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Introduction to Linear Algebra
MAT 211, Spring 2009

Exercises for Midterm 2

1. A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ is given by $T(x, y) = (x, x + y, 2y)$. Find the matrix of T with respect to the bases $\mathcal{A} = \{(1, 3), (4, -1)\}$ in $\mathbb{R}^2$ and $\mathcal{B} = \{(1, 0, 0), (0, 2, 0), (0, 0, -1)\}$ in $\mathbb{R}^3$.

Answer: $T_{\mathcal{A},\mathcal{B}} = \begin{pmatrix} 1 & 4 \\ 2 & 2/3 \\ -6 & -2 \end{pmatrix}$

2. Consider the linear subspace $V = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$ with the two bases:

$$\mathcal{A} = \{\overline{v}_1 = (1, -1, 0),\ \overline{v}_2 = (1, 1, -2)\} \quad \text{and} \quad \mathcal{B} = \{\overline{w}_1 = (1, 0, -1),\ \overline{w}_2 = (0, 1, -1)\}.$$

a) Find coordinates of $\overline{w}_1$ and $\overline{w}_2$ with respect to the basis $\mathcal{A}$.
b) Find coordinates of $\overline{v}_1$ and $\overline{v}_2$ with respect to the basis $\mathcal{B}$.
c) Find the matrices of the base change $\mathcal{B} \to \mathcal{A}$ and $\mathcal{A} \to \mathcal{B}$.

Answer: a) (1/2,1/2) and (-1/2,1/2)
b) (1,-1) and (1,1)
c) $S_{\mathcal{B}\to\mathcal{A}} = \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}$, $S_{\mathcal{A}\to\mathcal{B}} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

3. Let $T : \mathcal{P}_3 \to \mathcal{P}_2$ be a transformation defined by the formula $p(x) \mapsto xp''(x) + p'(x)$.
a) Show that T is linear.
b) Find the matrix of T with respect to the standard bases in $\mathcal{P}_3$ and $\mathcal{P}_2$.
c) Find the matrix of T with respect to the basis $\{1,\ x + 1,\ (x + 1)^2,\ (x + 1)^3\}$ in $\mathcal{P}_3$ and $\{1,\ x + 1,\ (x + 1)^2\}$ in $\mathcal{P}_2$.
d) Find bases in the kernel and image of T.

Answer: a) use the definition of a linear transformation

b) $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 9 \end{pmatrix}$

c) $\begin{pmatrix} 0 & 1 & -2 & 0 \\ 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 9 \end{pmatrix}$

d) $\mathrm{Im}T = \mathcal{P}_2,\ \mathrm{Ker}T = \mathrm{span}\{1\}$

4. Show that all upper triangular 2×2 matrices form a subspace of the vector space M_2 of all square 2×2 matrices. Find a basis of this subspace.

Answer: The sum of two upper triangular matrices is obviously an upper triangular matrix and the product of an upper triangular matrix by a real number is an upper triangular matrix. It means that the set of upper triangular matrices is closed with respect to linear operations and is a subspace.

A basis is $\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$.

5. Let $T : U_2 \to U_2$ be the linear transformation in the space of upper triangular 2×2 matrices defined by the formula

$$T : M \mapsto \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}^{-1} M \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}.$$

Find the matrix of T with respect to the basis

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$$

of U_2. Is T an isomorphism?

Answer: The matrix is $T_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. T is an isomorphism.

6. Let $W = \mathrm{span}\{(1, 1, 0, 0),\ (0, 0, 1, 1)\} \subset \mathbb{R}^4$. Find a basis in the orthogonal complement $W^{\perp}$.

Answer: $\{(1, -1, 0, 0), (0, 0, 1, -1)\}$

7. Find the orthogonal projection of the vector $\overline{v} = (1, 1, 1) \in \mathbb{R}^3$ onto the subspace of $\mathbb{R}^3$ which is spanned by the vectors $\overline{u}_1 = (1, -1, 0)$ and $\overline{u}_2 = (1, 1, -2)$.

Answer: $\overline{0}$

8. Let W be a subspace of $\mathbb{R}^4$ which is defined by

$$W = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \,|\, x_1 + x_3 + x_4 = 0,\ x_2 - x_3 = 0\}.$$

Find an orthonormal basis in W and an orthonormal basis in $W^{\perp}$. Find the orthogonal projection of the vector $\overline{v} = (3, -2, 0, 3)$ onto W.

Answer: $\{\frac{1}{\sqrt{2}}(1, 0, 0, -1),\ \frac{1}{\sqrt{10}}(1, 0, 0, -1)\},\ \{\frac{1}{\sqrt{3}}(1, 0, 0, 1),\ \frac{1}{\sqrt{15}}(1, 3, -2, 1)\},\ \sqrt{10}(1, -2, -2, 1)$

9. Let $\mathcal{P}_2$ be a vector space of polynomials of degree ≤ 2 with the inner product defined by the formula $< p, q >= \int_{-1}^{1} p(x)q(x)\,dx$.

a) Find an orthogonal basis of $\mathcal{P}_2$ applying the Gram-Schmidt orthogonalization to the standard basis $\{1, x, x^2\}$ of $\mathcal{P}_2$.
b) Verify the Cauchy-Schwarz inequality for $p(x) = 1 - 2x$ and $q(x) = x^2$.
c) Verify the triangle inequality for $p(x) = 1 - 2x$ and $q(x) = x^2$.
d) Find a polynomial $r(x)$ of degree 1 which is orthogonal to $p(x) = 1 - 2x$. Verify the Pythagorean theorem for p and r.

Answer: a)$\{1,\ x,\ x^2 - 1/3\}$

b)$\frac{2}{3} \leq \sqrt{\frac{14}{3}}\sqrt{\frac{2}{5}}$

c)$\sqrt{\frac{8}{3}} \leq \sqrt{\frac{14}{3}} + \sqrt{\frac{2}{5}}$

d)$r(x) = x + 2/3$ (for example). $\frac{56}{9} = \frac{14}{3} + \frac{14}{9}$

10. Show that the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by the formula

$$T(x, y, z) = \frac{1}{3}(x - 2y - 2z,\ -2x + y - 2z,\ -2x - 2y + z)$$

is orthogonal. Is T invertible? If so, find the inverse.

Answer: The standard matrix of T is $A = \begin{pmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{pmatrix}$. This matrix is orthogonal since its columns (rows) are orthonormal. T is invertible, the inverse is a transformation given by A^T.