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Introduction to Linear Algebra
MAT 211
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Solutions for Midterm 2

Problem 1. A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$T(x, y) = (-x + 2y,\ 3x - y).$$

Find a matrix of T with respect to the basis $\mathcal{B} = \{(2, 1),\ (1, 1)\}$ in $\mathbb{R}^2$. Is T an isomorphism? Explain!

Solution. A matrix of T with respect to the basis $\mathcal{B} = \{(2, 1),\ (1, 1)\}$ is

$$T_{\mathcal{B}} = \begin{pmatrix} | & | \\ T(2,1) & T(1,1) \\ | & | \end{pmatrix},$$

where the coordinates of the column vectors are given in the basis $\mathcal{B}$. We calculate

$$T(2, 1) = (0, 5) = -5 \cdot (2, 1) + 10 \cdot (1, 1)$$
$$T(1, 1) = (1, 2) = -1 \cdot (2, 1) + 3 \cdot (1, 1)$$

and get the matrix:

$$T_{\mathcal{B}} = \begin{pmatrix} -5 & -1 \\ 10 & 3 \end{pmatrix}.$$

Another way to solve the problem is to consider the composition

$$\mathbb{R}^2 \xrightarrow[S_{\mathcal{B}\to St}]{id} \mathbb{R}^2 \xrightarrow[T_{St}]{T} \mathbb{R}^2 \xrightarrow[S_{St\to\mathcal{B}}]{id} \mathbb{R}^2$$

of T and the identity transformations of $\mathbb{R}^2$. The matrix of the composition is the product of three matrices:

$$T_{\mathcal{B}} = S_{St\to\mathcal{B}} \cdot T_{St} \cdot S_{\mathcal{B}\to St} = S^{-1}\, T_{\mathcal{B}}\, S,$$

where $S = S_{\mathcal{B}\to St}$ is the transition matrix from basis $\mathcal{B}$ to the standard basis and $T_{St} = \begin{pmatrix} -1 & 2 \\ 3 & -1 \end{pmatrix}$ is the standard matrix of T. So

$$T_{\mathcal{B}} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} -1 & 2 \\ 3 & -1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -5 & -1 \\ 10 & 3 \end{pmatrix}.$$

T is an isomorphism since $\det T = -5 \neq 0$.

Answer: $\begin{pmatrix} -5 & -1 \\ 10 & 3 \end{pmatrix}$, T is an isomorphism.

Problem 2. Show that the set of traceless 2×2 matices $U = \{A \in M_2 \,|\, \mathrm{tr} A = 0\}$ is a subspace of the space of 2×2 matices M_2. Find a basis of U.

Solution. Let A and B be two arbitrary matrices from U and k be a real number. Then $\mathrm{tr}\,(A+B) = \mathrm{tr}\,A + \mathrm{tr}\,B = 0 + 0 = 0$ and $\mathrm{tr}\,(kA) = k\,\mathrm{tr}\,A = 0 \cdot 0 = 0$. This shows that U is closed under matrix addition and multiplication by a scalar. Hence U is a subspace of M_2.

Any traceless matrix A can be written (in a unique way) as

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix},$$

where a, b and c are arbitrary real numbers. So

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and a basis of U is

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}.$$

Answer: $\boxed{\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}}$

Problem 3. Let $\mathcal{P}_2$ be the vector space of polynomials of degree ≤ 2 equipped with the inner product

$$< p, q >= \int\limits_{-1}^{1} p(x)q(x)\,dx.$$

Let W be a subspace of $\mathcal{P}_2$ generated by the polynomial $x-1$. Find an **orthogonal** basis of the orthogonal complement of W.

Solution. Let $W = \text{span}\{x-1\}$. Tnen $W^{\perp} = \{p(x) = a+bx+cx^2 \in \mathcal{P}_2 \mid\ < p(x),\, x-1 >= 0\}$. Since

$$< p(x),\, x-1 >= \int\limits_{-1}^{1} (a+bx+cx^2)(x-1)\,dx = 2\int\limits_{0}^{1} -a+(b-c)x^2\,dx = 2\left(-a+\frac{b-c}{3}\right),$$

the condition $< p(x),\, x-1 >= 0$ implies $a = \dfrac{b-c}{3}$ and the orthogonal complement $W^{\perp}$ consists of polynomials of form

$$p(x) = \frac{b-c}{3} + bx + cx^2 = b\left(\frac{1}{3}+x\right) + c\left(-\frac{1}{3}+x^2\right).$$

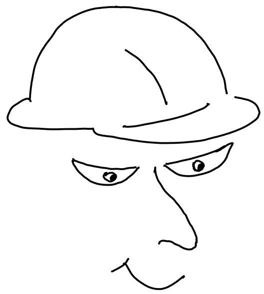
So $W^{\perp} = \text{span}\left\{\dfrac{1}{3}+x,\ -\dfrac{1}{3}+x^2\right\} = \text{span}\left\{1+3x,\ -1+3x^2\right\}$. Since

$$< 1+3x,\, -1+3x^2 >= \int\limits_{-1}^{1} (1+3x)(-1+3x^2) = 2\int\limits_{0}^{1} -1+3x^2 = 0,$$

the polynomials $1+3x$ and $-1+3x^2$ are orthogonal, and an orthogonal basis of $W^{\perp}$ is $\{1+3x,\ -1+3x^2\}$.

Answer: $\boxed{\{1+3x,\ -1+3x^2\}}$

Problem 4.



The former secret agent is now a contractor. His company works in the Euclidean space $\mathbb{R}^3$. He has got a tool called the Gram-Schmidt orthogonalization and a basis $\{(-1,0,1),\ (1,1,3),\ (1,3,-1)\}$ of $\mathbb{R}^3$. Help him to construct an **orthonormal** basis of $\mathbb{R}^3$ out of the given basis!

We use Gram-Schmidt orthogonalization to make an orthogonal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. Here are formulae:

$$\mathbf{u}_1 = \mathbf{v}_1$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \mathrm{pr}_{\mathbf{u}_1}\mathbf{v}_2 = \mathbf{v}_2 - \frac{< \mathbf{u}_1, \mathbf{v}_2 >}{< \mathbf{u}_1, \mathbf{u}_1 >}\mathbf{u}_1$$

$$\mathbf{u}_3 = \mathbf{v}_3 - \mathrm{pr}_{\mathbf{u}_1}\mathbf{v}_3 - \mathrm{pr}_{\mathbf{u}_2}\mathbf{v}_3 = \mathbf{v}_3 - \frac{< \mathbf{u}_1, \mathbf{v}_3 >}{< \mathbf{u}_1, \mathbf{u}_1 >}\mathbf{u}_1 - \frac{< \mathbf{u}_2, \mathbf{v}_3 >}{< \mathbf{u}_2, \mathbf{u}_2 >}\mathbf{u}_2.$$

In our case, $\mathbf{v}_1 = (-1,0,1),\ \ \mathbf{v}_2 = (1,1,3),\ \ \mathbf{v}_3 = (1,3,-1)$. We substitute the given data:

$$\mathbf{u}_1 = (-1,0,1)$$

$$\mathbf{u}_2 = (1,1,3) - \frac{(-1,0,1)\cdot(1,1,3)}{(-1,0,1)\cdot(-1,0,1)}(-1,0,1) = (1,1,3) - \frac{2}{2}(-1,0,1) = (2,1,2)$$

$$\mathbf{u}_3 = (1,3,-1) - \frac{(-1,0,1)\cdot(1,3,-1)}{(-1,0,1)\cdot(-1,0,1)}(-1,0,1) - \frac{(2,1,2)\cdot(1,3,-1)}{(2,1,2)\cdot(2,1,2)}(2,1,2)$$

$$= (1,3,-1) + (-1,0,1) - \frac{1}{3}(2,1,2) = (-\frac{2}{3}, \frac{8}{3}, -\frac{2}{3}) = -\frac{2}{3}(1,-4,1).$$

The constructed basis $\{\mathbf{u}_1 = (-1,0,1), \mathbf{u}_2 = (2,1,2), \mathbf{u}_3 = -\frac{2}{3}(1,-4,1)\}$ is orthogonal. Let us normalize it.

$$\mathbf{e}_1 = \frac{\mathbf{u}_1}{||\mathbf{u}_1||} = \frac{\mathbf{u}_1}{\sqrt{< \mathbf{u}_1, \mathbf{u}_1 >}} = \frac{(-1,0,1)}{\sqrt{(-1,0,1)\cdot(-1,0,1)}} = \frac{1}{\sqrt{2}}(-1,0,1)$$

$$\mathbf{e}_2 = \frac{\mathbf{u}_2}{||\mathbf{u}_2||} = \frac{\mathbf{u}_2}{\sqrt{< \mathbf{u}_2, \mathbf{u}_2 >}} = \frac{(2,1,2)}{\sqrt{(2,1,2)\cdot(2,1,2)}} = \frac{1}{3}(2,1,2)$$

$$\mathbf{e}_3 = \frac{\mathbf{u}_3}{||\mathbf{u}_3||} = \frac{\mathbf{u}_3}{\sqrt{< \mathbf{u}_3, \mathbf{u}_3 >}} = \frac{-\frac{2}{3}(1,-4,1)}{\sqrt{-\frac{2}{3}(1,-4,1)\cdot(-\frac{2}{3})(1,-4,1)}} = -\frac{1}{3\sqrt{2}}(1,-4,1).$$

Finally, the basis $\{\mathbf{e}_1 = \frac{1}{\sqrt{2}}(-1,0,1), \mathbf{e}_2 = \frac{1}{3}(2,1,2), \mathbf{e}_3 = -\frac{1}{3\sqrt{2}}(1,-4,1)\}$ is orthonormal.

Answer: $\boxed{\{\frac{1}{\sqrt{2}}(-1,0,1),\ \frac{1}{3}(2,1,2),\ -\frac{1}{3\sqrt{2}}(1,-4,1)\}}$