2002-I-1 Suppose Ω is a horizontal strip $\{z \mid \operatorname{Im} z \in (a,b)\}$. Suppose f is holomorphic in Ω and $f(z+1) = f(z)$ for all $z \in \Omega$. Prove that f has an expansion

$$f(z) = \sum_{-\infty}^{\infty} c_n e^{2\pi n z}$$

which converges uniformly in $\{\operatorname{Im} z \in (a+\epsilon, b-\epsilon)\}$ for all $\epsilon > 0$.

2002-I-4 Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers tending toward 0. By definition, the infinite product $\prod_{n=0}^{\infty}$ converges iff $\sum_{n=0}^{\infty} \log(1+a_n)$ converges to a finite value. The product is said to converge *absolutely* if the corresponding sum does.

(a) Show that $\prod_{n=0}^{\infty}(1+a_n)$ is absolutely convergent iff $\sum |a_n| < \infty$.

(b) Show that for $\operatorname{Re}(s) > 1$, the infinite product $\prod_{p \text{ prime}} \frac{1}{1-p^{-s}}$ is absolutely convergent.

(c) For $\operatorname{Re}(s) > 1$, let $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. Show that the infinite product from (b) equals $\zeta(s)$ whenever $\operatorname{Re}(s) > 1$. (Hint: expand $(1-p^{-s})^{-1}$ in a geometric series for each p.)

2003-I-1 Let f be a holomorphic function on the punctured plane $0 < |z| < \infty$. Assume that there exists a positive constant C and a real constant N such that

$$|f(z)| \leq C|z|^N \text{ for } 0 < |z| < \frac{1}{2}.$$

Show that $z = 0$ is either a pole or a removable singularity for f and find a bound for $\operatorname{ord}_0 f$, the order of the function f at 0.

2003-I-2 (a) State any form of Picard's Theorem.

(b) Let $n \geq 2$ be an integer. Show that there are no nowhere vanishing and nonconstant entire functions f and g that satisfy

$$f^n + g^n = 1.$$

(c) Assume now that $n > 2$. What are all the solutions f and g in the ring of entire functions that satisfy

$$f^n + g^n = 1?$$

Hint: transform the problem to the setting of meromorphic functions on the complex line.

2003-II-1 Suppose f is a meromorphic function on the plane which is holomorphic on the unit disk and has one simple pole only on the unit circle at the point z_0. Let $\sum_{n=0}^{\infty} a_n z^n$ be the power series for f at the origin. Show that $\lim_{n\to\infty} a_n z_0^n$ exists.

2004-I-3 Suppose $\Omega = \{z \in \mathbb{C} \mid |z-1| > 1 \text{ and } |z+1| > 1\}$. let U be a harmonic function on Ω with boundary values $U = 0$ on $\{|z-1| = 1\}$ and $U = 1$ on $\{|z+1| = 1, z \neq 0\}$. Find the value of $U(4)$.

2004-I-6 Let g denote a holomorphic function on a subset of the complex plane given by $|z| < r$, where r is a fixed real number satisfying $r > 1$. Suppose that $g(z) \leq 1$ holds for all $|z| \leq 1$.

(a) Show that for all $t \in \mathbb{C}$ with $|t| < 1$, the equation

$$z = tg(z)$$

has a unique solution $z = s(t)$ in the disc $|z| < 1$.

(b) Show that $t \to s(t)$ is a holomorphic function on the disc $|t| < 1$.

2005-I-2 Suppose T is an equilateral triangle of side length 1 and let $f\colon T \to D = \{z \mid |z| < 1\}$ be a conformal map which sends the center c of the triangle to the origin in the disk (this is a 1-to-1, onto holomorphic map between the interiors which you may assume extends continuously to the boundary). What is the radius of convergence of the power series of f around the point c?

2006-I-2 Let

$$f(z) = \sum_{n=0}^{\infty} z^{2^n}, |z| < 1.$$

Prove that for every ζ satisfying $\zeta^{2^l} = 1$ for some $l \in \mathbb{N}$,

$$\lim_{r\to 1^-} f(r\zeta) = \infty.$$

2006-I-4 Let $p > 2$ be a prime number. Find the absolute value of

$$S = \sum_{n=0}^{p-1} \exp\left\{\frac{2\pi i n^2}{p}\right\}.$$

2006-II-2 Show that there is no meromorphic fumnction satisfying

$$f(z+1) = f(z), \quad f(z+i) = f(z)$$

for all $z \in \mathbb{C}$, whose only poles are simple poles at the points $m + ni$, $m, n \in \mathbb{Z}$.

2007-I-2 Find an explicit holomorphic bijection $\zeta = f(z)$ between the crescent $\{z \in \mathbb{C} \mid |z| \leq 1, |z+1| \geq \sqrt{2}\}$ and the disk $\{\zeta \in \mathbb{C} \mid |\zeta| \leq 1\}$.

2007-II-1 Let $f(z)$ be a continuous complex-valued function on the closed unit disk $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$. If f is holomorphic in the interior $|z| < 1$ of D, and real-valued on the boundary circle $|z| = 1$, show that f must be constant.

2008-I-6 Let Ω denote the open subset $\{x+iy \mid -1 < y < 1\}$ of the complex plane, and let $f\colon \Omega \to \Omega$ denote a holomorphic function which satisfies. $f(-1) = -1$ and $f(1) = 1$. Show that f is the identity map.

2008-I-6 Let Ω denote the open subset $\{x+iy \mid -1 < y < 1\}$ of the complex plane, and let $f\colon \Omega \to \Omega$ denote a holomorphic function which satisfies $f(-1) = -1$ and $f(1) = 1$. Show that f is the identity map.

2008-II-3 Compute the real integral $\int_{-\infty}^{\infty} \frac{1}{1+x^4}\, dx$.

2008-I-3 Let $S = [-1, 1]$ denote the closed line segment from -1 to 1 and suppose f is the Riemann map from the unit disk to the domain

$$\Omega = \{z : dist(z, S) < 1\}.$$

What is the smoothness of $\partial\Omega$ and what is the radius of convergence of the power series of f around the origin?

2010-I-1 Let $f\colon \mathbb{C} - \{0\} \to \mathbb{C}$ be a holomorphic function on the punctured complex plane, and suppose that $f(2z) = f(z)$ for all $z \neq 0$. Prove that f is constant.

2010-II-6 A fractional linear transformation maps the annulus $r < |z| < 1$ (where $r > 0$) onto the domain bounded by the two circles $|z - \frac{1}{4}| = \frac{1}{4}$ and $|z| = 1$. Find r.