EXAM

Final Exam

Math 101

December 15, 2000

ANSWERS

Problem 1.

(a) Use the definition of the derivative to compute $h'(x)$ if $h(x) = \dfrac{1}{x^2}$.

Answer:

$$\begin{aligned}
h'(x) &= \lim_{t\to 0} \frac{h(x+t) - h(x)}{t} \\
&= \lim_{t\to 0} \frac{\frac{1}{(x+t)^2} - \frac{1}{x^2}}{t} \\
&= \lim_{t\to 0} \frac{x^2 - (x+t)^2}{t(x+t)^2(x^2)} \\
&= \lim_{t\to 0} \frac{x^2 - x^2 - 2xt - t^2}{t(x+t)^2(x^2)} \\
&= \lim_{t\to 0} \frac{-2x - t}{(x+t)^2(x^2)} \\
&= \frac{-2x}{(x^2)(x^2)} \\
&= -\frac{2}{x^3}.
\end{aligned}$$

(b) Use the definition of the integral to compute $\displaystyle\int_{-1}^{3} (2x+1)\,dx$.

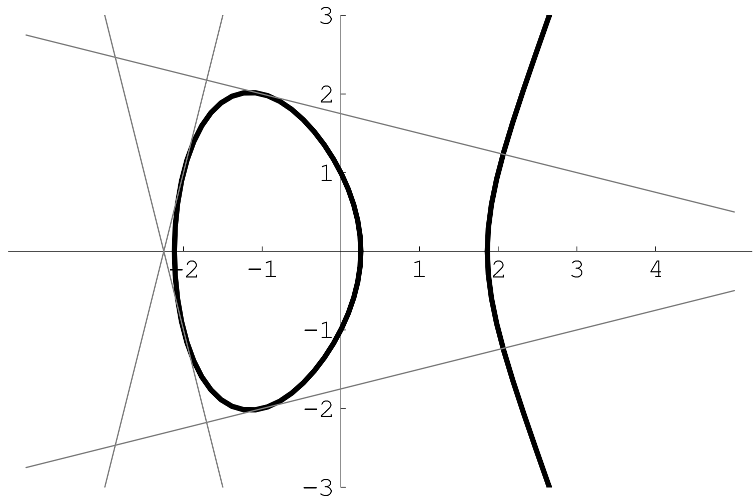
Answer:

$$\begin{aligned}
\int_{1}^{3} (2x+1)\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} (2x_i^* + 1)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(2\left(-1 + \frac{4}{n}i\right) + 1\right)\left(\frac{4}{n}\right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(-1 + \frac{8}{n}i\right)\left(\frac{4}{n}\right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(-\frac{4}{n} + \frac{32}{n^2}i\right) \\
&= \lim_{n\to\infty} -4 + \left(\frac{32}{n^2}\right)\left(\frac{n(n+1)}{2}\right) \\
&= 12.
\end{aligned}$$

Problem 2. Below is a picture of the *elliptic curve* defined by the equation

$$y^2 - 1 = x^3 - 4x$$

and four lines tangent to the curve. Pick two of these lines and give their equations. Clearly indicate which of your equations describe which lines.



Answer:
We will find the equations of all four lines.
First, we differentiate $y^2 - 1 = x^3 - 4x$ implicitely to find $\frac{dy}{dx}$:

$$2y\frac{dy}{dx} = 3x^2 - 4 \Rightarrow \frac{dy}{dx} = \frac{3x^2 - 4}{2y}.$$

So the slopes of the tangent lines to this curve at $(-2,1)$, $(-1,2)$, $(-1,-2)$, and $(-2,-1)$ are 4, $-\frac{1}{4}$, $\frac{1}{4}$, and -4 respectively. So, the equations of these four lines (starting with the one tangent to the curve at $(-2,1)$ and going clockwise) are

$$y - 1 = 4(x+2)$$
$$y - 2 = -\frac{1}{4}(x+1)$$
$$y + 2 = \frac{1}{4}(x+1)$$
$$y - 1 = -4(x+2)$$

Problem 3. Let f be defined by

$$f(x) = \frac{x^2 - x - 2}{3x^2 - 4x - 4}.$$

Give the equations of all horizontal and vertical asymptotes of the graph of f.

Answer:
Since

$$\lim_{x\to\infty} \frac{x^2 - x - 2}{3x^2 - 4x - 4} = \lim_{x\to-\infty} \frac{x^2 - x - 2}{3x^2 - 4x - 4} = \frac{1}{3}$$

the line $y = \frac{1}{3}$ is a horizontal asymptote of the graph of f.
Now write

$$\frac{x^2 - x - 2}{3x^2 - 4x - 4} = \frac{(x+1)(x-2)}{(3x+2)(x-2)},$$

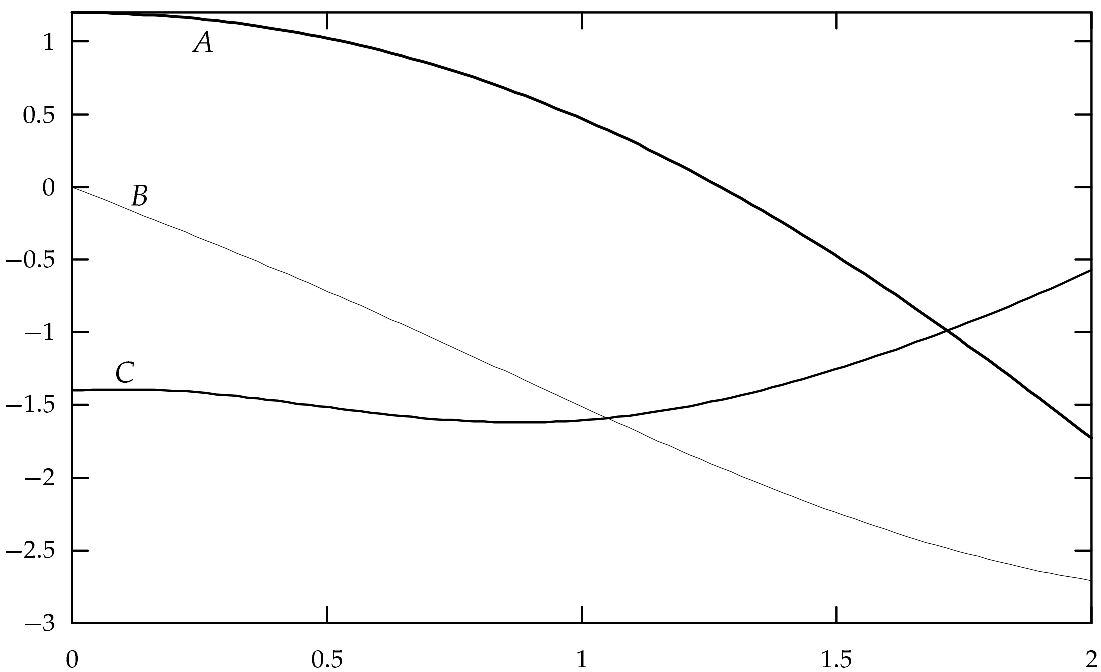
to see that

$$\lim_{x\to-\frac{2}{3}^+} f(x) = \lim_{x\to-\frac{2}{3}^+} \frac{(x+1)(x-2)}{(3x+2)(x-2)} = +\infty.$$

Hence $x = -\frac{2}{3}$ is a vertical asymptote of the graph of f.

Note: $\lim_{x\to 2} f(x) = \frac{3}{8} \neq \pm\infty$ and so $x = 2$ is *not* a vertical asymptote of f.

Problem 4. Below, portions of the graphs of f, f', and f'' are sketched. Which are which? Give brief, but conclusive, evidence supporting your answer.



Answer:
As I have them labelled, A is the graph of f, B is the graph of f' and C is the graph of f''. Here is how I arrived at this conclusion:

The curve C has a couple of critical points (around .1 and .8) and neither of the other curves have zeros there. Therefore, the derivative of the function whose graph is C is not graphed here—C must be the graph of f''. Since C lies below the line $y = 0$, the graph of f must be concave down. This implies that A must be the graph of f. That leaves B for the graph of f' and all of its features are consistent with our analysis (f has a critical point at zero and $f'(0) = 0$; f is decreasing and $f'(x) < 0$ for $x > 0$; f' is decreasing and $f''(x) < 0$; ...)

Problem 5. Match:

(a) $\lim_{h \to 0} \frac{\sin(2xh + h^2)}{h} = 2x.$

(b) $\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} \sin^3(t)\, dt = \sin^3(x).$

(c) $D_x \left(\tan^2(x) - \sec^2(x) \right) = 0.$

(d) $D_x f(\sqrt{x})) = \frac{f'(\sqrt{x})}{2\sqrt{x}}.$

(e) $\int_{\frac{1}{8}}^{\frac{1}{3}} \frac{1}{x^2} \sqrt{1 + \frac{1}{x}}\, dx = \frac{38}{3}.$

(f) $f'\left(\frac{\pi}{2}\right) = -\frac{33}{8}$ if $f(x) = \left(\frac{x}{\pi}\right)^3 \cos(x) - 4x.$

(g) $D_x \int_2^4 2t\, dt = 0.$

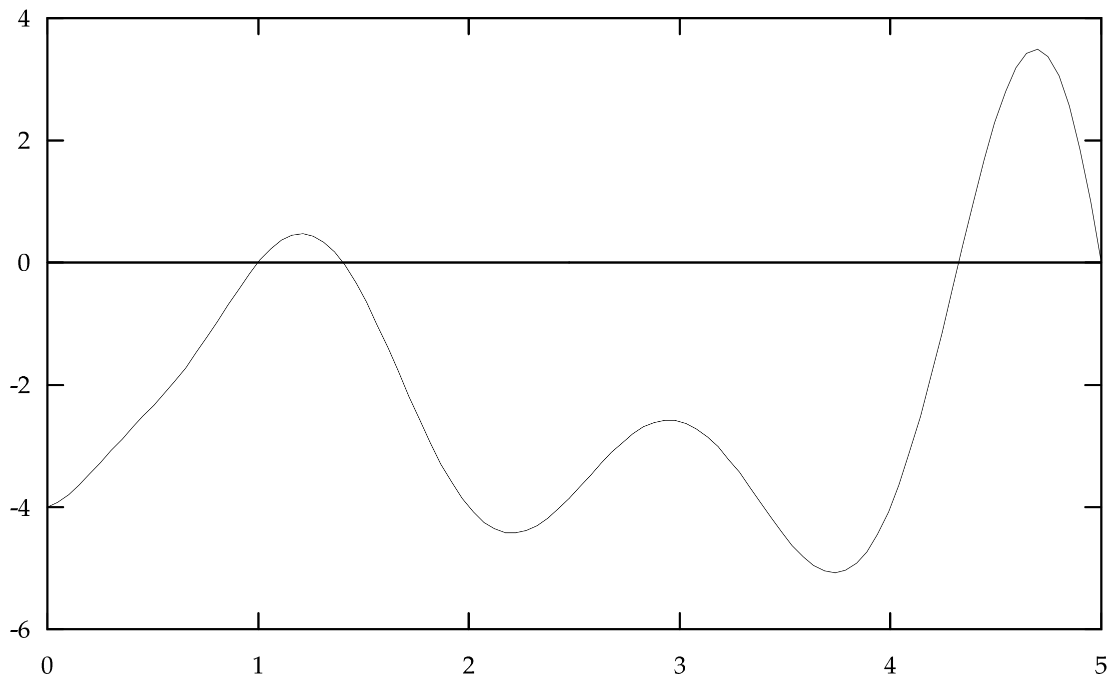
(h) $\int \left(x^3 \cos(x) + 3x^2 \sin(x) \right) dx = x^3 \sin(x).$

(i) $D_x \sqrt{f(x)} = \frac{f'(x)}{2\sqrt{f(x)}}.$

The answers (out of order, of course) are:

$$0, \quad 0, \quad 2x, \quad \frac{38}{3}, \quad -\frac{33}{8}, \quad x^3 \sin(x), \quad \sin^3(x), \quad \frac{f'(x)}{2\sqrt{f(x)}}, \quad \frac{f'(\sqrt{x})}{2\sqrt{x}}.$$

Problem 6. Below is the graph of a function f.



(a) Is $\int_0^5 f(x)\,dx$ positive or $\boxed{\text{negative}}$?

Answer:

There is more area trapped beneath the x axis than above, so $\int_0^5 f(x)\,dx$ is negative.

(b) Is $\int_0^5 f'(x)\,dx$ $\boxed{\text{positive}}$ or negative?

Answer:

$\int_0^5 f'(x)\,dx \stackrel{FTC}{=} f(5) - f(0) = 0 - (-4) = 4$ is positive.

(c) Let $g(x) = \int_0^x f(t)\,dt$. Is the graph of g $\boxed{\text{concave up}}$, concave down, or neither near $x = 1$?

Answer:

$g''(x) = f'(x) > 0$ for x near 1 (since the slopes of the tangent lines to the pictured curve are positive near 1). So, the graph of g is concave up near $x = 1$.

(d) Let $h(x) = \int_0^x f'(t)\,dt$. Which is true? $h < f$ $h = f$ $\boxed{h > f}$

Answer:

$h(x) = \int_0^x f'(t)\,dt \stackrel{FTC}{=} f(x) - f(0) = f(x) - (-4) = f(x) + 4 > f(x)$.

Problem 7. Fill in the boxes: (Show your work.)

$$\text{if } -2 \le x \le 4 \text{ then } \boxed{} \le 5 - 30x + 51x^2 - 32\,x^3 + 6x^4 \le \boxed{}.$$

Answer:
Let $f(x) = 5 - 30x + 51x^2 - 32x^3 + 6x^4$ for $x \in [-2, 4]$. The extreme value theorem gaurantees that f has a minimum and a maximum on $[-2, 4]$. Using Fermat's critical point theorem, the extrema may occur at -2, 4, or a critical point of f.

We differentiate and find the critical points:

$$\begin{aligned} f'(x) &= -30 + 102x - 96x^2 + 24x^3 \\ &= 6(5 + 17x - 16x^2 + 4x^3) \\ &= 6(x-1)(4x^2 - 12x + 5) \\ &= 6(x-1)(2x-5)(2x-1). \end{aligned}$$

The critical points of f are $x = \frac{1}{2}, 1, \frac{5}{2}$. Checking, we find that

$$\begin{aligned} f(-2) &= 621 \\ f\left(\frac{1}{2}\right) &= -\frac{7}{8} \\ f(1) &= 0 \\ f\left(\frac{5}{2}\right) &= -\frac{135}{8} \\ f(4) &= 189. \end{aligned}$$

So, f has an absolute minimum of $-\dfrac{135}{8}$ and an absolute maximum of 621 on the interval $[-2, 4]$. That is,

$$\text{if } \quad -2 \le x \le 4 \text{ then } -\frac{135}{8} \le 5 - 30x + 51x^2 - 32\,x^3 + 6x^4 \le 621.$$

Problem 8. A rectangular box (with a top) is to be constructed from 1200 square units of material. If the base of the box must be a rectangle twice as wide as it is long, how should the box be constructed so as to maximize its volume?

Answer:
Let the base have dimensions $x \times 2x$ and the height be h. Call the volume V and the surface area SA. We wish to maximize

$$V = 2x^2 h$$

subject to the constraint

$$SA = 2x^2 + 2x^2 + 2xh + 2xh + xh + xh = 1200 \Leftrightarrow 4x^2 + 6xh = 1200 \Rightarrow h = \frac{200}{x} - \frac{2}{3}x.$$

So we can express V as a function of x by

$$V(x) = 400x - \frac{4}{3}x^3 \text{ for } x > 0.$$

In order to maximize V, we find the critical points:

$$V'(x) = 400 - 4x^2 = 0 \Rightarrow x = 10.$$

Since $V'(x) > 0$ for $x < 10$ and $V'(x) < 0$ for $x > 10$, we can be sure that the abolute maximum volume occurs when $x = 10$. The box, then, should be constructed with a 10×20 base and a height of $h = 20 - \frac{2}{3}10 = \frac{40}{3}$.

Problem 9. Let $g(x) = \dfrac{x+1}{(x-1)^2}$. Find all inflection points of the graph of g.

Answer:

$$g'(x) = \frac{(x-1)^2 - 2(x+1)(x-1)}{(x-1)^4} = \frac{x-1-2x-2}{(x-1)^3} = -\frac{x+3}{(x-1)^3}$$

so

$$g''(x) = -\frac{(x-1)^3 - 3(x+3)(x-1)^2}{(x-1)^6} = -\frac{x-1-3x-9}{(x-1)^4} = \frac{2x+10}{(x-1)^4}.$$

Notice that $g''(x) < 0$ for $x < -5$ and $g''(x) > 0$ for $x > -5$. So, the point $\big(-5, g(-5)\big) = \big(-5, -\frac{1}{9}\big)$ is the only inflection point of the graph of g.

Problem 10. The *average value of a function f on the interval* $[a, b]$ is defined to be

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

If a particle is travelling with a velocity of $v(t) = t\sin\left(t^2\right)$ feet per second at time t seconds, what is the particle's average velocity between $t = 0$ and $t = \sqrt{\pi}$ seconds?

Answer:
We compute the particle's average velocity between $t = 0$ and $t = \sqrt{\pi}$ seconds:

$$\begin{aligned}
\frac{1}{\sqrt{\pi}-0}\int_0^{\sqrt{\pi}} t\sin(t^2)\,dt &= \left(\frac{1}{\sqrt{\pi}-0}\right)\left(-\frac{1}{2}\cos(t^2)\right)\Big]_0^{\sqrt{\pi}} \\
&= \left(\frac{1}{\sqrt{\pi}-0}\right)\left(-\frac{1}{2}\cos(\pi) - -\frac{1}{2}\cos(0)\right) \\
&= \left(\frac{1}{\sqrt{\pi}-0}\right)\left(-\frac{1}{2}(-1) + \frac{1}{2}(1)\right) \\
&= \frac{1}{\sqrt{\pi}}.
\end{aligned}$$