EXAM

Midterm 2

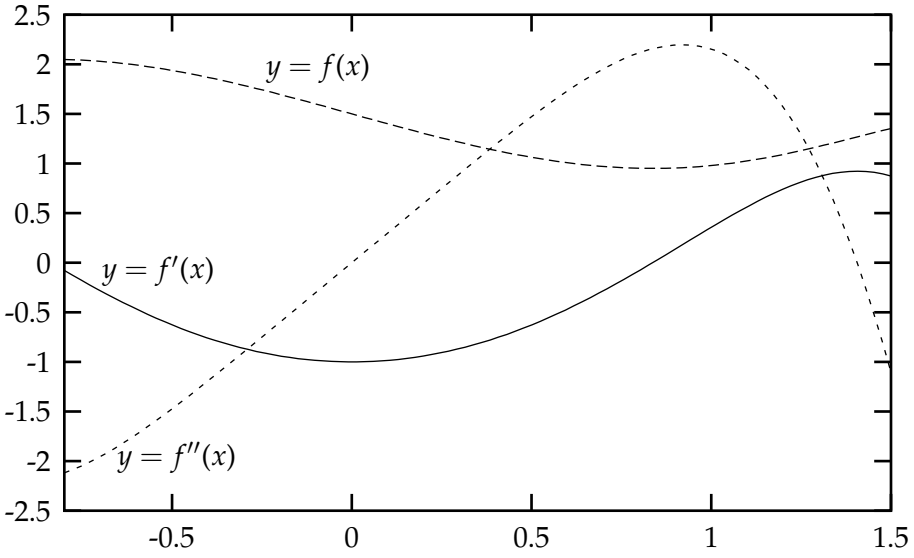
Math 131

Monday, November 10, 2003

ANSWERS

Problem 1. **[10 points]** Below, the graphs of f, f', and f'' are sketched. Which are which? Label clearly.

Answer:



There are a few ways to decide conclusively which curve is which. The one labelled $y = f(x)$ lies entirely above the x axis, so it can't be a derivative or else one of the other curves would be always increasing. Once the graph of f has been identified, $y = f'(x)$ can be picked out by noticing that $y = f(x)$ has a horizontal tangent around $x = 1$. That determines the graph of f' as the curve that crosses the x axis around $x = 1$. That leaves the last curve to be $y = f''(x)$, which fits for independent reasons, such as showing that $f''(x) > 0$ for $x \in (0, 1.4)$, where the $y = f(x)$ curve is concave up.

Problem 2. **[20 points]** Suppose that oil is leaking from a ruptured tanker at a rate of 100 cubic meters per hour and is creating a circular oil slick 0.02 meters deep. How fast is the radius of the spill growing when the radius is 1000 meters? (The volume of a circular cylinder of thickness h and radius r is $V = \pi r^2 h$.

Answer:
We have $V = \pi r^2 h = (0.02)\pi r^2$, and differentiating with respect to time yields

$$\frac{dV}{dt} = (0.04)\pi r \frac{dr}{dt}.$$

Substituting 100 for $\frac{dV}{dt}$ and 1000 for r gives us

$$100 = 40\pi \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{5}{2\pi}.$$

So, the radius of the spill is growing at a rate of $\dfrac{5}{2\pi}$ meters per hour.

Problem 3. [2 points each] True or False.

Answer:

(a) $\frac{d}{dx}\sqrt{f(x)} = \frac{f'(x)}{2\sqrt{f(x)}}$. **True**.

(b) $\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g'(x)$. **False**. $\left(\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x)\right)$.

(c) If f is differentiable at x then f is continuous at x. **True**.

(d) If f is continuous at x then f is differentiable at x. **False**.

(e) $\frac{d}{dx}f^{-1}(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$. **True**.

Problem 4. [20 points] Estimate the number $\sqrt[3]{65}$ using a tangent line approximation.

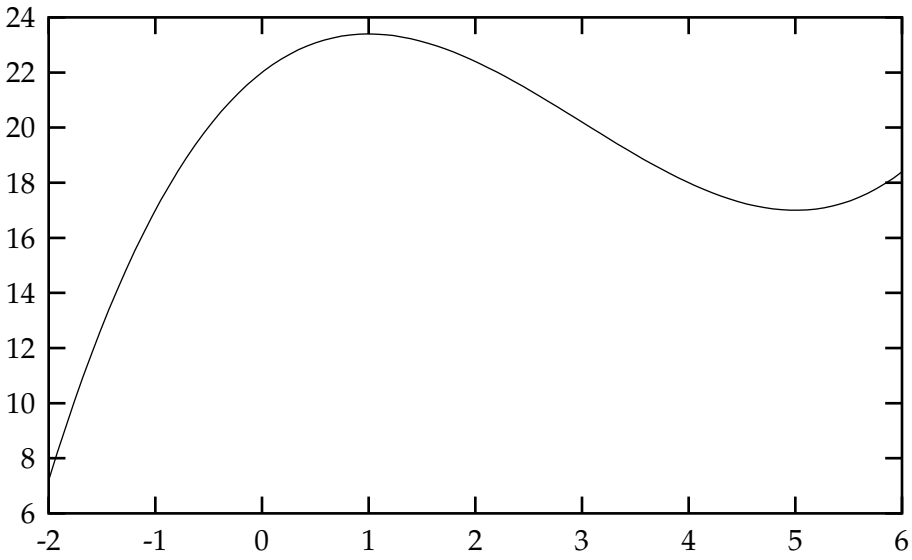
Answer:
Let $f(x) = \sqrt[3]{x}$. Then, the tangent line approximation for f near 64 gives

$$f(64+h) \approx f(64) + f'(64)h \text{ if } h \text{ is small.}$$

We have $f(64) = 4$, $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$, and $f'(64) = \left(\frac{1}{3}\right)\left(\frac{1}{16}\right) = \frac{1}{48}$. So, using the approximation above with $h = 1$ gives us

$$\sqrt[3]{65} = f(65) \approx f(64) + f'(64)(1) = 4 + \left(\frac{1}{48}\right)(1) = \frac{193}{48} = 4.020833\ldots.$$

Problem 5. The graph of a derivative $y = f'(x)$ is sketched below.



Use the picture to answer the following questions about the function f.

(a) **[10 points]** On which interval is f concave down?

Answer:
f is concave down where f' is decreasing, which is on the interval $(1, 5)$.

(b) **[10 points]** Circle one:

$f(2) > f(3)$ $f(2) < f(3)$ there's not enough information given to say which

Answer:
Since $f'(x) > 6 > 0$ for all $x \in (-2, 6)$, f is increasing. In particular, $f(2) < f(3)$.

Problem 6. **[20 points]** Consider the curve defined by $y^3 + xy + e^{y-1} = 7$. Find the equation of the tangent line to this curve at $(5, 1)$.

Answer:
We differentiate implicitely to get

$$3y^2\frac{dy}{dx} + y + x\frac{dy}{dx} + e^{y-1}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{3y^2 + x + e^{y-1}}.$$

So, at $(5, 1)$, we have

$$\frac{dy}{dx} = -\frac{1}{3+5+1} = -\frac{1}{9}.$$

So, the equation of the tangent line to $y^3 + xy + e^{y-1} = 7$ at $(5, 1)$ is

$$y - 1 = -\frac{1}{9}(x - 5).$$