Name ______________________________

Problem 1 (15pt). Find derivatives of the following functions. Simplify your answer whenever it is possible.

2pt a) $y = \sin \frac{1}{x^2}$, $y' = \left(\cos \frac{1}{x^2}\right) \cdot \frac{-2}{x^3} = \frac{-2\cos\frac{1}{x^2}}{x^3}$

2pt b) $y = x \cdot 3^x$, $y' = 3^x + x \cdot 3^x \ln 3 = 3^x(1 + x\ln 3)$

2pt c) $y = \dfrac{x^4 - 2x^2 + 1}{x^2 - 1}$ $= \frac{(x^2-1)^2}{x^2-1} = x^2 - 1$, $y' = 2x$

3pt d) $y = \cot \sqrt{x^2+1}$, $y' = -\frac{1}{\sin^2\sqrt{x^2+1}} \frac{x}{\sqrt{x^2+1}} = -\frac{x}{\sqrt{x^2+1}\,\sin^2\sqrt{x^2+1}}$

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(3pt) e) $y = \arctan \frac{\pi}{x}$, $y' = \frac{1}{1+\frac{\pi^2}{x^2}} \cdot \left(-\frac{\pi}{x^2}\right) = -\frac{\pi}{x^2+\pi^2}$

(3pt) f) $y = \left(\frac{1}{x}\right)^{\ln x} = x^{-\ln x}$

$\ln y = \ln x^{-\ln x}$

$\ln y = -(\ln x)^2$

$\frac{y'}{y} = -2 \frac{\ln x}{x}$

$y' = -2 \left(\frac{1}{x}\right)^{\ln x} \cdot \frac{\ln x}{x}$

$y' = \frac{-2 \ln x}{x^{1+\ln x}}$

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Problem 2 (15pt). Show that the function $y = e^{4x} + 2e^{-x}$ is a solution of the differential equation $y''' - 13y' - 12y = 0$.

1pt $y' = 4e^{4x} - 2e^{-x}$

2pt $y'' = 16e^{4x} + 2e^{-x}$

2pt $y''' = 64e^{4x} - 2e^{-x}$

10pt

$$\underbrace{64e^{4x} - 2e^{-x}}_{y'''} - 13\underbrace{(4e^{4x} - 2e^{-x})}_{y''} - 12\underbrace{(e^{4x} + 2e^{-x})}_{y} = 0$$

$$64e^{4x} - 2e^{-x} - 52e^{4x} + 26e^{-x} - 12e^{4x} - 24e^{-x} = 0$$

$$0 = 0 \quad \checkmark$$

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Problem 3 (15pt). Use a linearization to find an approximate value of $\frac{1}{e^{0.001}}$. Is the true value greater than or less than your approximation? Explain!

$\frac{1}{e^{0.01}} = e^{-0.01}$

A partial credit for this

A linearization formula

$f(a+h) \approx f(a) + f'(a) \cdot h$

written for $f(x) = e^x$, $a = 0$, $h = -0.01$

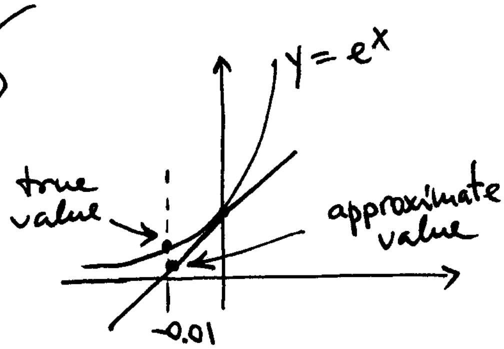
becomes

$$e^{0+(-0.01)} \approx e^0 + \underbrace{\left.\frac{d\, e^x}{dx}\right|_{x=0}}_{e^0=1} \cdot (-0.01)$$

10pt

So $e^{-0.01} \approx 1 + 1 \cdot (-0.01) = \boxed{0.99}$

5pt

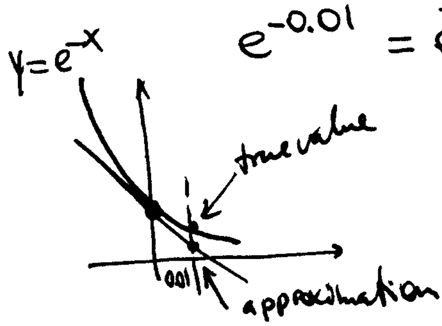


The true value of $\frac{1}{e^{0.01}}$ is <u>greater</u> than the approximation:

$\frac{1}{e^{0.01}} = e^{-0.01} > 0.99$

OR $f(x) = e^{-x}$

$$e^{-0.01} = e^{-(0+0.01)} \approx e^0 + \left.\frac{d}{dx} e^{-x}\right|_{x=0} \cdot (0.01) =$$



$= 1 - 0.01 = 0.99$

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Problem 4 (15pt). Find the equation of the tangent line to the curve $x^2y^3 + x\ln y = 1$ at the point $x = 1$, $y = 1$.

Point $(1,1)$ belongs to the curve since $1^2\cdot 1^3 + 1\cdot \ln 1 = 1$

The eq. of the tangent line at $(1,1)$ is

5pt $\quad y - 1 = \left.\dfrac{dy}{dx}\right|_{\substack{x=1\\y=1}} (x-1)$.

The derivative $\dfrac{dy}{dx}$ can be obtained by the implicit differentiation:

$$x^2y^3 + x\ln y = 1$$

$\frac{d}{dx}$

5pt $\quad 2xy^3 + 3x^2y^2y' + \ln y + x\dfrac{y'}{y} = 0$, where $y' = \dfrac{dy}{dx}$

$$y'\left(3x^2y^2 + \frac{x}{y}\right) = -2xy^3 - \ln y$$

2pt $\quad y' = -\dfrac{2xy^3 + \ln y}{3x^2y^2 + \frac{x}{y}}$

1pt $\quad \left.\dfrac{dy}{dx}\right|_{\substack{x=1\\y=1}} = \left.y'\right|_{\substack{x=1\\y=1}} = -\dfrac{2\cdot1\cdot1 + \ln 1}{3\cdot1\cdot1 + \frac{1}{1}} = -\dfrac{2}{4} = -\dfrac{1}{2}$

The eq. of the tang. line is

$$y - 1 = -\frac{1}{2}(x-1)$$

2pt $\quad \boxed{y = -\frac{1}{2}x + \frac{3}{2}}$

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Problem 5 (20pt). Show that the curve $x = t^3 - t$, $y = t^2$ has two different tangent lines at the point $(x, y) = (0, 1)$ and find their slopes.

First, find the values of t corresponding to the point $x=0, y=1$:

$$\begin{cases} 0 = t^3 - t \\ 1 = t^2 \end{cases} \Rightarrow \begin{cases} t(t-1)(t+1) = 0 \\ (t-1)(t+1) = 0 \end{cases} \Rightarrow \boxed{t = \pm 1}$$

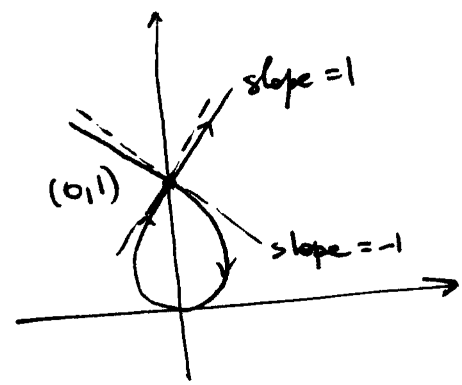
10pt

Two values of t show that the curve passes through $(0,1)$ twice.

The slopes are

5 pt

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{3t^2-1}$$



5 pt

$$\left\{ \begin{array}{l} \left.\dfrac{dy}{dx}\right|_{t=1} = \left.\dfrac{2t}{3t^2-1}\right|_{t=1} = \dfrac{2}{3-1} = 1 \\ \left.\dfrac{dy}{dx}\right|_{t=-1} = \left.\dfrac{2t}{3t^2-1}\right|_{t=-1} = \dfrac{-2}{3-1} = -1 \end{array} \right.$$

$\boxed{\text{Answer: The slopes are } -1 \text{ and } +1.}$

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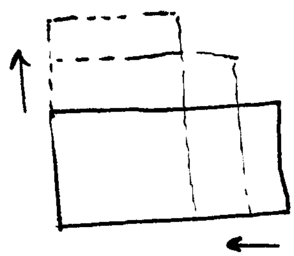
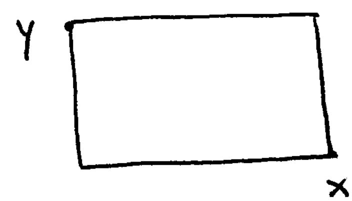
Problem 6 (20pt). How fast is the area of a magic rectangle changing if one side of it is 20 ft long and is decreasing at a rate of 1 ft/sec and the other side is 15 ft long and is increasing at a rate 2 ft/sec?





Let $x(t), y(t)$ be the sides of the rectangle.
Then the area is

5pt for area

$$A(t) = x(t)\cdot y(t)$$

Given : $\left.\frac{dx}{dt}\right|_{x=20} = -1$ ft/sec

$\left.\frac{dy}{dt}\right|_{y=15} = 2$ ft/sec

Find : $\left.\frac{dA}{dt}\right|_{\substack{x=20\\y=15}} = ?$

5pt for differentiation

$$A(t) = x(t)y(t) \Rightarrow \frac{dA}{dt} = \frac{dx}{dt}y(t) + x(t)\frac{dy}{dt}$$

10pt for substitution of correct values

$$\left.\frac{dA}{dt}\right|_{\substack{x=20\\y=15}} = -1\cdot 15 + 20\cdot 2 = 25 \text{ ft/sec}$$

Answer: the area is increasing at a rate 25 ft/sec