

2002-I-6 Define forms θ_1 and θ_2 on the torus $T^2 = S^1 \times S^1$ by pulling back forms from the two projections onto circles. Let M be the compact oriented three-manifold with boundary T^2 . Show that $\int_{T^2} \theta_1 \wedge \theta_2$ is non-zero, and as a result that θ_1 and θ_2 cannot both extend to closed forms on M .

2002-II-1 Let M and N be smooth, connected n -dimensional manifolds, and let $f: M \rightarrow N$ be an immersion. (That is, assume that the derivative of f always sends non-zero tangent vectors to non-zero tangent vectors.) If M is compact and non-empty, show that N is compact, and that f is a covering map.

2002-II-5 Let θ be a closed 1-form on a compact manifold M without boundary. Further suppose that $\theta \neq 0$ at each point of M . Prove that $H_{\text{de Rham}}^1(M) \neq 0$.

2003-I-4 Let X be a smooth connected orientable n -manifold, and let $Y \subset X$ be a smooth, non-empty, compact orientable $(n - 1)$ -dimensional submanifold. Y admits a neighborhood in X diffeomorphic to $Y \times \mathbb{R}$, \mathbb{R} the real numbers (do not prove this).

Show that if $X \setminus Y$ is *connected*, then $H_{DR}^1(X) \neq 0$, where H_{DR}^1 denotes the first de Rham cohomology group. (Hint: consider a suitable differential form with compact support in a neighborhood of Y).

2003-II-5 Is the unit three-sphere S^3 diffeomorphic to the product of two other smooth manifolds of dimensions > 0 ?

2003-II-6 Let M be a smooth connected n -manifold whose universal cover is diffeomorphic to the n -sphere S^n .

(a) Prove the following statement: *if $\psi: S^n \rightarrow S^n$ is any diffeomorphism without fixed points, then it is smoothly homotopic to the antipodal map.*

(b) Show that if n is odd, then M is orientable.

(c) Show that if n is even, then $|\pi_1(M)| \leq 2$.

2004-I-5 Recall that a connected sum $M\#N$ of two n -dimensional manifolds M, N is constructed as follows: cut out the interior of a closed n -dimensional ball from each manifold, and glue the resulting spaces together along their $(n - 1)$ -sphere boundaries.

- (a) Let $\mathbb{R}P^n$ denote n -dimensional real projective space. If $n > 2$ then compute the fundamental group of the connected sum $\mathbb{R}P^n\#\mathbb{R}P^n$.
- (b) If $n > 2$ then show that the universal covering space for the connected sum $\mathbb{R}P^n\#\mathbb{R}P^n$ is homeomorphic to the product $S^{n-1} \times \mathbb{R}$, and describe the action of the fundamental group from $\mathbb{R}P^n\#\mathbb{R}P^n$ on the universal cover.

2004-II-1 Let X denote the product of the 2-dimensional sphere with the circle; let Y denote the 3-dimensional torus; and let $f: X \rightarrow Y$ denote an infinitely differentiable map. Show that for any 3-form ω on Y we have that

$$\int_X f^*(\omega) = 0.$$

2004-II-3 Let M denote a C^∞ n -dimensional manifold.

- (a) Let ϕ denote a C^∞ n -form on M which is nowhere zero. Show that every $x_0 \in M$ has a neighborhood on which one can find local C^∞ coordinates x^1, x^2, \dots, x^n such that

$$\phi = dx^1 \wedge dx^2 \wedge \dots \wedge dx^n.$$

- (b) Let ψ denote a closed C^∞ $(n - 1)$ -form on M which is nowhere zero. Show that local C^∞ coordinates x^1, x^2, \dots, x^n can be found near each point $x_0 \in M$ so that

$$\psi = dx^2 \wedge dx^3 \wedge \dots \wedge dx^n.$$

2005-I-5 Let $GL(n, \mathbb{C})$ denote the set of invertible $n \times n$ matrices with complex coefficients. This may be thought of as a subset of Euclidean $2n^2$ -space, and so comes equipped with a standard topology.

- (a) Show that $GL(n, \mathbb{C})$ is connected.
- (b) Show that the fundamental group of $GL(n, \mathbb{C})$ is infinite.

2005-I-6 Any degree-one map $f: M \rightarrow N$ between closed manifolds induces a surjection on the fundamental group.

2005-II-6 Let M be a 3-manifold with boundary such that the boundary $\partial M = T^2 = S^1 \times S^1$. Define 1-forms θ_1 and θ_2 by pulling back the standard volume 1-forms on each circle by the projections onto the circle. Show that it is impossible to extend both θ_1 and θ_2 to closed forms on M .

2006-I-5 (a) For which values of t is the subset

$$\{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_n^2 - x_{n+1}^2 = t\}$$

a submanifold in \mathbb{R}^{n+1} ?

(b) Determine the topological type of this manifold for each value of t .

2006-II-4 (a) Find a basis of the de Rham cohomology group $H^1(T^2)$, where T^2 is the two-dimensional torus.

(b) Find of basis of the de Rham cohomology group $H^1(K)$, where K is the Klein bottle.

2006-II-6 Let M be a compact, orientable, connected n -dimensional submanifold of \mathbb{R}^{n+1} without the boundary. Prove that $\mathbb{R}^{n+1} \setminus M$ has exactly two components. (Hint: What is $H^0(\mathbb{R}^{n+1} \setminus M)$?)

2007-I-5 Let M and N be two smooth manifolds, and let f and g be smooth maps from M to N which are *smoothly homotopic*, in the sense that there exists a smooth map $F: M \times \mathbb{R} \rightarrow N$ such that $f(x) = F(x, 0)$ and $g(x) = F(x, 1)$ for all $x \in M$. If ω is any closed differential form on N , prove that $f^*\omega - g^*\omega$ is an exact differential form on M .

2007-II-3 If Σ is any surface (2-manifold), let $T\Sigma$ denote its tangent bundle, and let $V\Sigma \subset T\Sigma$ be the set of *non-zero* tangent vectors.

- (a) If Σ is orientable, and if there is a continuous vector field ξ on Σ such that $\xi \neq 0$ everywhere, show that $V\Sigma$ is homeomorphic to $\Sigma \times S^1 \times \mathbb{R}$.
- (b) If $\Sigma = S^2$, show that $V\Sigma$ is homeomorphic to $\mathbb{R}P^3 \times \mathbb{R}$.
- (c) Be a fundamental-group argument, use (a) and (b) to show that every continuous vector field on S^2 must have a zero.

2007-II-5 A 2-form ω is said to be *decomposable* if there exist 1-forms ϕ and ϑ such that $\omega = \phi \wedge \vartheta$. If ω is a *closed* decomposable 2-form on an n -manifold M , and if $p \in M$ is a point at which $\omega \neq 0$, use the Frobenius Theorem to prove that

$$\omega = dx^1 \wedge dx^2$$

in some coordinate system (x^1, \dots, x^n) on a neighborhood of p .

2008-I-1 Show that 2-dimensional real projective space P^2 can be topologically embedded in 4-dimensional Euclidean space \mathbb{R}^4 , i.e. that P^2 is homeomorphic to a subspace of \mathbb{R}^4 .

2008-II-1 Let P^2 denote 2-dimensional real projective space and let $p, q \in P^2$. Show the the subset $X = (P^2 \times q) \cup (p \times P^2)$ of $P^2 \times P^2$ is not a retract of $P^2 \times P^2$.

2008-II-5 Let G denote the cyclic group of order 3 and let M denote a smooth manifold. Let $G \times M \rightarrow M$ denote a smooth fixed point free group action with orbit space denoted by N ; note that N is also a smooth manifold and has M a 3-sheeted covering space (you do not have to show this). Let $H^1(M)$, $H^1(N)$ denote the de Rham cohomology (vector) spaces for M , N in dimension 1.

Suppose that $H^1(M) = \mathbb{R}$. Then show that $H^1(N) \neq 0$.

2009-I-1 Let M^k be a smooth compact k -manifold, and let $F: M \rightarrow S^n$ be a smooth map, where $n > k$. Prove that F is homotopic to a constant map.

2009-I-2 Find the fundamental group of the space of complex polynomials $az^2 + bz + c$ of degree two with distinct roots.

2009-II-3 A symplectic form $\omega \in \Omega^2(M^{2n})$ is a closed 2-form such that

$$\omega^{\wedge n} = \underbrace{\omega \wedge \cdots \wedge \omega}_n \neq 0,$$

at all points. If M is compact, then show that $[\omega] \neq 0 \in H^2(M)$.

2009-II-4 Let $f: S^{2n-1} \rightarrow S^n$ ($n = 2, 3, \dots$) be a smooth map, and let $\omega \in \Omega^n(S^n)$ be such that $\int_{S^n} \omega = 1$. Show that $f^*\omega$ is exact, and if $f^*\omega = d\alpha$, then

$$\int_{S^{2n-1}} \alpha \wedge d\alpha$$

is well-defined (independent of the choice of ω and α). (Remark: this is the Hopf invariant of f).

2010-I-2 Consider the differential 1-form $\phi = dx^1 + x^2 dx^3$ on \mathbb{R}^3 . Is it possible to find a smooth coordinate system (y^1, y^2, y^3) on a neighborhood of the origin such that $\phi = f dy^1$ in these new coordinates, for some smooth function $f(y^1, y^2, y^3)$? Support your answer with a proof.

2010-I-3 Let $f: S^1 \times S^1 \rightarrow \mathbb{R}P^2$ be a smooth map from the 2-torus to the projective plane. Prove that f cannot be an immersion.

2010-II-2 Let M be a smooth manifold and let $X, Y \subset M$ be two smooth submanifolds. We say that X and Y are transverse if, at each point $p \in X \cap Y$, the tangent spaces of X and Y together span the tangent space of M :

$$T_p M = T_p X + T_p Y.$$

If X and Y are transverse submanifolds of M , prove that $Z = X \cap Y$ is also a submanifold of M .

2010-II-3 Let S^4 denote the unit sphere in \mathbb{R}^5 , and let G be the finite group of homeomorphisms of the 4-manifold generated by

$$(x_1, x_2, x_3, x_4, x_5) \rightarrow (-x_2, x_1, -x_4, x_3, x_5).$$

Notice that G fixes the two points $p_+, p_- \in S^4$ given by

$$p_{\pm} = (0, 0, 0, 0, \pm 1).$$

- Show that G acts freely on $M = S^4 - \{p_+, p_-\}$.
- Calculate the fundamental group of the quotient space M/G .
- Prove that the quotient space S^4/G is simply connected.