EXAM

Midterm 1

Math 131

October 9, 2003

ANSWERS

Problem 1. Evaluate. Justify your answers.

(a) $\lim_{x\to 2} \frac{x^4-16}{x^2-4} =$

Answer:

$$\lim_{x\to 2} \frac{x^4-16}{x^2-4} = \lim_{x\to 2} \frac{(x^2-4)(x^2+4)}{x^2-4} = \lim_{x\to 2} x^2+4 = 8.$$

(b) $\lim_{x\to 6} \frac{x-6}{\sqrt{2x-3}} =$

Answer:

$$\lim_{x\to 6} \frac{x-6}{\sqrt{2x-3}} = \frac{0}{3} = 0.$$

(c) $\lim_{t\to\infty} \frac{3t^3+t-5}{4t^3+t^2+6} =$

Answer:

$$\lim_{t\to\infty} \frac{3t^3+t-5}{4t^3+t^2+6} = \lim_{t\to\infty} \frac{3+\frac{1}{t^2}-\frac{5}{t^3}}{4+\frac{1}{t}+\frac{6}{t^3}} = \frac{3}{4}.$$

(d) $\lim_{x\to\infty} \frac{x^2-x}{x+10\sqrt{x}} =$

Answer:

$$\lim_{x\to\infty} \frac{x^2-x}{x+10\sqrt{x}} = \lim_{x\to\infty} \frac{x-1}{1+\frac{10}{\sqrt{x}}} = \infty.$$

Problem 2. The function f defined by $f(x) = 3x - 1$ is continuous at $x = 2$. Intuitively, this means that $f(x)$ will be near $f(2) = 5$ whenever x is near 2. More precisely, we have

$$\lim_{x \to 2} f(x) = 5 \text{ and } f(2) = 5.$$

Your problem: find a number δ so that

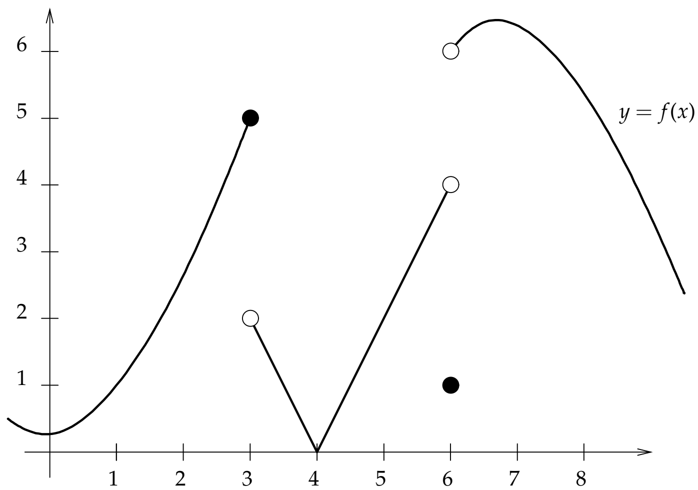
$$\text{if } 2 - \delta < x < 2 + \delta \text{ then } 4.8 < f(x) < 5.2.$$

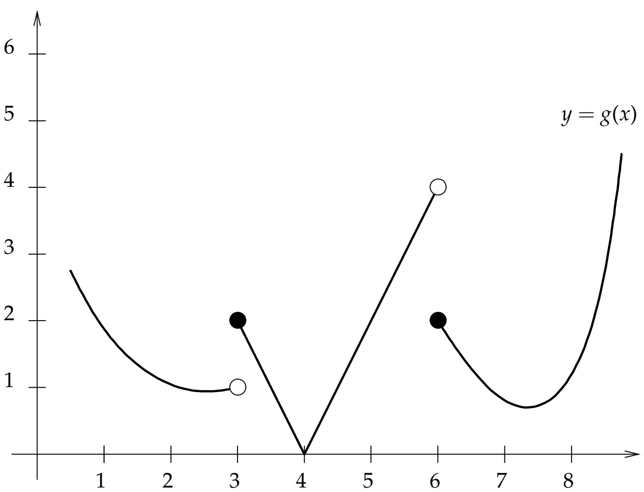
Answer:
Set

$$\begin{aligned} 4.8 < f(x) < 5.2 &\Leftrightarrow |f(x) - 5| < .2 \\ &\Leftrightarrow |(3x - 1) - 5| < .2 \\ &\Leftrightarrow |3x - 6| < .2 \\ &\Leftrightarrow 3|x - 2| < .2 \\ &\Leftrightarrow |x - 2| < \frac{.2}{3} = \frac{1}{15} \\ &\Leftrightarrow 2 - \frac{1}{15} < x < 2 + \frac{1}{15}. \end{aligned}$$

So, for $\delta = \frac{1}{15}$, we have if $\frac{1}{15} < x < 2 + \frac{1}{15}$, then $4.8 < f(x) < 5.2$.

Problem 3.





Use the picture to find:

(a) $\lim_{x \to 3^+} g(x) = 2$.

(b) $f(3) = 5$.

(c) $\lim_{x \to 3} f(x) =$ does not exist.

(d) $\lim_{x \to 6} (f + g)(x) = 8$.

(e) $(f + g)(6) = 3$.

(f) $\lim_{x \to 4} \frac{f(x)}{g(x)} = 1$.

Hint: The answers are (out of order) 1, 2, 3, 5, 8, and "does not exist."

Problem 4. True or False. No justification is required.

(a) If f is a continuous function and $\lim_{x\to 1} f(x) = 5$ then $f(1) = 5$.

Answer:

True, by the definition of continuity.

(b) Let f be continuous on $[a, b]$. Then, the intermediate value theorem says that for any $c \in (a, b)$, $f(c)$ is necessarily between $f(a)$ and $f(b)$.

Answer:

False. For $f(x) = x^2$, one has $f(-1) = 1$, $f(2) = 4$, but $f(0) = 0$ which is not between 1 and 4.

(c) $\lim_{t\to 0} t \sin\left(\frac{1}{t}\right) = 0$.

Answer:

True. By the squeeze theorem.

(d) The equation $x^5 + x - 2 = 2$ has a solution in the interval $(0, 3)$.

Answer:

True. By the intermediate value theorem.

(e) $\lim_{x\to 0} \frac{1}{x} = +\infty$.

Answer:

False. $\lim_{x\to 0^-} \frac{1}{x} = -\infty$ and $\lim_{x\to 0^+} \frac{1}{x} = +\infty$, so the two sided limit $\lim_{x\to 0} \frac{1}{x}$ does not exist.

(f) Let g be defined by $g(x) = \sqrt[5]{8\pi}$. Then the tangent line to the graph of g is horizontal at every point.

Answer:

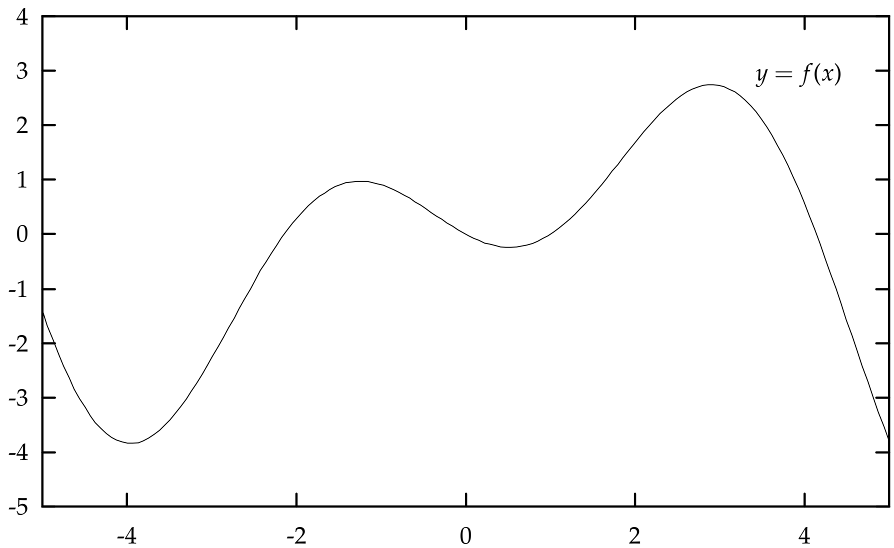
True. The graph of g is just a horizontal line.

Problem 5. Let $f(x) = \dfrac{3}{x+1}$. Use the definition of the derivative to compute $f'(1)$. Show your work.

Answer:

$$\begin{aligned}
f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \\
&= \lim_{h\to 0} \frac{\frac{3}{(1+h)+1} - \frac{3}{2}}{h} \\
&= \lim_{h\to 0} \frac{\frac{(3)(2)}{2(h+2)} - \frac{3(h+2)}{2(h+2)}}{h} \\
&= \lim_{h\to 0} \frac{6 - 3h - 6}{2h(h+2)} \\
&= \lim_{h\to 0} \frac{-3h}{2h(h+2)} \\
&= \lim_{h\to 0} \frac{-3}{2(h+2)} \\
&= -\frac{3}{4}.
\end{aligned}$$

Problem 6.



Use the sketch of $y = f(x)$ above to order the following numbers from smallest to largest:

$$f'(-4),\ f'(-2),\ f'(0),\ f'(4).$$

Answer:

$$f'(4) < f'(0) < f'(-4) < f'(-2)$$