2002-II-4 Let $f$ and $g$ be diffeomorphism of $\mathbb{R}^3$, both fixing the origin, such that

$$df_0 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

and

$$dg_0 = \begin{pmatrix} 2 & -1 & -3 \\ 0 & 2 & -3 \\ 0 & 0 & 3 \end{pmatrix}$$

Show that there does not exist a diffeomorphism $\psi$ of $\mathbb{R}^3$, also fixing the origin, such that $\psi \circ f = g \circ \psi$.

2005-II-4 Let $A\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map such that the characteristic polynomial of $A$ is $\chi_A(t) = t^4 + 1$. How many $A$-invariant subspaces are there in $\mathbb{R}^4$? (A subspace $W$ is called $A$-invariant if $AW \subset W$).

2006-II-1 Suppose that $A$ and $B$ are, respectively, $3 \times 2$ and $2 \times 3$ matrices with complex coefficients, satisfying

$$AB = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Find BA.

2007-I-3 Let $A$ and $B$ be $n \times n$ invertible matrices with complex entries such that

$$AB = \lambda BA$$

for some scalar $\lambda$. Prove that $\lambda$ is an $n^{\text{th}}$ root of unity. If $\lambda$ is a primitive $n^{\text{th}}$ root of unity, prove that the characteristic polynomials of $A$ and $B$ both have the form

$$\chi(t) = t^n + \text{constant}.$$

2008-I-4 Let $V$ denote a finite dimensional real vector space equipped with the inner product $\langle\ ,\ \rangle$; and let $\|\ \|$ denote the norm which comes from the inner product. Let $T\colon V \to V$ denote a linear transformation which satisfies $\|T(v)\| \leq \|T^*(v)\|$ for all $v \in V$. Show that $T$ is normal (i.e. show that $TT^* = T^*T$). (**Hint:** $tr(TT^* - T^*T) = 0$.)

2009-II-6 Let $V$ be a finite dimensional vector complex Hilbert space, i.e., a finite-dimensional complex vector space with Hermitian inner product. Prove that every linear operator $A\colon V \to V$ admits a decomposition $A = RU$, where the operator $R$ is Hermitian and non-negative, and the operator $U$ is unitary. Is such a decomposition unique?