EXAM

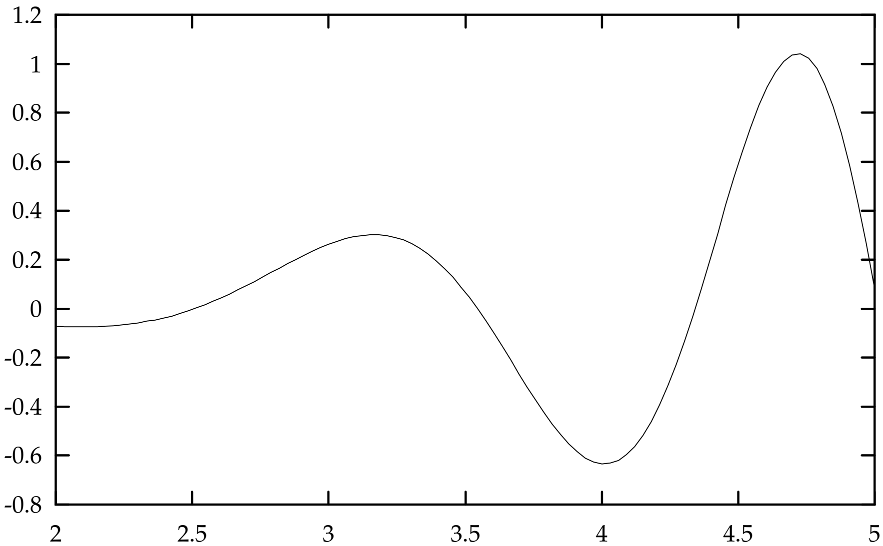
Sample Midterm 2

Math 131

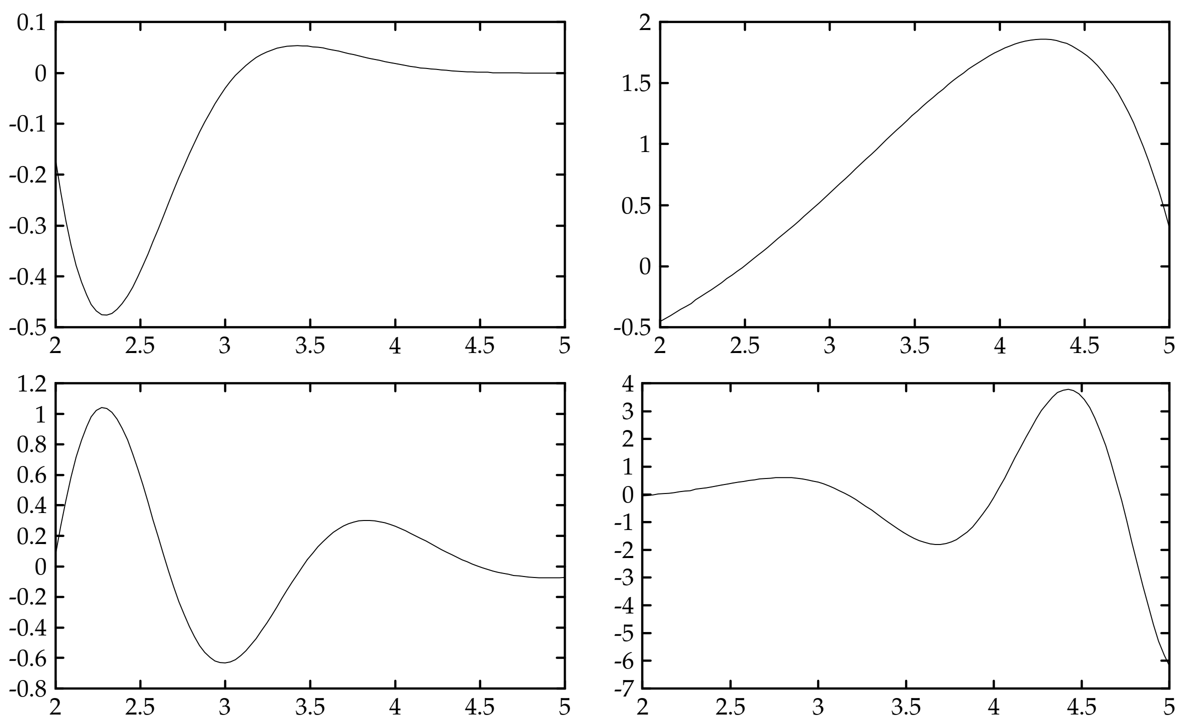
November 4, 2003

ANSWERS

Problem 1. Below is a sketch of the graph of a function f.



Which is the graph of f'?



Answer:
The graph of f' is the one on the bottom right. There are many ways to tell. For one, f is decreasing from, say 3.2 to 3.9. So, $f'(x) < 0$ for $x \in (3.2, 3.9)$; and there's only one graph that has that property.

Problem 2. Let f and g be two functions satisfying $f(2) = 5$, $f'(2) = 2$, $g(2) = -3$, and $g'(2) = 4$. Find

Answer:

(a) $\lim_{h\to 0} \frac{g(2+h) - g(2)}{h} = g'(2) = 4.$

(b) $\lim_{x\to 2} f(x) = 5$. This requires a sentence of explanation. Because f is differentiable at 2, f is continuous at 2. Thus, $\lim_{x\to 2} f(x) = f(2) = 5$.

(c) $\left(\frac{f}{g}\right)'(2) = \frac{g(2)f'(2) - f(2)g'(2)}{(g(2))^2} = \frac{(-3)(2) - (5)(4)}{(-3)^2} = -\frac{26}{9}.$

(d) $\lim_{h\to 0} \frac{f(2+h)g(2+h) + 15}{h} = (fg)'(2) = f'(2)g(2) + f(2)g'(2) = -6 + 20 = 14.$

Problem 3. Compute

Answer:

(a) $g'(x)$ if $g(x) = \dfrac{x^5 \cos(x)}{1 + x + x^8}$.

$$g'(x) = \frac{(1 + x + x^8)\left(5x^4 \cos(x) - x^5 \sin(x)\right) - \left(x^5 \cos(x)\right)(1 + 8x^7)}{(1 + x + x^8)^2}.$$

(b) $\dfrac{d^2y}{dx^2}$ if $y = e^{\sin(x)}$.

$$\frac{dy}{dx} = \cos(x)e^{\sin(x)} \text{ and } \frac{d^2y}{dx^2} = -\sin(x)e^{\sin(x)} + \cos^2(x)e^{\sin(x)}.$$

(c) $\displaystyle\lim_{x\to 0} \frac{\ln(1 + x)}{x}$.

$$\lim_{x\to 0} \frac{\ln(1 + x)}{x} = \lim_{x\to 0} \frac{\ln(1 + x) - \ln(1)}{x} = \ln'(1) = \frac{1}{1} = 1.$$

(d) $\displaystyle\lim_{x\to 0} \frac{\cos(x) - 1}{\sin(x)}$.

$$\lim_{x\to 0} \frac{\cos(x) - 1}{\sin(x)} = \lim_{x\to 0} \frac{\cos(x) - 1}{x} \frac{x}{\sin(x)} = (0)\left(\frac{1}{1}\right) = 0.$$

Problem 4. Suppose that air is being pumped into a spherical ballon at a rate of 20cm^3 per second.

(a) How fast is the radius growing when the volume is $1000\pi\text{cm}^3$?

Answer:

Note that for all times, we have $V = \frac{4}{3}\pi r^3$. Differentiating with respect to time t, gives

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

Note that when the volume is equal to $1000\pi\text{cm}^3 = \frac{4}{3}\pi r^3$, we have $r^3 = 750 \Rightarrow r = \sqrt[3]{750}\text{cm}$. Substituting $20cm^3$ per second in for $\frac{dV}{dt}$ and $r = \sqrt[3]{750}$ gives

$$20\text{cm}^3/\text{sec} = 4\pi \left(\sqrt[3]{750}\right)^2 \text{cm}^2 \frac{dr}{dt}.$$

Thus,

$$\frac{dr}{dt} = \frac{5}{\pi \left(\sqrt[3]{750}\right)^2}\text{cm/sec}.$$

(b) How fast is the surface area growing at this time?

Answer:

We have $S = 4\pi r^2$ and differentiating with respect to time t gives

$$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}.$$

Substituting the numbers

$$\frac{dr}{dt} = \frac{5}{\pi \left(\sqrt[3]{750}\right)^2}\text{cm/sec and } r = \sqrt[3]{750}\text{cm}$$

give

$$\frac{dS}{dt} = \frac{40}{\sqrt[3]{750}}\text{cm}^2/\text{sec}.$$

Problem 5. Consider a function A with the following properties:

- $A(1) = \frac{\pi}{4}$,
- $\lim_{x\to\infty} A(x) = \frac{\pi}{2}$ and $\lim_{x\to-\infty} A(x) = -\frac{\pi}{2}$
- $A'(x) = \frac{1}{1+x^2}$ for all $x \in (-\infty, \infty)$.

(a) There isn't a nice formula for $A(x)$ so, it is not possible to determine $A(1.2)$ exactly. Use an approximation by differentials to estimate $A(1.2)$.

Answer:

We have

$$A(1.2) \simeq A(1) + A'(1)(0.2) = \frac{\pi}{4} + \left(\frac{1}{1+1^2}\right)(0.2) = \frac{\pi}{4} + 0.1 = .885398\ldots.$$

(b) On what interval is the graph of A concave down?

Answer:

The graph of A will be concave down when $A''(x) = \frac{-2x}{(1+x^2)^2} < 0$. That is, when $x > 0$.

(c) Multiple choice: circle the correct one. $\frac{d}{dx}\left(A\left(2\frac{\sin(x)}{\cos(x)}\right)\right) =$

$$\frac{2}{1+3\sin^2(x)} \qquad \frac{2\sec(x)\tan(x)}{1+x^2} \qquad 4x^2 \qquad \frac{1}{1+4\tan^2(x)}$$

Answer:

$$\begin{aligned}\frac{d}{dx}\left(A\left(2\frac{\sin(x)}{\cos(x)}\right)\right) &= \left(\frac{1}{1+\left(2\frac{\sin(x)}{\cos(x)}\right)^2}\right)\left(2\frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)}\right)\\ &= \frac{2}{\cos^2(x)+4\sin^2(x)}\\ &= \frac{2}{1+3\sin^2(x)}.\end{aligned}$$