,b]\times R \to \mathbb{R}$ *is continuous. Then* $F\colon R \to \mathbb{R}$ *defined by* $F(y) = \int_a^b f(x,y)\,dx$ *is continuous.*

Proof:

Given $y \in R$, and $\epsilon > 0$, we can find (by Lemma 10) $\delta > 0$ such that for all $y' \in R$, if $|y-y'| < \delta$, then for all $x \in [a,b]$,

$$\left|\int_a^b f(x,y)\,dx - \int_a^b f(x,y')\,dx\right| < \epsilon.$$

This shows that F is continous. 11

We can then inductively define the integral $\int_R f$, where $R \subset \mathbb{R}^n$ is a closed rectangle, and $f\colon R \to \mathbb{R}^n$ is continuous, by

$$\int_R f = \int_{x^1=a}^{b} \int_{R'} f,$$

where $R = [a, b] \times R'$; in other words,

$$\int_R f = \int_{x^1=a^1}^{b^1} \cdots \int_{x^n=a^n}^{b^n} f,$$

where $R = \prod_{i=1}^{n} [a^i, b^i]$.

We would like to be able to say that we can interchange the order of integration, so that, for example,

$$\int_{x=a}^{b} \int_{y=c}^{d} f(x, y) = \int_{y=c}^{d} \int_{x=a}^{b} f(x, y). \tag{6}$$

To this end, we will prove the following:

12 Lemma

Suppose that $f\colon [a, b] \times [c, d] \to \mathbb{R}$ is continuous. Let $Q\colon [a, b] \times [c, d] \to \mathbb{R}$ be defined by

$$Q(r, s) = \int_{y=c}^{s} \int_{x=a}^{r} f(x, y). \tag{7}$$

Then $D_1 Q(r, s) = \int_{y=c}^{s} f(r, y)$.

Proof:

We will show that $\forall \epsilon > 0\ \exists \delta > 0$ such that if $0 < |h| < \delta$ then

$$\left| \frac{1}{h} \big(Q(r + h, s) - Q(r, s) \big) - \int_{y=c}^{s} f(r, y) \right| < \epsilon.$$

If $s = c$ then the left hand side is always 0. Otherwise, given $\epsilon > 0$, we let δ be such that

$$|x - r| < \delta \implies |f(x, y) - f(r, y)| < \frac{\epsilon}{|s - c|}$$

(such a δ must exist by Lemma 10). Then for all $y \in [c, s]$, and $0 < |h| < \delta$,

$$\left| \frac{1}{h} \int_{x=r}^{r+h} f(x, y) - f(r, y) \right| < \frac{\epsilon}{s - c},$$

so

$$\left| \frac{1}{h} \int_{y=c}^{s} \int_{x=r}^{r+h} f(x, y) - \int_{y=c}^{s} f(r, y) \right| < \epsilon.$$

12

We can then show:

13 Theorem
Suppose that $f\colon [a, b] \times [c, d] \to \mathbb{R}$ is continuous. Then

$$\int_{x=a}^{b} \int_{y=c}^{d} f(x, y) = \int_{y=c}^{d} \int_{x=a}^{b} f(x, y). \tag{8}$$

Proof:
Let $P(r, s) = \int_{x=a}^{r} \int_{y=c}^{s} f(x, y)$, and let $Q(r, s)$ be as in Lemma 12. Then $P(a, s) = Q(a, s) = 0$ for all s, and $D_1 P(a, s) = \int_{y=c}^{s} f(r, y)$ by the Fundamental Theorem of Calculus, so $D_1 P(r, s) = D_1 Q(r, s)$ by Lemma 12. Therefore $P(r, s) = Q(r, s)$ for all $(r, s) \in [a, b] \times [c, d]$, which shows (8). 13