2002-I-5 Let $\mathcal{H}$ be a Hilbert space and A a bounded linear transformation on $\mathcal{H}$. Let $P = \operatorname{Id} + A^*A$.

(a) Show that $\|u\| \leq \|Pu\|$.

(b) Show that P^{-1} exists as a bounded linear transformation on H.

2002-II-6 Let $S^1 = \mathbb{R}/\mathbb{Z}$. For $f, g \in L^1(S^1)$, let $f * g(x) = \int_0^1 f(x-y)g(y)\,dy$. Show that there does not exist $f \in L^1(S^1)$ with the property that $f * g = g$ for al $g \in L^1(S^1)$.

2003-I-3 Let X and Y be Banach spaces and let $L(X,Y)$ be the space of bounded linear maps from X to Y.

(a) Define the operator norm on $L(X,Y)$ and show that it makes $L(X,Y)$ into a Banach space.

(b) Show that the set of invertible maps INV in $L(X,Y)$ is an open subset of $L(X,Y)$.

2003-II-2 Let X be a real vector space with inner product $\langle , \rangle$.
Show that closed balls about the origin in X are strictly convex; that is, show that if $x \neq y$ are in a ball of radius r, then any point on the open line segment between them is in a ball of smaller radius.
Is this true for any vector space with a norm?

2003-II-3 Show that if (X,d) is a complete, non-empty metric space and $T\colon X \to X$ is a mapping such that the n-th iterate T^n is a contraction mapping, then T has a unique fixed point. (Recall that a map S is a contraction if there is a $0 < \lambda < 1$ so that $d(Sx, Sy) < \lambda d(x,y)$ for all $x, y \in X$). (N.B.: T is not necessarily a contraction).

2004-I-1 Prove that $C^\infty(I)$ is not dense in $Lip(I)$. Here I denotes the closed interval $I = \{x \in \mathbb{R} : -1 \leq x \leq 1\}$ and $Lip(I)$ denotes the Banach space consisting of all real-valued functions $f\colon I \to \mathbb{R}$ such that

$$|f_{Lip}| < \infty$$

where

$$|f_{Lip}| = \sup_{x\in I} |f(x)| + \sup_{x,y\in I, x\neq y} |f(x) - f(y)|/|x-y|.$$

(Hint: Define $g\colon I \to \mathbb{R}$ by $g(x) = x$ if $x > 0$, and $g(x) = 0$ if $x \leq 0$. Show that g can not be approximated by $f \in C^\infty(I)$ in the $| \quad |_{Lip}$ norm.)

2004-II-4 Let $f\colon X \to Y$ denote a linear map between normed linear spaces.

(a) Prove or give a counterexample: If f is bounded then its null-space is closed.

(b) Prove or give a counterexample: If the null-space of f is closed then f is bounded.

2005-I-1 Suppose that $\{f_n\}$ is a sequence of orthogonal functions on $[0,1]$ with $\int_0^1 |f_n(x)|^2\,dx \leq 1$ for all n (orthogonal means $\int f_k f_j\,dx = 0$ whenever $j \neq k$. Let $S_n = \sum_{k=1}^n f_k$.

(a) Show that $\int |S_n|^2 \leq n$.

(b) Show that $\lim_{n\to\infty} \frac{1}{n^2} S_{n^2}(x) = 0$ Lebesgue almost everywhere.

2005-II-1 Suppose f is bounded, continuous, increasing, and concave up on $[0,1]$. Define

$$f_n(x) = \frac{f(x) - f(x - \frac{1}{n})}{1/n},$$

(where we define $f(t) = f(0)$ for $t < 0$).

(a) Show $(f(1-\frac{1}{n}) - f(0)) \leq \int_0^1 f_n(x)\,dx \leq (f(1) - f(0))$.

(b) Show that $g(x) = \lim_n f_n(x)$ exists for almost every $x \in [0,1]$.

(c) Show that $\int_0^1 g(x)dx = f(1) - f(0)$.

2005-II-2 Let $L\colon X \to Y$ be a continuous linear map from one Banach space to another. Assume that there exists a positive constant k such that $\|L(x)\| \geq k\,\|x\|$. Prove that the range of L is closed in Y.

2006-I-5 Let A be a bounded operator on a seperable Hilbert space $\mathcal{H}$ and let A^* be its adjoint operator, defined by $\langle Ax, y\rangle = \langle x, A^*y\rangle$ for all $x, y \in \mathcal{H}$.

(a) For an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ for $\mathcal{H}$, set $a_{ij} = \langle Ae_i, e_j\rangle$. Show that

$$\sum_{i,j=1}^{\infty} |a_{ij}|^2 = \sum_{i=1}^{\infty} \|Ae_i\|^2 = \sum_{i=1}^{\infty} \|A^*e_i\|^2,$$

understood as the equality in $[0, \infty]$.

(b) Show that if for some orthonormal basis $\{e_n\}$ for $\mathcal{H}$

$$\sum_{i,j=1}^{\infty} |a_{ij}|^2 < \infty,$$

then the series is convergent for all orthonormal bases for $\mathcal{H}$ and the sum is independent of the choice of the basis.

2007-I-6. (a) If $A_n \subset [0,1]$ is measurable for each $n \in \mathbb{N}$, and if $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, show that $\{x \mid x \in A_n \text{ for infinitely many } n\}$ has measure zero.

(b) Let us call a real number x *trendy* if it is the limit of a sequence of rational numbers p_j/q_j which converge so quickly that

$$\left| x - \frac{p_j}{q_j} \right| < \frac{1}{q_j^3}.$$

Use part (a) to prove that the set of trendy numbers has measure 0.

2007-II-6. Let $f(x)$ be a real-valued C^1 function on the interval $[0,1]$. Prove that

$$|f(y)| \leq \left[\int_0^1 |f(x)|^2 dx\right]^{1/2} + \left[\int_0^1 |f'(x)|^2 dx\right]^{1/2}$$

for all $y \in [0,1]$. (Hint: Consider $g(x) = f(x) - a$, where $a = \int_0^1 f(x)dx$.

2008-I-2 The usual Cantor set C is defined by $C = \bigcup_{i=1}^{\infty} C_i$, where C_1 is the unit interval $[0, 1]$, each C_i is a finite union of closed subintervals of $[0, 1]$, and C_{i+1} is obtained from C_i by deleting the middle third of each of the closed intervals in C_i. A subspace $X \subset [0, 1]$ is called a *Cantor* subspace if it is homeomorphic to C.

(a) Show that C is a Lebesgue measurable subset of $[0, 1]$; calculate its Lebesgue measure.

(a) Show that for any number $0 < \alpha < 1$ there is a Cantor subspace $X \subset [0, 1]$ which is Lebesgue measurable with Lebesgue measure equal to α.

2008-I-5 Let $(X, \mathcal{A}, \mu)$ be a measure space, i.e. X is a space and $\mathcal{A}$ is the set of measurable subsets of X and $\mu(A)$ is the measure of each set $A \in \mathcal{A}$. Let A_n, $n \in \mathbb{N}$, denote a collection of sets in $\mathcal{A}$ (where $\mathbb{N} = \{1, 2, 3, \ldots\}$) such that $\sum_{n=1}^{\infty} \mu(A_n) < \infty$. For each $x \in X$ let N_x denote the cardinality of $\{n \in \mathbb{N} \mid x \in A_n\}$. Suppose that

$$\int_X 2^{N_x} \, d\mu(x) < \infty.$$

Prove that

$$\sum_{F \subset \mathbb{N}, |F| < \infty} \mu\Big(\bigcap_{n \in F} A_n\Big) < \infty.$$

2008-II-4 Suppose that $f \colon \mathbb{R}^n \to \mathbb{R}$ is an infinitely differentiable real valued function with compact support on n-dimensional Euclidean space.

(a) Show that there is at most one possible solution $u \colon \mathbb{R}^n \to R$ of the equation $\nabla u = f$ (where $\nabla = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$), if we insist that u is also infinitely differentiable and has compact support.

(b) Show that any solution u of $\nabla u = f$ which is infinitely differentiable and has compact support must equal $f * N$, where N is the Newtonian potential. (Recall that $N(x) = \frac{log|x|}{2\pi}$ if $n = 2$ and $N(x) = \frac{|x|^{2-n}}{(2-n)\omega_n}$ if $n \neq 2$, where ω_n equals the surface are of the unit sphere in $\mathbb{R}^n$. Recall also that $f * N(x) = \int_{\mathbb{R}^n} f(y) N(x - y) \, dy$.)

2009-I-4 Suppose that u is a harmonic function on all of $\mathbb{R}^n$.

(a) Show that if $u \in L^1(\mathbb{R}^n)$, then $u \equiv 0$.

(b) Say $1 < p < \infty$. Show that if $u \in L^p(\mathbb{R}^n)$, then $u \equiv 0$.

2009-II-1

(a) Say $1 < p < \infty$. Exhibit, with proof, a function $f \in L^1(0,1)$ which is not in $L^p(0,1)$.

(b) Exhibit, with proof, a function $F \in L^1(0,1)$ which is not in $L^p(0,1)$ for any $p > 1$. (Hint: try to define F piecewise, using the functions of (a).)

2009-II-2 Suppose that $f\colon \mathbb{R} \to \mathbb{R}^2$ is a continuous, 1-1 map so that

$$1 \leq \frac{|f(x) - f(y)|}{|x-y|^\alpha} \leq C,$$

for all $x, y \in \mathbb{R}$, some $0 < \alpha \leq 1$ and some $C < \infty$. Show that $X = f(\mathbb{R})$ has zero Lebesgue measure area. (Hint: show that every disk contains a subdisk of comparable size which misses X).

2010-I-4 Suppose that $a, b, c \in \mathbb{R}$, with $b^2 - ac > 0$. Let f be a given smooth function on $\mathbb{R}^2$. Show that the differential equation

$$a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} = f$$

has a smooth solution u on $\mathbb{R}^2$.

2010-II-4 Let $D \subset \mathbb{R}^n$ be a bounded domain. Define the Hölderspace $C^{0,\alpha}$ to consist of all continuous functions for which the norm

$$\|u\|_{C^{0,\alpha}(D)} = \sup_{x \in D} |u(x)| + \sup_{x,y \in D} x \neq y \frac{|u(x) - u(y)|}{|x-y|^\alpha}$$

is finite.

(a) Show that $C^{0,\alpha}$ is a Banach space.

(b) Show that the inclusion $C^{0,\alpha}(D) \hookrightarrow C^{0,\beta}(D)$ of Hölder spaces is compact if $\beta < \alpha$, that is, the image of every bounded sequence has a convergent subsequence.