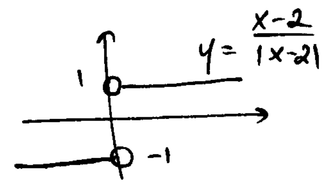
Name ______________________________

Problem 1 (25pt). Calculate the following limits or explain why they do not exist:

2pt a) $\lim_{x\to 0} \frac{x-2}{|x-2|} = \frac{0-2}{|0-2|} = \frac{-2}{2} = \boxed{-1}$

2pt b) $\lim_{x\to 2} \frac{x-2}{|x-2|}$ $\boxed{\text{DNE}}$ since $\frac{x-2}{|x-2|} = \begin{cases} 1, & x > 2 \\ -1, & x < 2 \end{cases}$



$\lim_{x\to 2^-} \frac{x-2}{|x-2|} = -1 \neq \lim_{x\to 2^+} \frac{x-2}{|x-2|} = 1$

3pt c) $\lim_{x\to 1} \frac{x-\sqrt{x}}{\sqrt{x}-1} = \lim_{x\to 1} \frac{\sqrt{x}(\sqrt{x}-1)}{\sqrt{x}-1} = \boxed{1}$

3pt d) $\lim_{x\to -\infty} \frac{x^3-1}{3x^3+2x} = \lim_{x\to -\infty} \frac{1-\frac{1}{x^3}}{3+\frac{2}{x^2}} = \boxed{\frac{1}{3}}$

3pt e) $\lim_{x\to \infty} \frac{x^2-1}{x^4+2x-1} = \lim_{x\to\infty} \frac{\frac{1}{x^2}-\frac{1}{x^4}}{1+\frac{2}{x^3}-\frac{1}{x^4}} = \frac{0}{1} = \boxed{0}$

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3pt f) $\lim_{x\to 0^-} e^{1/x}\cos x = 0 \cdot 1 = \boxed{0}$

$\frac{1}{x} \xrightarrow[x\to 0^-]{} -\infty$, $e^{\frac{1}{x}} \xrightarrow[x\to 0^-]{} 0$, $\cos x \xrightarrow[x\to 0^-]{} 1$

3pt g) $\lim_{x\to 0^+} e^{1/x}\cos x = \boxed{+\infty}$

$\frac{1}{x} \xrightarrow[x\to 0^+]{} +\infty$, $e^{\frac{1}{x}} \xrightarrow[x\to 0^+]{} +\infty$, $\cos x \xrightarrow[x\to 0^+]{} 1$

3pt h) $\lim_{x\to +\infty} \sin(\arctan x) = \sin\frac{\pi}{2} = \boxed{1}$

3pt i) $\lim_{x\to +\infty} \arctan(\sin x)$ $\boxed{\text{DNE}}$ since $\lim_{x\to +\infty} \sin x$ DNE

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Problem 2 (30pt). Let $f(x) = \dfrac{x^3}{x^2-1}$.

a) Find the domain and all the zeros of $f(x)$.

2pt Domain: $x \neq \pm 1$

1pt Zeros: $x = 0$

2pt b) Is $f(x)$ even or odd or neither? Explain!

$$f(-x) = \frac{(-x)^3}{(-x)^2-1} = -\frac{x^3}{x^2-1} = -f(x)$$. So $f(x)$ is $\boxed{\text{odd}}$

5pt c) Find all the discontinuities of $f(x)$ and specify their types.

Discontinuity at $x=1$ and $x=-1$. Both are infinite since

$$\lim_{x\to 1^+} \frac{x^3}{x^2-1} = \lim_{x\to -1^+} \frac{x^3}{x^2-1} = +\infty, \quad \lim_{x\to 1^-} \frac{x^3}{x^2-1} = \lim_{x\to -1^-} \frac{x^3}{x^2-1} = -\infty$$

5+5+5 pts d) Find all vertical, horizontal and oblique asymptotes of $f(x)$.

$\boxed{x=1}$ and $\boxed{x=-1}$ are vertical asymptotes (see c))

$$\lim_{x\to\pm\infty} \frac{x^3}{x^2-1} = \pm\infty$$ so $\boxed{\text{no}}$ horizontal asymptotes

For oblique as., put $y = kx + b$

$$k = \lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{x^3}{x^3-x} = 1 \; ; \; b = \lim_{x\to+\infty} (f(x) - kx) =$$

$$= \lim_{x\to+\infty} \left(\frac{x^3}{x^2-1} - x\right) = \lim_{x\to+\infty} \frac{x}{x^2-1} = 0$$. So $\boxed{y = x}$ is an oblique as.

as $x \to +\infty$ (and as $x \to -\infty$ by the symmetry)

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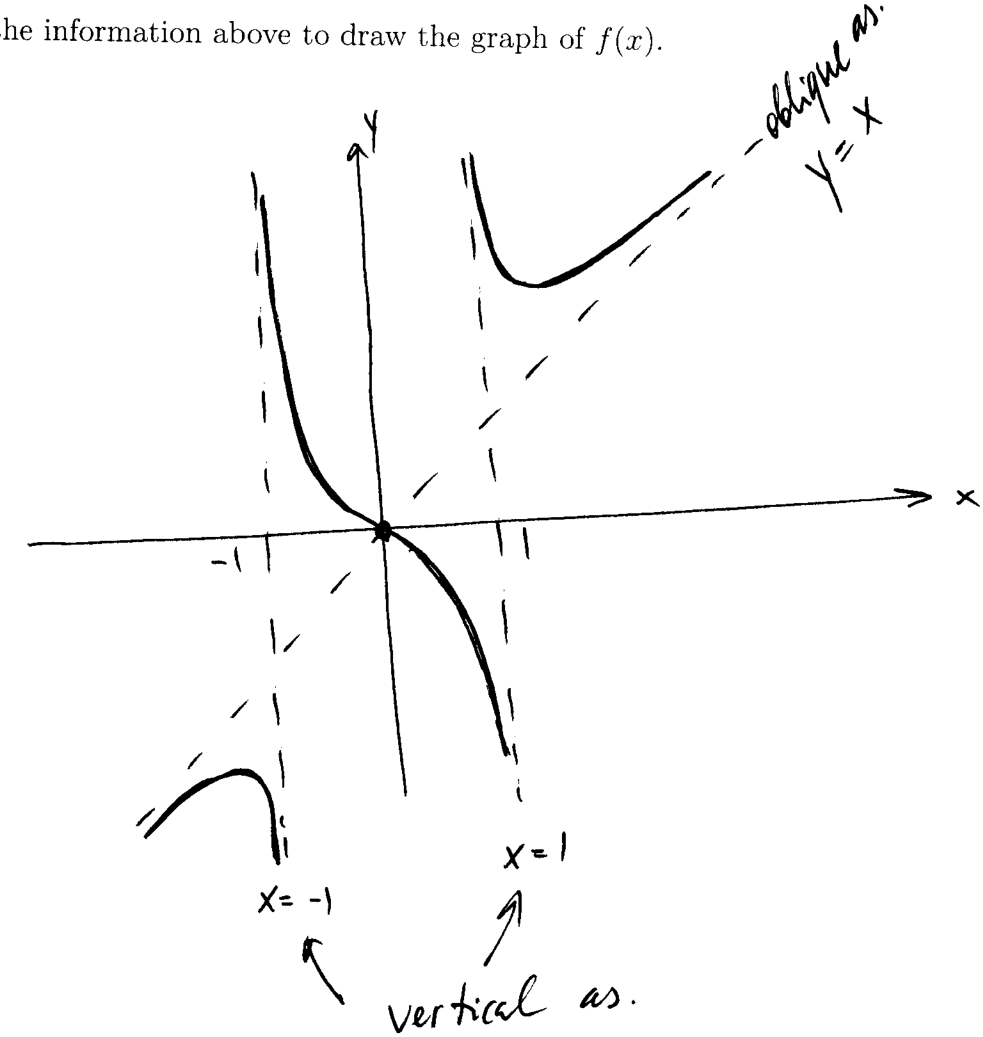
Problem 2 (continued).

5pt

e) Use the information above to draw the graph of $f(x)$.



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Problem 3 (5pt). For which value of a constant a the function

$$f(x) = \begin{cases} x+1, & x \leq 1 \\ 3 - ax^2, & x > 1 \end{cases}$$

is continuous? Draw the graph of f for such a.

3+2 pts

$f(x)$ is continuous $\forall\, x \neq 1$.

For $x = 1$,

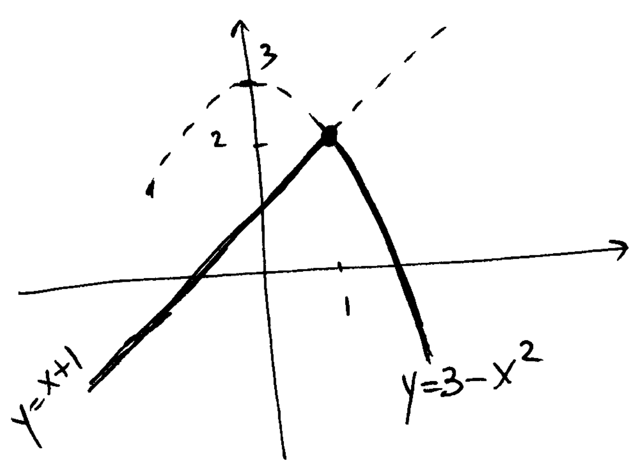
$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1} (x+1) = f(1) = 2$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (3 - ax^2) = 3 - a$$

For continuity at $x = 1$, one must have

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1) \quad \text{that is}$$

$$2 = 3 - a \Rightarrow \boxed{a = 1}$$



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Problem 4 (20pt). Let $f(x) = \sqrt{x+1}$.

5pt

a) Find the difference quotient for $f(x)$ at point $x = 8$.

$$\frac{f(8+h)-f(8)}{h} = \frac{\sqrt{8+h+1}-\sqrt{8+1}}{h} = \boxed{\frac{\sqrt{9+h}-3}{h}}$$

5pt

b) Find the derivative of $f(x)$ at $x = 8$ as the limit of the difference quotient.

$$f'(8) = \lim_{h\to 0} \frac{f(8+h)-f(8)}{h} = \lim_{h\to 0} \frac{\sqrt{9+h}-3}{h} =$$

$$= \lim_{h\to 0} \frac{h}{h(\sqrt{9+h}+3)} = \boxed{\frac{1}{6}}$$

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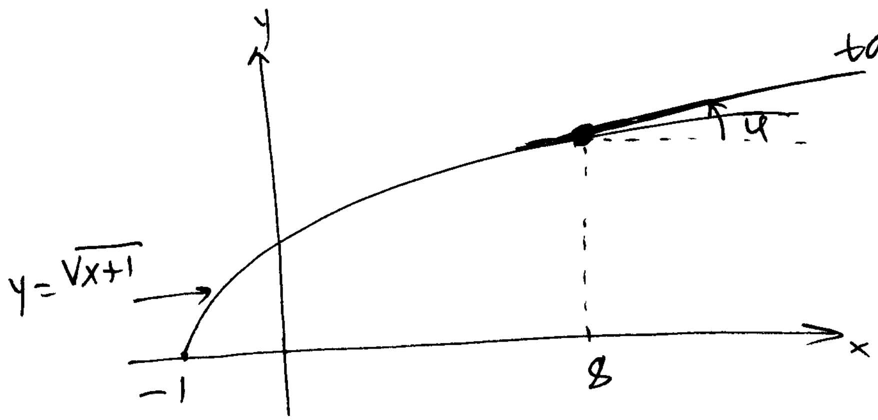
Problem 4 (continued).

5pt

c) Explain a geometrical meaning of $f'(8)$ (draw a picture).



tang. line to $y=\sqrt{x+1}$ at $x=8$ has slope $\frac{1}{6}$

$\tan\varphi = \frac{1}{6}$

5 pt

d) Find the equation of the tangent line to the graph of $y = \sqrt{x+1}$ at point $x = 8$.

$$y - y(8) = y'(8)(x-8)$$

$$y(8) = \sqrt{8+1} = 3$$

$$y'(8) = \frac{1}{6} \quad (\text{see } b))$$

$$y - 3 = \frac{1}{6}(x-8)$$

$$\boxed{y = \frac{1}{6}x + \frac{5}{3}}$$ eq. of tang. line

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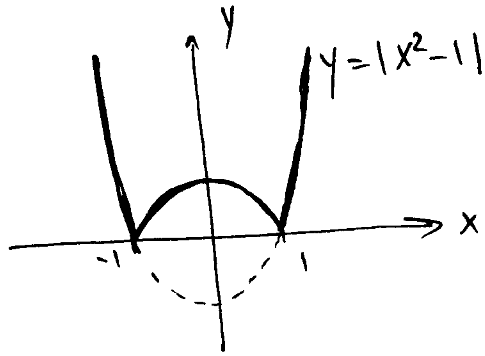
Problem 5 (20pt). Let $f(x) = |x^2 - 1|$. Draw the graph of $f(x)$. Where does $f(x)$ fail to be differentiable? Find $f'(x)$ and draw its graph.

5pt



5pt f is not differentiable at $x=-1$ and $x=1$

$$f(x) = \begin{cases} x^2-1, & x \le -1 \text{ and } x \ge 1 \\ -x^2+1, & -1 < x < 1 \end{cases}$$

5pt

$$f'(x) = \begin{cases} 2x, & x<-1 \text{ and } x>1 \\ -2x, & -1 < x < 1 \\ \text{DNE}, & x=-1 \text{ and } x=1 \end{cases}$$

5pt