HYPERSURFACES OF LOW DEGREE ARE RATIONALLY SIMPLY-CONNECTED

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ABSTRACT. For a general hypersurface of degree d in projective n-space, if $n \geq d^2$ the spaces of 2-pointed rational curves on the hypersurface are rationally connected; thus the hypersurfaces are *rationally simply connected*. This paper proves stronger versions of theorems in [HS05].

1. INTRODUCTION

A smooth projective variety is *rationally connected* if every pair of points is contained in a rational curve, cf. [Kol96], [Deb01]. This is analogous to the notion of path-connectedness in topology. A path-connected topological space is simply-connected if the space of based paths is path-connected. By analogy, Mazur suggested definining a rationally connected variety to be *rationally simply-connected* if the space of based, 2-pointed rational curves of fixed homology class is rationally connected. Unfortunately, this notion is a bit too strong. The condition should be imposed only if the homology class is suitably positive and if the basepoints are suitably general. Also, since the space of 2-pointed rational curves is typically not compact, it is compactified using the Kontsevich moduli space, cf. [FP97].

Theorem 1.1. *In characteristic* 0*, a general hypersurface of degree d in $\mathbb{P}^n$ is rationally simply connected if $n \geq d^2$ and $d \geq 2$. Precisely, for every $e \geq 2$, the evaluation morphism $ev : \overline{\mathcal{M}}_{0,2}(X, e) \to X \times X$ is surjective and a general fiber is irreducible, reduced and rationally connected. This also holds for $(n, d) = (\geq 2, 1)$.*

Tsen's theorem proves that a Fano hypersurface defined over the function field of a curve has a rational point, cf. [Tse36]. Rational connectedness, introduced by Campana [Cam92] and Kollár-Miyaoka-Mori [KMM92b], is the natural notion for generalizing Tsen's theorem, cf. [GHS03], [dJS03] and [GHMS05]. The analogue of Tsen's theorem over a higher-dimensional base is the Tsen-Lang theorem, cf. [Lan52]. To generalize the Tsen-Lang theorem, the correct notion should be some version of *higher rational connectedness*

In truth, the best definition of rational simple-connectedness is not yet clear. At issue is a precise definition of sufficiently positive homology class. However, if $\text{Pic}(X) = \mathbb{Z}$ there is only one possible definition: a collection of

Date: February 27, 2006.

homology classes $\beta \in H_2(X, \mathbb{Z})$ is *sufficiently positive* if for one ample invertible sheaf L, and hence every ample invertible sheaf, there exists an integer $e_0 = e_0(L)$ such that β is in the collection if $c_1(L) \cdot \beta \geq e_0$. This is the notion that appears in Theorem 1.1.

Principle 1.2 (de Jong)**.** A proper, smooth variety defined over the function field of a surface over an algebraically closed field of characteristic 0 has a rational point if the base-change to the algebraic closure of the function field is rationally simply-connected (in a slightly stronger sense than used here) and if a certain Brauer obstruction vanishes.

In proving the principle, there are technical difficulties related to singularities of spaces of rational curves, the geometric analogue of weak approximation, and *the* Brauer obstruction (as opposed to *a* Brauer obstruction). In a forthcoming paper with A. J. de Jong, de Jong's strategy for proving Principle 1.2 is given, and Principle 1.2 is proved under some additional hypotheses.

Acknowledgments: A re-investigation of [HS05] leading to this paper was motivated by Theorem 11.1 of A. J. de Jong. I am grateful to de Jong for many useful conversations.

2. Sketch of the proof

The proof of Theorem 1.1 uses ideas from [HS05]. It is an induction argument. The induction step uses the fiber-by-fiber connected sum of 2 families of marked rational curves. Three new results go into the proof. First, very twisting morphisms are systematically studied leading to a simpler version of the induction argument, Proposition 3.10. The other two results concern the base case for the induction. Proposition 4.6 proves Theorem 1.1 in case $e = 2$. The last result is existence of a very twisting morphism $\zeta : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,1}(X, 1)$. Some notation is necessary to state the result.

Notation 2.1. Let $n \geq 2$. The scheme $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^n, 1)$ is the partial flag variety $\text{Flag}(1, 2; n + 1)$. There are morphisms $\text{ev} : \overline{\mathcal{M}}_{0,1}(\mathbb{P}^n, 1) \to \mathbb{P}^n$ and $\text{pr} : \overline{\mathcal{M}}_{0,1}(\mathbb{P}^n, 1) \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, 1) = \text{Grass}(2, n + 1)$.
Denote by h the divisor class $\text{ev}^* C_1(\mathcal{O}_{\mathbb{P}^n}(1))$. Denote by x the divisor class $\text{pr}^* C_1(\mathcal{O}_{\text{Grass}}(1))$, where $\mathcal{O}_{\text{Grass}}(1)$ is the invertible sheaf giving the Plücker embedding.
The NEF cone of $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^n, 1)$ is the set of divisor classes $\mathbb{Z}_{\geq 0}\{x, h\}$.

The degree of a very twisting morphism ζ is large: If $n = d^2$ then $\deg(\zeta^* h) \geq d^2 - d - 1$ and $\deg(\zeta^* x) \geq 2(d^2 - d - 1)$. So it is unreasonable to try to directly construct a very twisting morphism. The strategy is to instead consider a nodal, reducible, genus 0 curve C, and a morphism $\zeta : C \to \overline{\mathcal{M}}_{0,1}(X, 1)$ whose restriction to every irreducible component of C is a twisting morphism of minimal degree.

The curve C deforms to a smooth genus 0 curve, i.e., to $\mathbb{P}^1$. If C has many irreducible components, a deformation of ζ to a morphism from $\mathbb{P}^1$ may be a very twisting morphism. Proposition 5.7 gives a criterion for this. The essential case is $n = d^2$. Then the criterion is surjectivity of the derivative map between the Zariski tangent spaces of two parameter spaces. Sections 8, 9, 10, and particularly Proposition 10.1, prove the derivative map is surjective. This is the heart of the article and consists of an involved deformation theory analysis.

3. Twisting and very twisting morphisms

Hypothesis 3.1. Let K be an algebraically closed field. All schemes, algebraic spaces and Deligne-Mumford stacks are defined over K, and all morphisms and 1-morphisms commute with the morphisms to Spec K.

Every rank r, locally free $\mathcal{O}_{\mathbb{P}^1}$-module is isomorphic to a direct sum of invertible sheaves by Grothendieck's lemma [Har77, Exer. V.2.6], i.e., $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r)$ for a sequence of integers $a_1 \leq \cdots \leq a_r$.

Definition 3.2. The *negativity of* $\mathcal{E}$ is the largest integer n (possibly 0) such that $a_i < 0$ for all $i \leq n$. The *nullity of* $\mathcal{E}$ is the largest integer z (possibly 0) such that $a_i = 0$ for all $n < i \leq n+z$. The *positivity of* $\mathcal{E}$ is the difference $p := r - n - z$.
If $n(\mathcal{E}) = 0$, then $\mathcal{E}$ is *generated by global sections.* If $n(\mathcal{E}) + z(\mathcal{E}) = 0$ and $p(\mathcal{E}) > 0$, then $\mathcal{E}$ is *ample.*

Let $\pi : Y \to Z$ be a morphism of finite type Deligne-Mumford stacks. Denote by Z^0 the maximal open substack of Z that is smooth over K. Denote by Y^0 the maximal open substack of $\pi^{-1}(Z^0)$ on which π is smooth. Denote by T_π the locally free sheaf on Y^0 that is the dual of the sheaf of relative differentials of π, i.e., T_π is the vertical tangent bundle of π.

Definition 3.3. A non-constant morphism $f : \mathbb{P}^1 \to Y$ is π*-relatively free*, resp. π*-relatively very free*, if

- (i) $f(\mathbb{P}^1) \subset Y^0$,
- (ii) $f^*\pi^*T_{Z^0}$ is generated by global sections, and
- (iii) f^*T_π is generated by global sections (resp. ample).

For a π-relatively free morphism f, the *nullity of* f is $z(f^*T_\pi)$, and the *positivity of* f is $p(f^*T_\pi)$.

Lemma 3.4. *The smooth locus of the 1-morphism* $\mathit{Hom}(\mathbb{P}^1, \pi) : \mathit{Hom}(\mathbb{P}^1, Y) \to \mathit{Hom}(\mathbb{P}^1, Z)$ *contains the locus of* π*-relatively free morphisms.*

Proof. Associated to a π-relatively free morphism f there is a short exact sequence,

$$0 \longrightarrow f^*T_\pi \longrightarrow f^*T_Y \xrightarrow{d\pi} f^*\pi^*T_Z \longrightarrow 0.$$

Because the outer terms are generated by global sections, in particular $h^1(\mathbb{P}^1, f^*T_\pi) = h^1(\mathbb{P}^1, f^*T_Y) = h^1(\mathbb{P}^1, f^*\pi^*T_Z) = 0$. Therefore both $[f] \in \mathrm{Hom}(\mathbb{P}^1, Y)$ and $[\pi \circ f] \in \mathrm{Hom}(\mathbb{P}^1, Z)$ are smooth points, and $H^0(\mathbb{P}^1, d\pi) : H^0(\mathbb{P}^1, f^*T_Y) \to H^0(\mathbb{P}^1, f^*\pi^*T_Z)$ is surjective. Because this is the derivative of $\mathrm{Hom}(\mathbb{P}^1, \pi)$ at $[f]$, the Jacobian criterion implies $\mathrm{Hom}(\mathbb{P}^1, \pi)$ is smooth at $[f]$. □

Remark 3.5. The smooth locus of $\mathbb{P}^1 \times \mathrm{Hom}(\mathbb{P}^1, Y) \to Y$ contains $\mathbb{P}^1 \times \{f\}$ for every π-relatively free morphism f. For every open subset $U \subset Y$ whose closure intersects $\mathrm{Image}(f)$, every sufficiently small deformation f_ϵ of f is π-relatively free and $\mathrm{Image}(f_\epsilon)$ intersects U.

The following proposition is the relative version of the criterion [Kol96, Theorem IV.5.8].

Proposition 3.6. *Let $\pi : Y \to Z$ be a proper morphism of irreducible, finite type Deligne-Mumford stacks, and denote by $|\pi|$ the induced morphism of coarse moduli spaces.*

(i) *If there exists a π-relatively free morphism f with positivity > 0, then every irreducible component of the geometric generic fiber of $|\pi|$ is uniruled.*

(ii) *If also char$(K) = 0$, the dimension of the MRC quotient of the geometric generic fiber of $|\pi|$ is at most the nullity of f, i.e., the fiber dimension of the MRC quotient morphism is at least the positivity of f.*

Proof. **(i):** This is essentially [GHMS05, Lem. 4.4], which in turn relies on the *Rigidity Lemma*, cf. [Mum70, p. 43]. Denote $Y_f = \mathbb{P}^1 \times_{\pi\circ f, Z, \pi} Y$ and denote the projection by $\pi_f : Y_f \to \mathbb{P}^1$. The morphism f uniquely determines a section σ of π_f. The projection π_f is smooth along $\sigma(\mathbb{P}^1)$. The sheaf $\sigma^*T_{\pi_f} = f^*T_\pi$ is generated by global sections.

Denote by $M \subset \mathrm{Hom}(\mathbb{P}^1, Y_f)$ the locally closed substack of the Hom stack parametrizing morphisms $\tau : \mathbb{P}^1 \to Y_f$ with $\pi_f \circ \tau = \mathrm{Id}_f$. By [Kol96, Prop. II.3.5], the smooth locus of $\mathrm{ev} : \mathbb{P}^1 \times M \to Y_f$ contains $\mathbb{P}^1 \times \{[\sigma]\}$, and the fiber dimension of ev is $h^0(\mathbb{P}^1, \sigma^*T_{\pi_f}(-1))$. The proof is unchanged in the case of Deligne-Mumford stacks.

Because the positivity of f is > 0, the fiber dimension of ev is > 0. By [GHMS05, Lem. 4.4], for every $t \in \mathbb{P}^1$ there exists a rational curve containing $\sigma(t)$ and contained in $|\pi_f|^{-1}(t)$. The hypothesis $K = \mathbb{C}$ from [GHMS05] is not used in the proof of [GHMS05, Lem. 4.4].

For every $t \in \mathbb{P}^1$ there is a rational curve containing $f(t)$ contained in a fiber of $|\pi|$. By Remark 3.5, the morphism $\mathbb{P}^1 \times \mathrm{Hom}(\mathbb{P}^1, Y) \to Y$ is smooth at $(t, [f])$. So there is a dense open subset U of Y^0 such that for every geometric point $y \in U$, there is a rational curve containing y and contained in a fiber of $|\pi|$.

(ii): By [Hir64], assume Y and Z are smooth and irreducible. The relative MRC quotient for Deligne-Mumford stacks is a datum (Q, ρ, U, ψ) of a smooth, irreducible algebraic space Q, a proper morphism $\rho : Q \to Z$ whose geometric generic fiber is not uniruled, an open subset $U \subset Y$ whose complement has codimension ≥ 2, and a generically smooth 1-morphism $\psi : U \to Q$ such that $\pi|_U$ is 2-equivalent to $\rho \circ \psi$. Existence of the relative MRC quotient for Deligne-Mumford stacks is proved in [Sta04].

Because $f : \mathbb{P}^1 \to Y$ is π-relatively free, it is free. By [Kol96, Prop. II.3.7], the morphism f can be deformed so that $f(\mathbb{P}^1)$ is contained in $U \cap Y^0$ and $f(\mathbb{P}^1)$ intersects the smooth locus of ψ. By the Jacobian criterion, at every point $y \in Y^0$ the derivative map $d\pi_y : T_Y \otimes \kappa(y) \to T_Z \otimes \kappa(\pi(y))$ is surjective. The derivative map $d\rho_{\psi(y)} : T_Q \otimes \kappa(\psi(y)) \to T_Z \otimes \kappa(\pi(y))$ factors $d\pi_y$. So $d\rho_{\psi(y)}$ is also surjective. By the Jacobian criterion, ρ is smooth at $\psi(y)$, i.e., ρ is smooth along $\psi(Y^0)$. In particular, ρ is smooth along $\psi(f(\mathbb{P}^1))$.

There is a sheaf homomorphism $T_\pi|_U \to \psi^* T_\rho$ whose restriction to the smooth locus of ψ is surjective. So there is a sheaf homomorphism $\alpha : f^*T_\pi \to f^*\psi^*T_\rho$ whose cokernel is a torsion sheaf. Because f^*T_π is generated by global sections, also Image(α) and $f^*\psi^*T_\rho$ are generated by global sections. Since the geometric generic fiber of ρ is not uniruled, by (i) the positivity of $\psi \circ f$ is 0. This implies that Image$(\alpha) = f^*\psi^*T_\rho$ and that $f^*\psi^*T_\rho$ is a quotient of $f^*T_\pi / H^0(\mathbb{P}^1, f^*T_\pi(-1)) \otimes \mathcal{O}_{\mathbb{P}^1}(1)$. Therefore the rank of $f^*\psi^*T_\rho$ is at most the nullity of f. In other words, the dimension of the geometric generic fiber of ρ is at most the nullity of f. □

The main definition of both [HS05] and this paper is the following. It is slightly different than in [HS05] because it is used differently here.

Definition 3.7. Let X be a quasi-projective scheme, and denote by X^0 the smooth locus of X. Let $r \geq 0$ and $e > 0$ be integers. Let $\zeta : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,r}(X^0, e)$ be a 1-morphism. The 1-morphism ζ is *twisting*, resp. *very twisting*, if

(i) the morphism $\text{ev} : \overline{\mathcal{M}}_{0,r}(X^0, e) \to (X^0)^r$ is unobstructed at every geometric point of the image of ζ,
(ii) the morphism ζ is ev-relatively free, resp. ev-relatively very free,
(iii) for every $i = 1, \ldots, r$, the degree of $\zeta^*\psi_i$ is nonpositive, and
(iv) the image under ζ of the geometric generic point of $\mathbb{P}^1$ is a stable map with irreducible domain.

Let T be a Deligne-Mumford stack and let $\zeta : \mathbb{P}^1 \times T \to \overline{\mathcal{M}}_{0,r}(X^0, e)$ be a 1-morphism. The morphism ζ is *twisting relative to* T, resp. *very twisting relative to* T, if for every geometric point of T the restriction of ζ over this point is twisting, resp. very twisting.

Remark 3.8. (i) If $r = 0$, the morphism ev is just the structure morphism to Spec (K). So in this case, ζ is ev-relatively free, resp. ev-relatively very free, iff it is free, resp. very free.

(ii) Condition (i) implies that ev is smooth along the image of ζ. If $\mathrm{char}(K) = 0$ and X is a general hypersurface of degree $d < \frac{n+1}{2}$ in $\mathbb{P}^n$, the smooth locus of ev is precisely the set of points where ev is unobstructed. But for some schemes the unobstructed locus is strictly smaller than the smooth locus, e.g., for a general hypersurface of degree n in $\mathbb{P}^n$ and $e = 4$.

(iii) The 1-morphism ζ is equivalent to a datum $((\pi : \Sigma \to \mathbb{P}^1, \sigma_1, \dots, \sigma_r), g : \Sigma \to X^0)$. The class $\zeta^*\psi_i$ is simply the divisor class of $\sigma_i^*\mathcal{O}_\Sigma(-\sigma_i(\mathbb{P}^1))$. Therefore condition (iii) states that the self-intersection $(\sigma_i(\mathbb{P}^1) \cdot \sigma_i(\mathbb{P}^1))_\Sigma$ is nonnegative for every $i = 1, \dots, r$.

(iv) Often the morphism $(\pi, g) : \Sigma \to \mathbb{P}^1 \times X^0$ is unramified and is étale locally a regular embedding, i.e., the sheaf homomorphism $d(\pi, g)^\dagger : (\pi, g)^*\Omega_{\mathbb{P}^1 \times X^0} \to \Omega_\Sigma$ is surjective and the kernel is locally free. In particular, this is true if $e = 1$. If $d(\pi, g)^\dagger$ is surjective and the kernel is locally free, denote by $N_{(\pi,g)}$ the dual of the kernel. The morphism ev is unobstructed along $\zeta(\mathbb{P}^1)$ iff $R^1\pi_*(N_{(\pi,g)}(-(\sigma_1(\mathbb{P}^1) + \cdots + \sigma_r(\mathbb{P}^1))))$ is 0 and then ζ^*T_{ev} is the locally free sheaf $\pi_*(N_{(\pi,g)}(-(\sigma_1(\mathbb{P}^1) + \cdots + \sigma_r(\mathbb{P}^1))))$.

(v) By Remark 3.5, if there exists a 1-morphism $\zeta : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,r}(X, e)$ that satisfies (i)–(iii), and if $\zeta(\mathbb{P}^1)$ intersects the closure of the open set parametrizing stable maps with irreducible domain, then a small deformation of ζ is a 1-morphism satisfying (i)–(iv).

If $r \geq 2$ consideration of the Picard group of Σ implies that $\deg(\zeta^*\psi_i) = 0$ for all $i = 1, \dots, r$. However if $r = 1$ – the main case of interest – then $-\deg(\zeta^*\psi)$ can be arbitrarily positive.

Proposition 3.9. *Let $\zeta_0 : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,1}(X^0, e)$ be a very twisting morphism. Denote $a_0 = -deg(\zeta_0^*\psi)$.*

(i) *If a_0 is odd, there exists an integer a_1 and for every integer $a \geq a_1$ there is a very twisting morphism $\zeta_a : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,1}(X^0, e)$ such that $a = -deg(\zeta_a^*\psi)$.*

(ii) *If a_0 is even, there exists an integer a_1 and for every even integer $a \geq a_1$ there is a very twisting morphism $\zeta_a : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,1}(X^0, e)$ such that $a = -deg(\zeta_a^*\psi)$.*

Proof. **(i):** Since ζ_0 is very twisting, $\zeta_0^*T_{\mathrm{ev}} \cong \mathcal{O}_{\mathbb{P}^1}(b_1) + \cdots + \mathcal{O}_{\mathbb{P}^1}(b_t)$ for integers $1 \leq b_1 \leq \cdots \leq b_t$. Define,

$$a_1 = 2a_0 \left\lceil \frac{a_0 + b_1}{2b_1} \right\rceil .$$

Let $a \geq a_1$ be an integer. There are 2 cases depending on whether a is even or odd.

If a is even, then $a = q(2a_0) + r$ for an integer q and an integer r satisfying $0 \leq r < 2a_0$. Because a is even, $r = 2r'$ for an integer r' satisfying $0 \leq r' < a_0$. Define $m = 2q$. Because $a \geq a_1$, $m \geq \frac{a_0+b_1}{b_1}$, and thus $mb_1 - r' > b_1 > 0$.

If a is odd, then $a + a_0 = q(2a_0) + r$ for an integer q and an integer r satisfying $0 \leq r < 2a_0$. Because $a + a_0$ is even, $r = 2r'$ for an integer r' satisfying $0 \leq r' < a_0$. Define $m = 2q - 1$. Because $a \geq a_1$, $m \geq \frac{a_0}{b_1}$, and thus $mb_1 - r' > 0$.

In each case, $a = ma_0 + 2r'$ where $0 \leq r' < a_0$ and where $mb_1 - r' > 0$. Let $h : \mathbb{P}^1 \to \mathbb{P}^1$ be a finite morphism of degree m. The morphism $\zeta_0 \circ h : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,1}(X, e)$ is very twisting. Moreover, $(\zeta_0 \circ h)^* T_{\text{ev}} \cong \mathcal{O}_{\mathbb{P}^1}(mb_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(mb_t)$ and $-\deg((\zeta_0 \circ h)^* \psi) = ma_0$.

Denote by $((\pi : \Sigma \to \mathbb{P}^1, \sigma : \mathbb{P}^1 \to \Sigma), g : \Sigma \to X)$ the datum giving rise to $\zeta_0 \circ h$. There exists a section $\sigma' : \mathbb{P}^1 \to \Sigma$ such that the divisor $\sigma'(\mathbb{P}^1) \subset \Sigma$ is in the linear equivalence class of $|\sigma(\mathbb{P}^1) + r'\pi^{-1}(0)|$. Denote by $\zeta_a : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,1}(X, e)$ the 1-morphism associated to the datum $((\pi : \Sigma \to \mathbb{P}^1, \sigma' : \mathbb{P}^1 \to \Sigma), g : \Sigma \to X)$.

For every $\mathsf{s} \in \mathbb{P}^1$, the obstruction group of ev at $\zeta_0(h(\mathsf{s}))$ is the hypercohomology group

$$\mathbf{H}^2(\pi^{-1}(\mathsf{s}), L^\vee_{(\pi,g)} \otimes \mathcal{O}_{\pi^{-1}(\mathsf{s})}(-\sigma(\mathsf{s}))),$$

where $L_{(\pi,g)}$ is the cotangent complex of the morphism $(\pi, g) : \Sigma \to \mathbb{P}^1 \times X$ and where $L^\vee_{(\pi,g)}$ is the object $RHom_{\mathcal{O}_\Sigma}(L_{(\pi,g)}, \mathcal{O}_\Sigma)$ in the derived category of quasi-coherent sheaves on Σ. By hypothesis, the obstruction group is 0. By construction, $\mathcal{O}_{\pi^{-1}(\mathsf{s})}(-\sigma'(\mathsf{s})) \cong \mathcal{O}_{\pi^{-1}(\mathsf{s})}(-\sigma'(\mathsf{s}))$. Therefore, the obstruction group of ev at $\zeta_a(\mathsf{s})$ is 0. So ζ_a satisfies (i) of Definition 3.7.

Moreover, $\zeta_a^* T_{\text{ev}}$ is the cohomology sheaf,

$$\mathcal{H}^1 R\pi_*(L^\vee_{(\pi,g)}(-\sigma'(\mathbb{P}^1))) = \mathcal{H}^1 R\pi_*(L^\vee_{(\pi,g)}(-\sigma(\mathbb{P}^1))) \otimes \mathcal{O}_{\mathbb{P}^1}(-r'),$$

i.e., $h^* \zeta_0^* T_{\text{ev}} \otimes \mathcal{O}_{\mathbb{P}^1}(-r')$. So $\zeta_a^* T_{\text{ev}} \cong \mathcal{O}_{\mathbb{P}^1}(mb_1 - r') \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(mb_t - r')$. By construction, $mb_1 - r' > 0$. Therefore $\zeta_a^* T_{\text{ev}}$ is ample. So ζ_a satisfies (ii) of Definition 3.7 for a very twisting morphism.

Finally, $(\sigma'(\mathbb{P}^1) \cdot \sigma'(\mathbb{P}^1))_\Sigma = (\sigma(\mathbb{P}^1) \cdot \sigma(\mathbb{P}^1))_\Sigma + 2r'$. Therefore,

$$-\deg(\zeta_a^* \psi) = -\deg(g^* \zeta_0^* \psi) + 2r' = ma_0 + 2r' = a.$$

So ζ_a satisfies (iii) of Definition 3.7 for a very twisting morphism. Because ζ_0 satisfies *(iv)* of Definition 3.7, also ζ_a satisfies *(iv)*. Therefore ζ_a is very twisting and $-\deg(\zeta_a^* \psi) = a$.

(ii): This is exactly as in the even case of (i). The odd case of (i) does not work because $a + a_0$ is not even. □

The main result about very twisting morphisms is the following theorem, which forms the induction step in the proof of Theorem 1.1.

Proposition 3.10. *Let X be a quasi-projective variety. Let $e_1, e_2 > 0$ be integers. Let r equal 1, resp. 2. Let $\zeta_1 : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,r}(X, e_1)$ be an ev-relatively very free morphism mapping a general point of $\mathbb{P}^1$ to a stable* *map with irreducible domain and with* $ev \circ \zeta_1$ *nonconstant (resp.* $ev_1 \circ \zeta_2$ *nonconstant). Let* $\zeta_2 : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,1}(X, e_2)$ *be a very twisting morphism.*

Assume that $\overline{\mathcal{M}}_{0,0}(X, \epsilon)$ *is irreducible for all positive integers* ϵ *that are sufficiently divisible. Assume one of the following,*

(i) $-deg(\zeta_1^*\psi) > 0$ *(resp.* $-deg(\zeta_1^*\psi_1) > 0$*), or*
(ii) *for every general degree* e_1 *morphism* $g : \mathbb{P}^1 \to X$ *there exists a twisting morphism* $\zeta_g : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,1}(X, e_2)$ *such that* $ev \circ \zeta_g = g$.

Then there exists an ev-relatively very free morphism $\zeta : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,r}(X, e_1 + e_2)$ *mapping a general point of* $\mathbb{P}^1$ *to a stable map with irreducible domain and with* $ev \circ \zeta_1$ *nonconstant (resp.* $ev_1 \circ \zeta_2$ *nonconstant).*

Moreover, if $r = 1$ *and* ζ_1 *is very twisting, then there exists such a* ζ *that is very twisting and has* $-deg(\zeta^*\psi) = 0$.

Proof. To ease notation, if $r = 1$ denote ev also by ev_1. By Proposition 3.9, assume that $-\mathrm{deg}(\zeta_2^*\psi) > 0$. If $r = 1$ and ζ_1 is very twisting, also assume that $-\mathrm{deg}(\zeta_1^*\psi) > 0$, i.e., (i) applies.

Let ϵ be a positive integer such that $\overline{\mathcal{M}}_{0,0}(X, \epsilon)$ is irreducible. After precomposing ζ_1 and ζ_2 with finite morphism $\mathbb{P}^1 \to \mathbb{P}^1$, assume that $\mathrm{deg}(\mathrm{ev} \circ \zeta_1) = \mathrm{deg}(\mathrm{ev} \circ \zeta_2) = \epsilon$. Denote by $\mathrm{Hom}(\mathbb{P}^1, X)_\epsilon$ the open subset of $\mathrm{Hom}(\mathbb{P}^1, X)$ parametrizing morphisms of degree ϵ; this is irreducible by hypothesis. There are morphisms,

$$\mathrm{Hom}(\mathbb{P}^1, \mathrm{ev}_1) : \mathrm{Hom}(\mathbb{P}^1, \overline{\mathcal{M}}_{0,r}(X, e_1)) \longrightarrow \mathrm{Hom}(\mathbb{P}^1, X)$$
$$\mathrm{Hom}(\mathbb{P}^1, \mathrm{ev}) : \mathrm{Hom}(\mathbb{P}^1, \overline{\mathcal{M}}_{0,1}(X, e_2)) \longrightarrow \mathrm{Hom}(\mathbb{P}^1, X)$$

By Lemma 3.4, $\mathrm{Hom}(\mathbb{P}^1, \mathrm{ev}_1)$ is smooth at $[\zeta_1]$ and $\mathrm{Hom}(\mathbb{P}^1, \mathrm{ev})$ is smooth at $[\zeta_2]$. So the image of each is a dense open subset of $\mathrm{Hom}(\mathbb{P}^1, X)_\epsilon$. Therefore, after deforming ζ_1 and ζ_2, assume that also $\mathrm{ev}_1 \circ \zeta_1 = \mathrm{ev} \circ \zeta_2$.

Denote by $((\pi_2 : \Sigma_2 \to \mathbb{P}^1, \sigma_{2,1}), g_2)$ the datum associated to ζ_2. If $r = 1$, denote by $((\pi_1 : \Sigma_1 \to \mathbb{P}^1, \sigma_{1,1}), g_1)$ the datum associated to ζ_1. If $r = 2$, denote by $((\pi_1 : \Sigma_1 \to \mathbb{P}^1, \sigma_{1,1}, \sigma_{1,2}), g_1)$ the datum associated to ζ_1. Denote the two cases in the statement as *Case (i)* and *Case (ii)* respectively. In Case (ii) a similar argument to the last paragaph proves, after deforming ζ_1 and ζ_2 further, there exists a dense open subset $U \subset \mathbb{P}^1$ such that for every closed point $\mathsf{s} \in U$,

(i) $\pi_1^{-1}(\mathsf{s})$ is irreducible, and
(ii) there exists a twisting morphism $\zeta_{\mathsf{s}} : \pi_1^{-1}(\mathsf{s}) \to \overline{\mathcal{M}}_{0,1}(X, e_2)$ such that $\zeta_{\mathsf{s}}(\sigma_{1,1}(\mathsf{s})) = \zeta_2(\mathsf{s})$.

In Case (i), denote $C = C_0 = \sigma_{1,1}(\mathbb{P}^1)$. In Case (ii), denote $C_0 = \sigma_{1,1}(\mathbb{P}^1)$. Let $\delta > 0$ be an integer such that $(C_0 \cdot C_0)_{\Sigma_1} + \delta > 0$. Let $\mathsf{s}_1, \dots, \mathsf{s}_\delta \in U$ be distinct points. For $i = 1, \dots, \delta$, denote $C_i = \pi_1^{-1}(\mathsf{s}_i)$. Denote $C = C_0 \cup C_1 \cup \dots \cup C_\delta$. In each case, the linear system $|C|$ on Σ_1 contains an irreducible curve that intersects $\sigma_{1,1}(\mathbb{P}^1)$ (resp. $\sigma_{1,1}(\mathbb{P}^1) \cup \sigma_{1,2}(\mathbb{P}^1)$) transversely in finitely many points. Let $\mathcal{D} \subset \mathbb{P}^1_t \times \Sigma_1$ be a divisor such that $\mathrm{pr}_{\mathbb{P}^1}^{-1}(0) \cap \mathcal{D} \subset \Sigma_1$ equals C and such that for general $\mathsf{t} \in \mathbb{P}^1_t$, $\mathrm{pr}_{\mathbb{P}^1}^{-1}(\mathsf{t}) \cap \mathcal{D} \subset \Sigma_1$ is an irreducible divisor that intersects $\sigma_{1,1}(\mathbb{P}^1)$ (resp. $\sigma_{1,1}(\mathbb{P}^1) \cup \sigma_{1,2}(\mathbb{P}^1)$) transversely in finitely many points (the subscript "t" in $\mathbb{P}^1_t$ is to distinguish $\mathbb{P}^1_t$ from the target of π_1).

Denote by $\zeta_C : C \to \overline{\mathcal{M}}_{0,1}(X, e_2)$ the unique morphism such that $\zeta_C \circ \sigma_{1,1} = \zeta_2$ and such that $\zeta_C|_{C_i} = \zeta_{\mathsf{s}_i}$ for $i = 1, \dots, \delta$ in Case (ii). There is a 1-morphism of relative Hom stacks,

$$\widehat{\mathrm{ev}} : \mathrm{Hom}_{\mathbb{P}^1_t}(\mathcal{D}, \mathbb{P}^1_t \times \overline{\mathcal{M}}_{0,1}(X, e_2)) \to \mathrm{Hom}_{\mathbb{P}^1_t}(\mathcal{D}, \mathbb{P}^1_t \times X).$$

There is a point $(0, [\zeta_C]) \in \mathrm{Hom}_{\mathbb{P}^1_t}(\mathcal{D}, \mathbb{P}^1_t \times \overline{\mathcal{M}}_{0,1}(X, e_2))$ lying over $0 \in \mathbb{P}^1_t$. By the same argument as in the proof of Lemma 3.4, the morphism $\widehat{\mathrm{ev}}$ is smooth at $(0, [\zeta_C])$.

The following morphism defines a section $\widehat{g}_1$ of $\mathrm{Hom}_{\mathbb{P}^1_t}(\mathcal{D}, \mathbb{P}^1_t \times X) \to \mathbb{P}^1_t$,

$$(\mathrm{pr}_{\mathbb{P}^1_t}, g_1 \circ \mathrm{pr}_{\Sigma_1}) : \mathcal{D} \to \mathbb{P}^1_t \times X.$$

Denote by $\mathrm{pr}_{\mathbb{P}^1_t} : H \to \mathbb{P}^1_t$ the fiber product,

$$H = \mathbb{P}^1_t \times_{\widehat{g}_1, \widehat{\mathrm{ev}}} \mathrm{Hom}_{\mathbb{P}^1_t}(\mathcal{D}, \mathbb{P}^1_t \times \overline{\mathcal{M}}_{0,1}(X, e_2)).$$

Again, $(0, [\zeta_C])$ defines a point of H lying over $0 \in \mathbb{P}^1_t$. And $\mathrm{pr}_{\mathbb{P}^1_t}$ is smooth at $(0, [\zeta_C])$ by base-change. Therefore the image of $\mathrm{pr}_{\mathbb{P}^1_t}$ is a dense open subset of $\mathbb{P}^1_t$. Let $\mathsf{t} \in \mathbb{P}^1_t$ be a general point, and define $D \subset \Sigma_1$ to be $\mathrm{pr}_{\mathbb{P}^1_t}^{-1} \cap \mathcal{D} \subset \Sigma_1$. Then there exists a morphism $\zeta_D : \mathcal{D} \to \overline{\mathcal{M}}_{0,1}(X, e_2)$ that is a deformation of ζ_C. If t and ζ_D are general, then ζ_D is a very twisting morphism. The curve $D \subset \Sigma_1$ is the image of a section $\sigma_{1,0} : \mathbb{P}^1 \to \Sigma_1$. Denote by $\widetilde{\zeta}_2 : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,1}(X, e_2)$ the 1-morphism $\widetilde{\zeta} = \zeta_D \circ \sigma_{1,0}$.

By hypothesis, $D \cap \sigma_{1,1}(\mathbb{P}^1)$ (resp. $D \cap (\sigma_{1,1}(\mathbb{P}^1) \cup \sigma_{1,2}(\mathbb{P}^1))$) is transverse. Denote by $\nu : \widetilde{\Sigma}_1 \to \Sigma_1$ the blowing up of Σ_1 at the finitely many intersection points. Denote by $\widetilde{\sigma_{1,i}} : \mathbb{P}^1 \to \widetilde{\Sigma}_1$ the strict transform of $\sigma_{1,i}$ for $i = 0, 1$ (resp. $i = 0, 1, 2$). In Case (i), $(D \cdot \sigma_{1,1}(\mathbb{P}^1))_{\Sigma_1} = (D \cdot D)_{\Sigma_1} = -\deg(\zeta_1^* \psi_1) > 0$. If $r = 2$, also $(D \cdot \sigma_{1,2}(\mathbb{P}^1))_{\Sigma_1} = 0$. In Case (ii), $(D \cdot \sigma_{1,1}(\mathbb{P}^1))_{\Sigma_1} = -\deg(\zeta_1^* \psi_1) + \delta > 0$ and $(D \cdot D)_{\Sigma_1} = -\deg(\zeta_1^* \psi_1) + 2\delta > 0$. If $r = 2$, also $(D \cdot \sigma_{1,2}(\mathbb{P}^1))_{\Sigma_1} = \delta$. In each case, for the strict transform $\widetilde{D} \subset \widetilde{\Sigma}_1$, $(\widetilde{D} \cdot \widetilde{D})_{\widetilde{\Sigma}} \geq 0$; it is precisely 0 except in Case (ii) if $r = 1$.

Denote by $\widetilde{\pi}_1 = \pi_1 \circ \nu$ and denote $\widetilde{g}_1 = g_1 \circ \nu$. If $r = 1$, denote by $\widetilde{\zeta}_1 : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,r+1}(X, e_1)$ the 1-morphism associated to the datum $((\widetilde{\pi}_1 : \widetilde{\Sigma}_1 \to \mathbb{P}^1, \widetilde{\sigma}_{1,0}, \widetilde{\sigma}_{1,1}), \widetilde{g}_1)$. If $r = 2$, denote by $\widetilde{\zeta}_1 : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,r+1}(X, e_1)$ the 1-morphism associated to the datum $((\widetilde{\pi}_1 : \widetilde{\Sigma}_1 \to \mathbb{P}^1, \widetilde{\sigma}_{1,0}, \widetilde{\sigma}_{1,1}, \widetilde{\sigma}_{1,2}), \widetilde{g}_1)$. In each case, observe that $\mathrm{ev}_0 \circ \widetilde{\zeta}_1 = \mathrm{ev} \circ \widetilde{\zeta}_2$.

The following terminology is from [BM96]. Denote by τ the following genus 0 stable A-graph. There are two vertices v_1 and v_2 of degree e_1 and e_2 respectively. If $r = 1$, there are two flags, f_0 and f_1, attached to v_1. If $r = 2$, there are three flags, f_0, f_1 and f_2, attached to v_1. There is one flag attached to v_2 and together with f_0 it forms an edge connecting v_1 to v_2. If $r = 1$, then f_1 is a tail attached to v_1. If $r = 2$, both f_1 and f_2 are tails attached to v_1. Denote by $\overline{\mathcal{M}}(X, \tau)$ the associated Behrend-Manin stack. In a natural way the pair $(\widetilde{\zeta}_1, \widetilde{\zeta}_2)$ determines a 1-morphism $\zeta : \mathbb{P}^1 \to \overline{\mathcal{M}}(X, \tau)$: $\widetilde{\zeta}_1$ corresponds to v_1, $\sigma_{1,i}$ corresponds to f_i for each i, and $\widetilde{\zeta}_2$ corresponds to v_2.

There is a canonical contraction of τ to a stable A-graph with a single vertex of degree $e_1 + e_2$ and r tails. Denote by $\iota : \overline{\mathcal{M}}(X, \tau) \to \overline{\mathcal{M}}_{0,r}(X, e_1 + e_2)$ the associated 1-morphism. There is a canonical combinatorial morphism that is the inclusion of the maximal subgraph of τ that contains only the vertex v_1. Denote by $\alpha : \overline{\mathcal{M}}(X, \tau) \to \overline{\mathcal{M}}_{0,r+1}(X, e_1)$ the associated 1-morphism. Finally, denote by $\beta : \overline{\mathcal{M}}_{0,r+1}(X, e_1) \to \overline{\mathcal{M}}_{0,r}(X, e_1)$ the isogeny obtained by removing the tail f_0. Observe that $\text{ev} \circ \beta \circ \alpha$ is 2-equivalent to $\text{ev} \circ \iota$.

Of course $\alpha \circ \zeta = \widetilde{\zeta}_1$ and $\beta \circ \alpha \circ \zeta = \zeta_1$. Denote by $\iota \circ \zeta$ by ζ as well. The hypotheses that ζ_1 and ζ_2 are ev-relatively free imply that ι is unramified and is étale locally a regular embedding of codimension 1 at every point of $\zeta(\mathbb{P}^1)$. Denote by $\zeta^* N_\iota$ the locally free sheaf that is the dual of the pullback of the conormal sheaf of ι. The hypotheses on ζ_1 and ζ_2 imply that α is unobstructed at every point of $\zeta(\mathbb{P}^1)$ and β is unobstructed at every point of $\alpha(\zeta(\mathbb{P}^1))$. Denote by $\zeta^* T_\alpha$ the locally free sheaf that is the dual of the pullback of the sheaf of relative differentials of α. Denote by $\zeta^* \alpha^* T_\beta$ the locally free sheaf that is the dual of the pullback of the sheaf of relative differentials of β. Finally, ev is unobstructed at every point of $\beta(\alpha(\zeta(\mathbb{P}^1)))$ because ζ_1 is ev-relatively free. Denote by $\zeta^* \alpha^* \beta^* T_{\text{ev}}$ the locally free sheaf that is the dual of the pullback of the sheaf of relative differentials of ev. It follows that also $\text{ev} : \overline{\mathcal{M}}_{0,r}(X, e_1 + e_1) \to X^r$ is unobstructed at every point of $\zeta(\mathbb{P}^1)$. Denote by $\zeta^* T_{\text{ev}}$ the locally free sheaf that is the dual of the pullback of the sheaf of relative differentials of ev.

There is a filtration of $\zeta^* T_{\text{ev}}$,

$$\zeta^* T_{\text{ev}} = F^0 \supset F^1 \supset F^2 \supset F^3 \supset F^4 = (0),$$

with associated graded sheaves $F^0/F^1 = \zeta^* N_\iota$, $F^1/F^2 = \zeta^* \alpha^* \beta^* T_{\text{ev}} = \zeta_1^* T_{\text{ev}}$, $F^2/F^3 = \zeta^* \alpha^* T_\beta$, and $F^3/F^4 = \zeta^* T_\alpha = \widetilde{\zeta}_2^* T_{\text{ev}}$.

First, $\zeta^* N_\iota \cong \mathcal{O}_{\mathbb{P}^1}(-\widetilde{\zeta}_1^* \psi_0) \otimes \mathcal{O}_{\mathbb{P}^1}(-\widetilde{\zeta}_2^* \psi)$. As seen above, $-\deg(\widetilde{\zeta}_1^* \psi_0) = (\widetilde{D} \cdot \widetilde{D})_{\widetilde{\Sigma}_1} \geq 0$. And $-\deg(\widetilde{\zeta}_2^* \psi) \geq -\deg(\zeta_2^* \psi) > 0$. Therefore $\zeta^* N_\iota$ is an ample invertible sheaf. Next $\zeta_1^* T_{\text{ev}}$ is ample because ζ_1 is ev-relatively very free. Next, $\zeta^* \alpha^* T_\beta \cong \sigma_{1,0}^* \mathcal{O}_{\Sigma_1}(\sigma_{1,0}(\mathbb{P}^1))$ is an ample invertible sheaf of degree $(D \cdot D)_\Sigma > 0$. Therefore $\zeta^* \alpha^* T_\beta$ is an ample invertible sheaf. Finally,

$\zeta^* T_\alpha = \widetilde{\zeta}_2^* T_{\text{ev}}$ is ample because $\widetilde{\zeta}_2$ is very twisting. Because each of the associated graded sheaves is ample, also $\zeta^* T_{\text{ev}}$ is ample.

Also $\text{ev} \circ \zeta = \text{ev} \circ \zeta_1$ is free because ζ_1 is ev-relatively very free. Therefore ζ is ev-relatively very free. Of course ζ maps every point of $\mathbb{P}^1$ to a stable map with reducible domain. Because ζ_1 and ζ_2 are ev-relatively free, each of these stable maps deforms to a stable map with irreducible domain. So by Remark 3.5, a small deformation of ζ maps a general point of $\mathbb{P}^1$ to a stable map with irreducible domain. Because $\text{ev}_1 \circ \zeta = \text{ev}_1 \circ \zeta_1$, it is nonconstant.

Finally, if $r = 1$ and ζ_1 is very twisting, this is Case (i). By construction, the self-intersection of $\widetilde{\sigma}_{1,1}(\mathbb{P}^1) \subset \widetilde{\Sigma}_1$ is 0. Therefore ζ is very twisting and $-\deg(\zeta^*\psi) = 0$.

□

A similar argument proves the following result, which shows that if there exists a twisting morphism $\mathbb{P}^1 \to \overline{\mathcal{M}}_{0,2}(X, e)$, then for every $r \geq 2$ and every e' sufficiently divisible, there exists a twisting morphism $\mathbb{P}^1 \to \overline{\mathcal{M}}_{0,r}(X, e')$.

Proposition 3.11. *Assume that $\overline{\mathcal{M}}_{0,0}(X, \epsilon)$ is irreducible for all ϵ sufficiently divisible. Let $r \geq 0$, $s_0 \geq 0$, $s_1, \dots, s_r > 0$, and $e_0, e_1, \dots, e_r \geq 0$ be integers. Let $\zeta_0 : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,s_0}(X, e_0)$ and $\zeta_i : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,s_i}(X, e_i)$ be twisting morphisms for $i = 1, \dots, r$. Then there exists a twisting morphism $\zeta : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,s}(X, e)$ where $s = s_0 + s_1 + \dots + s_r - r$ and $e = e_0 + e_1 + \dots + e_r$.*

Sometimes existence of a very twisting morphism $\zeta : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,1}(X, e)$ implies existence of a twisting morphism $\zeta' : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,2}(X, e')$. This is explained in the following proposition.

Notation 3.12. Let $((\pi : \Sigma \to \mathbb{P}^1, \sigma : \mathbb{P}^1 \to \Sigma), g : \Sigma \to X)$ be a datum giving rise to a 1-morphism $\zeta : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,1}(X, e)$. Assume that the geometric generic fiber of π is irreducible, that $(\sigma(\mathbb{P}^1) \cdot \sigma(\mathbb{P}^1))_\Sigma = 0$, that $g \circ \sigma : \mathbb{P}^1 \to X$ is very free, and that g is unramified and étale locally a regular embedding. Denote by N the locally free sheaf on Σ that is the dual of the kernel of $dg^\dagger : g^*\Omega_X \to \Omega_\Sigma$. The linear system $|\sigma(\mathbb{P}^1)|$ is a base-point-free pencil. Denote by $\rho : \Sigma \to \mathbb{P}^1$ the associated morphism. Denote by $\tau_1, \tau_2 : \mathbb{P}^1 \to \Sigma$ sections of ρ whose images are general fibers of π. Denote by $\zeta' : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,2}(X, e')$ the 1-morphism associated to the datum $((\rho : \Sigma \to \mathbb{P}^1, \tau_1, \tau_2), g)$.

Definition 3.13. If ζ satisfies the hypotheses of Notation 3.12, define a *rotation of* ζ to be the 1-morphism ζ' associated to any choice of τ_1 and τ_2.

Proposition 3.14. *If ζ is very twisting and satisfies the hypotheses of Notation 3.12, then every rotation of ζ is twisting.*

Proof. The object $L^\vee_{(\pi,g)}$ in the derived category of quasi-coherent sheaves on Σ is represented by the complex,

$$\begin{array}{cc} & 0 \qquad\qquad 1 \\ L^\vee_{(\pi,g)}: & \pi^*T_{\mathbb{P}^1} \longrightarrow \mathcal{N} \end{array}$$

Denote by $\mathcal{O}_\Sigma(-D)$ the invertible sheaf $\mathcal{O}_\Sigma(-\sigma(\mathbb{P}^1)-\tau_1(\mathbb{P}^1)-\tau_2(\mathbb{P}^1))$. The object $L^\vee_{(\pi,g)}\otimes\mathcal{O}_\Sigma(-D)$ fits into a distinguished triangle,

$$\mathcal{N}(-D)[-1] \longrightarrow L^\vee_{(\pi,g)}(-D) \longrightarrow \pi^*T_{\mathbb{P}^1}(-D)[0] \longrightarrow \mathcal{N}(-D)[0]$$

Now $\pi^*T_{\mathbb{P}^1}(-D)$ has vanishing h^0, h^1 and h^2, because $R^i\pi_*\mathcal{O}_\Sigma(-D)$ is (0) for $i=1,2$. Therefore the hypercohomology of $L^\vee_{(\pi,g)}(-D)$ equals the hypercohomology of $\mathcal{N}(-D)[-1]$. Because ζ is very twisting, $\mathbf{h}^i(\Sigma,\mathcal{N}(-D)[-1]) = h^i(\Sigma, L^\vee_{(\pi,g)}(-D)) = 0$ for $i\geq 2$.

The object $L^\vee_(\rho,g)$ in the derived category of quasi-coherent sheaves on Σ is represented by the complex,

$$\begin{array}{cc} & 0 \qquad\qquad 1 \\ L^\vee_{(\rho,g)}: & \rho^*T_{\mathbb{P}^1} \longrightarrow \mathcal{N} \end{array}$$

And $\rho^*T_{\mathbb{P}^1}(-D)$ has vanishing h^0, h^1 and h^2: ρ_* is zero, and $R^1\rho^*$ is $\mathcal{O}_{\mathbb{P}^1}(-1)$. Therefore the hypercohomology of $L^\vee_{(\rho,g)}(-D)$ equals the hypercohomology of $\mathcal{N}(-D)[1]$. In particular, $h^i(\Sigma, L^\vee_{(\rho,g)}(-D)) = h^i(\Sigma,\mathcal{N}(-D)[-1]) = 0$ for $i\geq 2$.

Consider the object $C = R\rho_*L^\vee_{(\pi,g)}(-\tau_1(\mathbb{P}^1)-\tau_2(\mathbb{P}^1))$. For all $i\geq 3$, the cohomology sheaf $\mathcal{H}^iC$ is (0), i.e., C is quasi-isomorphic to a complex concentrated in degrees ≤ 2. Moreover, because this is a family of stable maps, for all $i\leq 0$, the cohomology sheaf $\mathcal{H}^iC$ is (0), i.e., C is quasi-isomorphic to a complex concentrated in degrees $[1,2]$.

Because $g\circ\sigma$ is very free, ζ' maps the geometric generic point of $\mathbb{P}^1$ to an unobstructed point of ev. Therefore $\mathcal{H}^2C$ is a torsion sheaf, which is just $\mathcal{H}^2R\rho_*L^\vee_{(\pi,g)}(-D)\otimes\mathcal{O}_{\mathbb{P}^1}(1)$. Because it is torsion, it is isomorphic to $\mathcal{H}^2C\otimes\mathcal{O}_{\mathbb{P}^1}(-1) = \mathcal{H}^2R\rho_*L^\vee_{(\pi,g)}(-D)$. There is a Leray spectral sequence computing the hypercohomology of $L^\vee_{(\pi,g)}(-D)$ in terms of the hypercohomology of $R\rho_*L^\vee_{(\pi,g)}(-D)$. In the spectral sequence, for every differential whose domain is $H^0(\mathbb{P}^1,\mathcal{H}^2R\rho_*L^\vee_{(\pi,g)}(-D))$, the target vector space is (0). Thus there is an injective homomorphism,

$$H^0(\mathbb{P}^1,\mathcal{H}^2R\rho_*L^\vee_{(\pi,g)}(-D)) \to \mathbf{H}^2(\Sigma, L^\vee_{(\pi,g)}(-D)).$$

Since $\mathbf{h}^2(\Sigma, L^\vee_{(\pi,g)}(-D)) = 0$, the torsion sheaf $\mathcal{H}^2R\rho_*L^\vee_{(\pi,g)}(-D) = (0)$. So also $\mathcal{H}^2C = (0)$, which proves that ev is unobstructed at every geomeric point of $\zeta'(\mathbb{P}^1)$. This is (i) of Definition 3.7.

Moreover, C is quasi-isomorphic to the complex concentrated in degree 1, namely $(\zeta')^* T_{\mathrm{ev}}[-1]$. To prove that $(\zeta')^* T_{\mathrm{ev}}$ is generated by global sections, it suffices to prove that

$$h^1(\mathbb{P}^1, (\zeta')^* T_{\mathrm{ev}} \otimes \mathcal{O}_{\mathbb{P}^1}(-1)) = 0.$$

This equals $\mathbf{h}^2(\mathbb{P}^1, C \otimes \mathcal{O}_{\mathbb{P}^1}(-1))$. Because C is concentrated in a single degree, the Leray spectral sequence computing hypercohomology of $L^\vee_{(\pi,g)}(-D)$ in terms of hypercohomology of $C \otimes \mathcal{O}_{\mathbb{P}^1}(-1)$ degenerates. In particular,

$$\mathbf{h}^2(\mathbb{P}^1, C \otimes \mathcal{O}_{\mathbb{P}^1}(-1)) = \mathbf{h}^2(\Sigma, L^\vee_{(\pi,g)}(-D)) = 0.$$

Therefore $(\zeta')^* T_{\mathrm{ev}}$ is generated by global sections. This is (ii) of Definition 3.7

Finally, $(\tau_i(\mathbb{P}^1) \cdot \tau_i(\mathbb{P}^1))_\Sigma = 0$ for $i = 1, 2$, because $\tau_i(\mathbb{P}^1)$ is a fiber of π. Therefore $\zeta^* \psi_i$ has degree 0 for $i = 1, 2$. This is (iii) of Definition 3.7. □

4. A very free family of pointed conics

Hypothesis 4.1. In this section, it is assumed that $\mathrm{char}(K) = 0$.

Let $X \subset \mathbb{P}^n$ be a general hypersurface of degree $d \leq n-2$. The following terminology is from [BM96]. Denote by τ the stable A-graph that has two degree 1 vertices v_1, v_2, and edge joining v_1 to v_2, and two tails f_1, f_2 attached to v_1 and v_2 respectively. Denote by $\overline{\mathcal{M}}(X, \tau)$ the associated Behrend-Manin stack, which is in fact a scheme. This stack parametrizes data $((L_1, x_1, y), (L_2, x_2, y))$ consisting of lines $L_1, L_2 \subset X$ a point $y \in L_1 \cap L_2$, and points $x_i \in L_i$ (possibly equal to y). There is an evaluation morphism $\mathrm{ev} : \overline{\mathcal{M}}(X, \tau) \to X^2$ sending a datum to (x_1, x_2).

Proposition 4.2. *The scheme $\overline{\mathcal{M}}(X, \tau)$ is an integral, projective scheme of dimension $3n - 2d - 1$. Every irreducible component of the singular locus $\overline{\mathcal{M}}(X, \tau)_{sing}$ has dimension $\leq 2n - d - 1$.*

Proof. Denote by τ_0 the graph obtained from τ by removing the tails f_1 and f_2. Then $\overline{\mathcal{M}}(X, \tau_0) = \overline{\mathcal{M}}_{0,1}(X, 1) \times_X \overline{\mathcal{M}}_{0,1}(X, 1)$. By [Kol96, Theorem V.4.3], $\overline{\mathcal{M}}_{0,1}(X, 1)$ is smooth, projective and irreducible of dimension $2n - d - 2$. By [HRS04, Thm. 2.1], $\mathrm{ev} : \overline{\mathcal{M}}_{0,1}(X, 1) \to X$ is flat of relative dimension $n - d - 1$, and every geometric fiber is connected. By straightforward computation, the singular locus of ev has codimension $\geq n - d$. Therefore $\mathrm{pr}_1 : \overline{\mathcal{M}}(X, \tau_0) \to \overline{\mathcal{M}}_{0,1}(X, 1)$ is flat of relative dimension $n - d - 1$, every geometric fiber is connected, and the singular locus of pr_1 has codimension $\geq n - d$ (being the preimage under the flat morphism pr_2 of the singular locus of ev). Thus $\overline{\mathcal{M}}(X, \tau_0)$ is an integral, projective scheme of dimension $3n - 2d - 3$, and the singular locus has dimension $\leq 2n - d - 3$. The morphism $\overline{\mathcal{M}}(X, \tau) \to \overline{\mathcal{M}}(X, \tau_0)$ is the fiber product of two $\mathbb{P}^1$-bundles. □

Remark 4.3. If $d = 1$ or $d = 2$, then $\overline{\mathcal{M}}(X, \tau)$ is smooth.

Proposition 4.4. *If $n \geq d^2$ and $d \geq 2$, the geometric generic fiber of $ev : \overline{\mathcal{M}}(X,\tau) \to X \times X$ is smooth, connected, nonempty and rationally connected of dimension $n+1-2d$.*

Proof. Let $(x_1, x_2) \in X \times X$ be a pair contained in no common line contained in X. Consider the morphism $\text{ev}_y : \text{ev}^{-1}(x_1,x_2) \to X$ by $((L,x_1,y),(L_2,x_2,y)) \mapsto y$. Because x_1, x_2 are contained in no common line contained in X, this morphism is a closed immersion. Denote by $Y_{x_1,x_2} \subset \mathbb{P}V$ the image closed subscheme.

Let V be a vector space of dimension $n+1$ such that $\mathbb{P}^n = \mathbb{P}V$. Let $L_1, L_2 \subset V$ be 1-dimensional subspaces corresponding to x_1 and x_2. For $i = 1,2$ denote by $\mathcal{E}_1$ the rank 2 locally $\mathcal{O}_{\mathbb{P}V}$-module $E_i = (L_i^\vee \otimes \mathcal{O}_{\mathbb{P}V}) \oplus \mathcal{O}_{\mathbb{P}V}(1)$. Denote by $\phi_i : V^\vee \otimes \mathcal{O}_{\mathbb{P}V} \to E_i$ the unique sheaf homomorphism such that $\text{pr}_1 \circ \phi_i : V^\vee \otimes \mathcal{O}_{\mathbb{P}V} \to L_i^\vee \otimes \mathcal{O}_{\mathbb{P}V}$ and $\text{pr}_2 \circ \phi_i : V^\vee \otimes \mathcal{O}_{\mathbb{P}V} \to \mathcal{O}_{\mathbb{P}V}(1)$ are the canonical surjections. There is an induced sheaf homomorphism,

$$\text{Sym}^d \phi_i : \text{Sym}^d(V^\vee) \otimes \mathcal{O}_{\mathbb{P}V} \to \text{Sym}^d(E_i) \cong \bigoplus_{k=0}^{d} (L_i^\vee)^{\otimes(d-k)} \otimes \mathcal{O}_{\mathbb{P}V}(k).$$

For each $k = 0, \ldots, d$, denote by $\phi_{i,k}$ composition of $\text{Sym}^d \phi_i$ with projection onto the k^{th} direct summand.

Let $F \in \text{Sym}^d(V^\vee)$ be a defining equation of X. It is straightforward that Y_{x_1,x_2} is the scheme of simultaneous zeroes of $\phi_{i,k}(F)$ for $i = 1, 1 \leq k \leq d$ and $i = 2, 1 \leq k \leq d-1$. Because $n+1-2d \geq 1$, Y_{x_1,x_2} is nonempty and connected, and every irreducible component has dimension $\geq n+1-2d$. In particular, this implies that $\text{ev} : \overline{\mathcal{M}}(X,\tau) \to X^2$ is surjective.

By Proposition 4.2 and Remark 4.3, the dimension of the singular locus of $\overline{\mathcal{M}}(X,\tau)$ is less than $2n-2 = \dim(X^2)$. Hence the morphism ev is generically smooth of relative dimension $n+1-2d$. So for $(x_1,x_2) \in X^2$ a general pair, Y_{x_1,x_2} is a complete intersection. By the adjunction formula, the dualizing sheaf is the restriction to Y_{x_1,x_2} of the invertible sheaf,

$$\bigwedge^{n+1}(V^\vee) \otimes \mathcal{O}_{\mathbb{P}V}(-n-1) \otimes \bigotimes_{k=1}^{d} \left((L_1^\vee)^{\otimes(d-k)} \otimes \mathcal{O}_{\mathbb{P}V}(k) \right) \otimes \bigotimes_{k=1}^{d-1} \left((L_2^\vee)^{\otimes(d-k)} \otimes \mathcal{O}_{\mathbb{P}V}(k) \right).$$

In other words, $\omega_Y \cong \mathcal{O}_Y(-n-1+d^2)$. Because $n \geq d^2$, $\omega_Y^\vee$ is ample, i.e., Y_{x_1,x_2} is a Fano manifold. By [KMM92a], [Cam92], Y_{x_1,x_2} is rationally connected. □

Remark 4.5. If $n \geq 3$ and $d = 1$, the proposition is still true. In this case, for any pair of distinct points $x_1, x_2 \in X$, the morphism $\text{ev}_y : \text{ev}^{-1}(x_1,x_2) \to X$ is the blowing up of X at x_1 and x_2.

Proposition 4.6. *Let $X \subset \mathbb{P}^n$ be a general hypersurface of degree d. If either $n \geq d^2$ and $d \geq 2$ or $n \geq 3$ and $d = 1$, there exists a ev-relatively very free 1-morphism $\zeta : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,2}(X,2)$ mapping a general point of $\mathbb{P}^1$ to a stable map with irreducible domain and with $ev_1 \circ \zeta$ nonconstant.*

Proof. The scheme X^2 is smooth, projective and rationally connected. Therefore there exists a family of very free rational curves dominating X^2, say $f : D \times \mathbb{P}^1 \to X^2$. For each geometric point $\mathsf{t} \in D$, denote by $\mathrm{pr}_{\mathsf{t}} : \overline{\mathcal{M}}(X,\tau)_{\mathsf{t}} \to \mathbb{P}^1$ the fiber product,

$$\overline{\mathcal{M}}(X,\tau)_{\mathsf{t}} = \mathbb{P}^1 \times_{f_{\mathsf{t}}, X^2, \mathrm{ev}} \overline{\mathcal{M}}(X,\tau).$$

By Proposition 4.2 and Remark 4.3, the image under ev of the singular locus of $\overline{\mathcal{M}}(X,\tau)$ has codimension ≥ 2. Therefore a general very free rational curve does not intersect this locus. By generic smoothness, for a general point $\mathsf{t} \in D$, $\overline{\mathcal{M}}(X,\tau)_{\mathsf{t}}$ is smooth (although pr_{t} is only generically smooth). By Proposition 4.4 and Remark 4.5, if t is general then the geometric generic fiber of pr_{t} is rationally connected, i.e., pr_{t} is a rationally connected fibration over a curve. By [GHS03], there exists a section σ_0 of pr_{t}, and $\sigma_0(\mathbb{P}^1)$ is contained in the smooth locus of pr_{t}. Denote by $\widehat{\sigma}_0 : \mathbb{P}^1 \to \overline{\mathcal{M}}(X,\tau)$ the composition of σ_0 with the projection.

Consider the morphism $\iota : \overline{\mathcal{M}}(X,\tau) \to \overline{\mathcal{M}}_{0,2}(X,2)$. Every smooth point of ev is an unobstructed point. Every unobstructed point of ev is in the open subset where ι is unramified and étale locally a regular embedding of codimension 1. In particular, the image of $\widehat{\sigma}_0$ is in this open set. Denote by N_ι the locally free sheaf on this open set that is the normal bundle of ι.

If $(x_1,x_2) \in X^2$ is general, $\mathrm{ev}^{-1}(x_1,x_2) = Y_{x_1,x_2}$ is contained in the locus where ι is unramified and étale locally a regular embedding of codimension 1. The restriction of the normal bundle of ι to Y_{x_1,x_2} is $\mathcal{O}_{\mathbb{P}V}(2)$. So for a very free curve in Y_{x_1,x_2}, not only is the relative tangent bundle T_{ev} ample, also N_ι is ample. Form a comb whose handle is σ_0, and whose teeth are very free curves in fibers of Y_{x_1,x_2}. By [Kol96, II.7.10, II.7.11], after attaching sufficiently many teeth, the comb deforms to a section σ of pr_{t} such that both σ^*T_{ev} is ample and σ^*N_ι is ample. Denote by $\widehat{\sigma} : \mathbb{P}^1 \to \overline{\mathcal{M}}(X,\tau)$ the induced morphism.

To distinguish it from $\mathrm{ev} : \overline{\mathcal{M}}(X,\tau) \to X^2$, denote by $\mathrm{ev}_a : \overline{\mathcal{M}}_{0,2}(X,2) \to X^2$ the evaluation morphism. On the open set where ι is unramified and étale locally a regular embedding of codimension 1, there is an exact sequence of locally free sheaves,

$$0 \longrightarrow T_{\mathrm{ev}} \longrightarrow \iota^* T_{\mathrm{ev}_a} \longrightarrow N_\iota \longrightarrow 0$$

Therefore, $\widehat{\sigma}^*\iota^*T_{\mathrm{ev}_a}$ is ample. So $\iota \circ \sigma : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,2}(X,2)$ is a ev_a-relatively very free morphism such that $\mathrm{ev}_a \circ \iota\widehat{\sigma}$ is nonconstant. Of course a general point of $\mathbb{P}^1$ is mapped to a stable map with reducible domain. But this stable map deforms to a stable map with irreducible domain. So by Remark 3.5, a small deformation ζ of $\iota \circ \widehat{\sigma}$ is a ev_a-relatively very free morphism mapping a general point of $\mathbb{P}^1$ to a stable map with irreducible domain and such that $\mathrm{ev}_a \circ \zeta$ is nonconstant. $\square$

5. Minimal twisting morphisms

Definition 5.1. Let X be a quasi-projective variety with smooth locus X^0. A twisting morphism $\zeta_0 : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,1}(X^0, 1)$ is a *minimal twisting morphism* if $\deg(\zeta_0^* h) = 1$ and $\deg(\zeta_0^* x) = 2$, cf. Notation 2.1. For a base scheme D, a twisting morphism relative to D, $\zeta_0 : D \times \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,1}(X^0, 1)$ is a *minimal twisting relative to D* if the restriction to every geometric point of D is a minimal twisting morphism.

Denote by $N \subset \mathrm{Hom}(\mathbb{P}^1, \overline{\mathcal{M}}_{0,1}(X^0, 1))$ the locally closed subscheme parametrizing minimal twisting morphisms.

Remark 5.2.
(i) In fact N is an open subset.
(ii) Let $((\pi : \Sigma \to \mathbb{P}^1, \sigma), g)$ be the datum associated to a twisting morphism ζ_0. Then ζ_0 is a minimal twisting morphism iff $g : \Sigma \to X$ is a closed immersion whose image is a smooth quadric surface and $g \circ \sigma : \mathbb{P}^1 \to X$ is a closed immersion whose image is a line.
(iii) Unless X is ruled by linear spaces over a non-uniruled variety, a minimal twisting morphism is minimal in the following sense: $\deg(\zeta_0^* x) \geq 2$ and $\deg(\zeta_0^* h) \geq 1$ for every twisting morphism ζ_0, cf. [HS05, Rmk. 5.12]. A variety is called *quadric type* if a minimal twisting family is minimal in this sense.
(iv) A smooth hypersurface is quadric type iff the degree is ≥ 2.

Notation 5.3. Let X be a quasi-projective variety and denote by X^0 the smooth locus of X. Denote by $\overline{\mathcal{M}}_{0,1}(X, 1)_{\mathrm{ev}}$ the maximal open subscheme of $\overline{\mathcal{M}}_{0,1}(X^0, 1)$ on which the obstruction group of $\mathrm{ev} : \overline{\mathcal{M}}_{0,1}(X^0, 1) \to X^0$ is zero. In other words, a pointed line $[L, x] \in \overline{\mathcal{M}}_{0,1}(X^0, 1)$ is in $\overline{\mathcal{M}}_{0,1}(X, 1)_{\mathrm{ev}}$ iff $T_X \otimes \mathcal{O}_L$ is generated by global sections. Denote by $T_{\mathrm{ev},X}$ the locally free sheaf on $\overline{\mathcal{M}}_{0,1}(X, 1)$ that is the dual of the sheaf of relative differentials of ev.

By definition, if $\zeta_0 : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,1}(X, 1)$ is a twisting family then $\zeta_0(\mathbb{P}^1) \subset \overline{\mathcal{M}}_{0,1}(X, 1)_{\mathrm{ev}}$ so that $\zeta_0^* T_{\mathrm{ev},X}$ is defined. The twisting family ζ_0 is very twisting iff $\zeta_0^* T_{\mathrm{ev},X}$ is ample. Typically this is not the case. Let X be of quadric type. Then for every connected component of N there exist nonnegative integers a, b such that $\zeta_0^* T_{\mathrm{ev},X} \cong \mathcal{O}_{\mathbb{P}^1}^a \oplus \mathcal{O}_{\mathbb{P}^1}(1)^b$. The family ζ_0 is very twisting iff $b > 0$ and $a = 0$.

Notation 5.4. Let X be a quasi-projective morphism, and denote by X^0 the smooth locus of X. Let $\mathsf{s} \in \mathbb{P}^1$ be a point, and let $[L, x] \in \overline{\mathcal{M}}_{0,1}(X^0, 1)_{\mathrm{ev}}$ be a point. Denote by $M_{\mathsf{s},[L,x]}$ the locally closed subscheme of $\mathrm{Hom}(\mathbb{P}^1, \overline{\mathcal{M}}_{0,1}(X^0, 1))$ that parametrizes minimal twisting families $\zeta_0 : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,1}(X^0, 1)$ such that $\zeta_0(\mathsf{s}) = [L, x]$. Denote by M the locally closed subscheme of $\mathbb{P}^1 \times \overline{\mathcal{M}}_{0,1}(X^0, 1)_{\mathrm{ev}} \times N$ that parametrizes triples $(\mathsf{s}, [L, x], \zeta_0)$ such that ζ_0 is a minimal twisting family with $\zeta_0(\mathsf{s}) = [L, x]$, i.e., M is the graph of the evaluation morphism $\mathrm{ev} : \mathbb{P}^1 \times N \to \overline{\mathcal{M}}_{0,1}(X^0, 1)_{\mathrm{ev}}$.

For every $\zeta_0 \in M_{\mathsf{s},[L,x]}$, $\zeta_0^* T_{\mathrm{ev},X} \otimes \kappa(\mathsf{s})$ equals T. For every connected component $M_{\mathsf{s},[L,x],i}$ of $M_{\mathsf{s},[L,x]}$, there is a pair of nonnegative integers a, b such that $\zeta_0^* T_{\mathrm{ev},X} \cong \mathcal{O}_{\mathbb{P}^1}^a \oplus \mathcal{O}_{\mathbb{P}^1}(1)^b$ for every ζ_0 in $M_{\mathsf{s},[L,x],i}$. The subbundle of $\zeta_0^* T_{\mathrm{ev},X}$ spanned by $\mathcal{O}_{\mathbb{P}^1}(1)^b$ restricts to a b-dimensional subspace of T. This defines a morphism from $M_{\mathsf{s},[L,x],i}$ to the Grassmannian of T.

Notation 5.5. For each connected component $M_{\mathsf{s},[L,x],i} \subset M_{\mathsf{s},[L,x]}$, denote by $q_{\mathsf{s},[L,x],i} : M_{\mathsf{s},[L,x],i} \to \mathrm{Grass}(b, T)$ the morphism defined above. The morphism $q_{\mathsf{s},[L,x],i}$ is called *spanning* if there is no proper subspace $T' \subset T$ such that the image of $q_{\mathsf{s},[L,x],i}$ is contained in $\mathrm{Grass}(b, T')$. Denote by $G(T)$ the disjoint union over all b of $\mathrm{Grass}(b, T)$. Denote by $q_{\mathsf{s},[L,x]} : M_{\mathsf{s},[L,x]} \to G(T)$ the disjoint union over all connected components $M_{\mathsf{s},[L,x],i}$ of the morphism $q_{\mathsf{s},[L,x],i}$. The morphism $q_{\mathsf{s},[L,x]}$ is *spanning* if at least one $q_{\mathsf{s},[L,x],i}$ is spanning.

Lemma 5.6. *Let X be a quasi-projective variety of quadric type, and denote by X^0 the smooth locus of X.*

(i) *The schemes M and N are smooth, and the projection morphism $pr_{1,2} : M \to \mathbb{P}^1 \times \overline{\mathcal{M}}_{0,1}(X^0, 1)_{ev}$ is smooth.*

(ii) *For each connected component $M_{\mathsf{s},[L,x],i} \subset M_{\mathsf{s},[L,x]}$, the morphism $q_{\mathsf{s},[L,x],i}$ is spanning iff the pullback morphism,*

$$q^*_{\mathsf{s},[L,x],i} : H^0(\mathit{Grass}(b, T), \mathcal{O}_{\mathit{Grass}}(1)) = T^\vee \to H^0(M_{\mathsf{s},[L,x],i}, q^*_{\mathsf{s},[L,x],i}\mathcal{O}_{\mathit{Grass}}(1)),$$

is injective.

(iii) *There exists an open subset $U \subset \mathbb{P}^1 \times \overline{\mathcal{M}}_{0,1}(X^0, 1)_{ev}$ such that for every $(\mathsf{s}, [L, x]) \in \mathbb{P}^1 \times \overline{\mathcal{M}}_{0,1}(X^0, 1)_{ev}$, $q_{\mathsf{s},[L,x]}$ is spanning iff $(\mathsf{s}, [L, x]) \in U$.*

Proof. **(i):** By [Kol96, Thm. II.1.7], the projection morphism $\mathrm{pr}_{1,2} : M \to \mathbb{P}^1 \times \overline{\mathcal{M}}_{0,1}(X^0, 1)_{\mathrm{ev}}$ is smooth if,

$$h^1(\mathbb{P}^1, \zeta_0^* T_{\overline{\mathcal{M}}_{0,1}(X^0,1)_{\mathrm{ev}}} \otimes \mathcal{O}_{\mathbb{P}^1}(-\mathsf{s})) = 0,$$

for every geometric point $(\mathsf{s}, [L_\mathsf{s}, x_\mathsf{s}], \zeta_0) \in M$. There is a short exact sequence,

$$0 \to \zeta_0^* T_{\mathrm{ev},X} \otimes \mathcal{O}_{\mathbb{P}^1}(-\mathsf{s}) \to \zeta_0^* T_{\overline{\mathcal{M}}_{0,1}(X^0,1)_{\mathrm{ev}}} \otimes \mathcal{O}_{\mathbb{P}^1}(-\mathsf{s}) \to \zeta_0^* \mathrm{ev}^* T_{X^0} \otimes \mathcal{O}_{\mathbb{P}^1}(-\mathsf{s}) \to 0.$$

Because $\mathrm{ev} \circ \zeta_0 : \mathbb{P}^1 \to X^0$ is a twisting line, h^1 of the third term is 0. Because $\zeta_0^* T_{\mathrm{ev},X}$ is generated by global sections, h^1 of the first term is 0. Therefore, by the long exact sequence in cohomology, h^1 of the middle term is 0.

The scheme X^0 is smooth by definition. The morphism $\mathrm{ev} : \overline{\mathcal{M}}_{0,1}(X^0, 1)_{\mathrm{ev}} \to X^0$ is smooth by definition. Hence $\overline{\mathcal{M}}_{0,1}(X^0, 1)_{\mathrm{ev}}$ is smooth. Hence the product $\mathbb{P}^1 \times \overline{\mathcal{M}}_{0,1}(X^0, 1)$ is smooth. Because $\mathrm{pr}_{1,2}$ is smooth, the scheme M is smooth. Because M is the graph of a morphism, projection $\mathrm{pr}_{1,3} : M \to \mathbb{P}^1 \times N$ is an isomorphism. Because $\mathbb{P}^1 \times N$ is smooth, the scheme N is smooth.

(ii): This follows from the definition by taking duals of T and T'.

(iii): It is not difficult to see that the set U is constructible. Therefore it suffices to prove it is stable under generization. Let $(R, \mathfrak{m})$ be a DVR containing K, and let $\mathsf{s} : \text{Spec }(R) \to \mathbb{P}^1$ and $[L, x] : \text{Spec }(R) \to \overline{\mathcal{M}}_{0,1}(X, 1)_{\text{ev}}$ be given. Denote by $\text{pr}_R : M_R \to \text{Spec }(R)$ the fiber product,

$$M_R = \text{Spec }(R) \times_{(\mathsf{s},[L,x]),\mathbb{P}^1 \times \overline{\mathcal{M}}_{0,1}(X,1),\text{pr}_{1,2}} M.$$

Because $\text{pr}_{1,2}$ is smooth, also pr_R is smooth. After replacing R by a finite unramified cover by a DVR, assume that for every connected component of M_R, the geometric generic fiber is also connected. Let $M_{R,i}$ be a connected component such that $q_{\mathsf{s},[L,x],i}$ is spanning.

There is a pullback homomorphism of R-modules,

$$q_R^* : T^\vee \otimes_K R \to H^0(M_{R,i}, q_R^* \mathcal{O}_{G(T)}(1)).$$

Because $q_{\mathsf{s},[L,x],i}$ is spanning, the kernel of the closed fiber of q_R^* is injective. Because pr_R is smooth, in particular it is flat. Therefore the target R-module is flat, hence torsion-free. So the kernel of q_R^* is a saturated submodule of $T^\vee$. Because the kernel of the closed fiber is zero, also the kernel of the generic fiber is zero. □

Proposition 5.7. *Let X be a quasi-projective variety of quadric type, and denote by X^0 the smooth locus of X. Denote by U the open set from Lemma 5.6.*

(i) *If U is nonempty, then there exists a very twisting family $\zeta' : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,1}(X, 1)$ such that $ev \circ \zeta' : \mathbb{P}^1 \to X^0$ is free.*
(ii) *If U is nonempty, then for a general deformation Y of X there exists a very twisting family $\zeta' : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,1}(Y^0, 1)$ such that $ev \circ \zeta' : \mathbb{P}^1 \to Y^0$ is free.*

Proof. **(i):** The idea is to construct a map ζ from a *comb* C to $\overline{\mathcal{M}}_{0,1}(X, 1)$ such that ζ deforms to a morphism $\zeta' : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,1}(X, 1)$ that is very twisting.

Define C_0 to be a copy of $\mathbb{P}^1$. By hypothesis, there exists a minimal twisting morphism $\zeta_0 : C_0 \to \overline{\mathcal{M}}_{0,1}(X, 1)$ such that the graph of ζ_0 intersects U. There are nonnegative integers a, b such that $\zeta_0^* T_{\text{ev},X} \cong \mathcal{O}_{C_0}^a \oplus \mathcal{O}_{C_0}(1)^b$. If $a = 0$, then ζ_0 is already very twisting. Therefore assume that $a > 0$.

Let $\mathsf{s}_1, \dots \mathsf{s}_a \in C_0$ be distinct points and denote by $[L_1, x_1], \dots, [L_a, x_a]$ their images under ζ_0. Of course $\zeta_0^{-1}(U)$ is infinite, so there exist points such that each pair $(\mathsf{s}_i, [L_i, x_i])$ is in U. For each i, denote $T_i = T_{\text{ev},X} \otimes \kappa([L_i, x_i])$. And denote by $T_i' \subset T_i$ the subspace spanned by the image of $\mathcal{O}_{C_0}(1)^b$ under restriction to the fiber at s_i. For $i = 1, \dots, a$, the quotient vector space T_i/T_i' is canonically isomorphic to $H^0(C_0, \zeta_0^* T_{\text{ev},X}/\mathcal{O}_{C_0}(1)^b)$. Denote this common vector space by T/T'.

Because each of $(\mathsf{s}_i, [L_i, x_i])$ is spanning, for each $i = 1, \dots, a$ there exists a curve C_i that is a copy of $\mathbb{P}^1$ and a minimal twisting morphism $\zeta_i : C_i \to \overline{\mathcal{M}}_{0,1}(X, 1)$ that is in $M_{\mathsf{s}_i,[L_i,x_i]}$, and an invertible subsheaf $\mathcal{L}_i \cong \mathcal{O}_{C_i}(1) \subset \zeta_i^* T_{\text{ev},X}$ such that the images of $\mathcal{L}_i \otimes \kappa(\mathsf{s}_i)$ in T/T' span T/T'.

Define $(\iota_i : C_i \hookrightarrow C)_{0 \leq i \leq a}$ to be the initial family of morphisms such that for $i = 1, \dots, a$, $\iota_i(\mathsf{s}_i) = \iota_0(\mathsf{s}_i)$. Then C is a connected, proper, nodal curve whose irreducible components are $C_0, \dots, C_a$. Moreover, the dual graph of C is a tree, the vertex of C_0 has valence a, and for $i = 1, \dots, a$ the vertex of C_i has valence 1 and is connected only to the vertex of C_0. The reducible curve C is a *comb*, the curve C_0 is the *handle*, and the curves $C_1, \dots, C_a$ are the *teeth*, cf. [Kol96, Defn. II.7.7].

By the universal property of C, there is a unique morphism $\zeta : C \to \overline{\mathcal{M}}_{0,1}(X^0, 1)_{\text{ev}}$ such that the restriction to each C_i is ζ_i. By [HS05, Lem. 4.5], $\zeta : C \to \overline{\mathcal{M}}_{0,1}(X^0, 1)_{\text{ev}}$ is twisting.

There exists a DVR R containing K and a proper, flat morphism $\rho : \mathcal{C} \to \text{Spec }(R)$ whose closed fiber is C, whose generic fiber is $\mathbb{P}^1$, and such that $\mathcal{C}$ is regular. Consider the relative Hom scheme,

$$H_R = \text{Hom}_{\text{Spec }(R)}(\mathcal{C}, \text{Spec }(R) \times \overline{\mathcal{M}}_{0,1}(X^0, 1)_{\text{ev}}).$$

The morphism ζ determines a point in the closed fiber of H_R. Because the restriction of $\zeta^* T_{\overline{\mathcal{M}}_{0,1}(X,1)}$ to every irreducible component of C is generated by global sections, a *leaf induction argument* proves that $h^1(C, \zeta^* T_{\overline{\mathcal{M}}_{0,1}(X,1)}) = 0$. By [Kol96, Thm. II.1.7], the projection morphism $H_R \to \text{Spec }(R)$ is smooth at $[\zeta]$. Therefore, after replacing R by a finite unramified cover by a DVR, there exists a section, i.e., there exists a morphism $\widetilde{\zeta} : \mathcal{C} \to \overline{\mathcal{M}}_{0,1}(X^0, 1)_{\text{ev}}$. Denote by $\zeta' : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,1}(X^0, 1)_{\text{ev}}$ the base-change of $\widetilde{\zeta}$ over the geometric generic point of Spec (R) (also base change K to the algebraic closure of the fraction field of R).

The claim is that ζ' is very twisting. By [HS05, Lem. 4.6], ζ' is twisting. Therefore it only remains to prove that $(\zeta')^* T_{\text{ev},X}$ is ample. Consider the locally free sheaf $\widetilde{\zeta}^* T_{\text{ev},X}$. Define $\mathcal{T} \subset \widetilde{\zeta}^* T_{\text{ev},X}(C_1 + \cdots + C_a)$ to be the kernel of the surjective sheaf homomorphism,

$$\widetilde{\zeta}^* T_{\text{ev},X}(C_1 + \cdots + C_a) \to \bigoplus_{i=1}^{a} \widetilde{\zeta}^* T_{\text{ev},X}(C_1 + \cdots + C_a)|_{C_i} / \mathcal{L}_i \otimes \mathcal{O}_{\mathcal{C}}(C_1 + \cdots + C_a)|_{C_i}.$$

In other words $\mathcal{T}$ is the *elementary transform up* of $\widetilde{\zeta}^* T_{\text{ev},X}$ along $C_1, \dots, C_a$ determined by $\mathcal{L}_1, \dots, \mathcal{L}_a$.

The sheaf $\mathcal{T}$ is locally free, and contains $\widetilde{\zeta}^* T_{\text{ev},X}$ as a subsheaf. The restriction of $\mathcal{T}$ to C_0 is the sheaf of meromorphic sections of $\zeta_0^* T_{\text{ev},X}$ that have simple poles in the direction of $\mathcal{L}_i \otimes \kappa(\mathsf{s}_i)$ for each $i = 1, \dots, a$. By construction, this sheaf is isomorphic to $\mathcal{O}_{C_0}(1)^{a+b}$. Also the restriction of $\mathcal{T}$ to C_i fits into a short exact sequence,

$$0 \longrightarrow \mathcal{L}_i \longrightarrow \zeta_i^* T_{\mathrm{ev},X} \longrightarrow \mathcal{T}|_{C_i} \longrightarrow \mathcal{L}_i(-\mathsf{s}_i) \longrightarrow 0.$$

In particular, $\mathcal{T}|_{C_i}$ is generated by global sections. Therefore, by [HS05, Lem 2.11], the sheaf $\mathcal{T}|_C$ is deformation ample. By [HS05, Lem. 2.9], the sheaf $\mathcal{T}$ on $\mathcal{C}$ is relatively deformation ample over Spec (R). By [HS05, Lem. 2.8], the restriction of $\mathcal{T}$ to the geometric generic fiber of $\mathcal{C}$ is ample. But, of course, the restriction of $\mathcal{T}$ to the geometric generic fiber is $(\zeta')^* T_{\mathrm{ev},X}$. Therefore ζ' is very twisting. Also, because $\zeta^* \mathrm{ev}^* T_X$ is generated by global sections, also $(\zeta')^* \mathrm{ev}^* T_X$ is generated by global sections, i.e., $\mathrm{ev} \circ \zeta' : \mathbb{P}^1 \to X^0$ is free.

(ii): Because $\mathrm{ev} \circ \zeta' : \mathbb{P}^1 \to X^0$ is free, it is a point at which the relative Kontsevich moduli space is smooth over the base of the deformation. Therefore $\mathrm{ev} \circ \zeta'$ deforms to Y. By [HS05, Prop. 4.8], the very twisting morphism ζ' also deforms to Y. ☐

Because of Proposition 5.7, to prove Theorem 1.1 it suffices to prove that there exists a hypersurface X, a point s, and a point $[L, x] \in \overline{\mathcal{M}}_{0,1}(X^0, 1)$ such that $q_{\mathsf{s},[L,x]}$ is spanning. The boundary case is when $n = d^2$. This case is the most difficult, and implies the result for all $n \geq d^2$. In this case the subspace of T has dimension 1, i.e., $q_{\mathsf{s},[L,x]}$ is a morphism $M_{\mathsf{s},[L,x]}E \to \mathbb{P}T$. To prove that $q_{\mathsf{s},[L,x]}$ is spanning, it suffices to prove there exists an element $\zeta_0 \in M_{\mathsf{s},[L,x]}$ at which $q_{\mathsf{s},[L,x]}$ is smooth – this will even prove that the image of $q_{\mathsf{s},[L,x]}$ contains a dense open subset of $\mathbb{P}T$. To see $q_{\mathsf{s},[L,x]}$ is smooth at ζ_0, it suffices to prove that the derivative of $q_{\mathsf{s},[L,x]}$ at ζ_0 is surjective. This is a deformation theory computation that is the heart of this note.

Before proceeding to this computation, the following proposition shows that the inequality $n \geq d^2$ in Theorem 1.1 is necessary.

Proposition 5.8. *Let K be an algebraically closed field of characteristic* 0. *Let (d, n) be a pair of positive integers such that $d \leq \frac{n}{2}$. Let $X \subset \mathbb{P}^n$ be a hypersurface of degree d, and denote by X^0 the smooth locus of X. If there exists a very twisting morphism $\zeta : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,1}(X^0, 1)$, then $n \geq d^2$.*

Proof. By [HS05, Defn. 4.3 (ii)], the stable map $\mathrm{ev} \circ \zeta : \mathbb{P}^1 \to X^0$ is unobstructed. This implies that the relative Kontsevich moduli space of the universal family of degree d hypersurfaces over $\mathbb{P}\mathrm{Sym}^d(K^{n+1})$ is smooth over $\mathbb{P}\mathrm{Sym}^d(K^{n+1})$ at $([X], [\mathrm{ev} \circ \zeta])$. So for a general deformation of X, the stable map deforms as well. By [HS05, Prop. 4.8], also the very twisting family deforms. So for a general hypersurface, there is also a very twisting morphism. Therefore assume that X is general (this is made precise in the next paragraph).

By [HRS04], if X is general then every Kontsevich moduli space is irreducible. Also, for every degree e there is a free rational curve of degree e: because $d \leq n$ there is a free line L on X, and a finite degree e morphism

$\mathbb{P}^1 \to L$ is free. The locus of free rational curves is open. Therefore, a general point of the Kontsevich moduli space parametrizes a free rational curve. By the same argument in the last paragraph, ζ can be deformed so that $\text{ev} \circ \zeta : \mathbb{P}^1 \to X$ is a free rational curve.

Consider the Picard group of $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^n, 1)$. There is the projection morphism $\text{pr} : \overline{\mathcal{M}}_{0,1}(\mathbb{P}^n, 1) \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, 1) = \text{Grass}(2, n+1)$. And there is the projection morphism $\text{ev} : \overline{\mathcal{M}}_{0,1}(\mathbb{P}^n, 1) \to \mathbb{P}^n$. Denote by x the first Chern class of $\text{pr}^*\mathcal{O}_{\text{Grass}}(1)$, and denote by h the first Chern class of $\text{ev}^*\mathcal{O}_{\mathbb{P}^n}(1)$. For $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^n, 1)$, $\text{Chow}^1 = \mathbb{Z}\{x, h\}$. There is a tautological class $\psi = x - 2h$. For any family of pointed lines parametrized by a base B, say $\zeta_{\mathbb{P}^n} = (\pi : \Sigma \to B, \sigma : B \to \Sigma, g : \Sigma \to \mathbb{P}^n)$, $C_1(\sigma^*\mathcal{O}_\Sigma(\sigma(B))) = -\zeta^*\psi$.

It is straightforward to compute that,

$$C_1(T_{\text{ev},\mathbb{P}^n}) = nx - (n-1)h = (n+1)x + n\psi.$$

Denote the universal family of pointed lines by,

$$\widetilde{\zeta} = (\widetilde{\pi} : \widetilde{\Sigma} \to \overline{\mathcal{M}}_{0,1}(\mathbb{P}^n, 1), \widetilde{\sigma} : \overline{\mathcal{M}}_{0,1}(\mathbb{P}^n, 1) \to \widetilde{\Sigma}, \widetilde{g} : \widetilde{\Sigma} \to \mathbb{P}^n).$$

It is straightforward to compute that,

$$C_1(\widetilde{\pi}_*(\widetilde{g}^*\mathcal{O}_{\mathbb{P}^n}(d) \otimes \mathcal{O}_{\widetilde{\Sigma}}(-\text{Image}(\widetilde{\sigma})))) = \frac{d(d+1)}{2}x - dh = d^2h + \frac{d(d+1)}{2}\psi.$$

Therefore, if $\zeta_X : B \to \overline{\mathcal{M}}_{0,1}(X, 1)_{\text{ev}}$ is a morphism and $\zeta_{\mathbb{P}^n}$ is the associated morphism to $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^n, 1)$, then,

$$C_1(\zeta_X^* T_{\text{ev},X}) = C_1(\zeta_{\mathbb{P}^n}^* T_{\text{ev},\mathbb{P}^n}) - C_1(\pi_*(g^*\mathcal{O}_{\mathbb{P}^n}(-d) \otimes \mathcal{O}_\Sigma(-\sigma(B)))) =$$
$$(n - \tfrac{d(d+1)}{2})\zeta^*x - (n-d-1)\zeta^*h = (n+1-d^2)\zeta^*h + (n - \tfrac{d(d+1)}{2})\zeta^*\psi.$$

In particular, this holds for the very twisting family $\zeta : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,1}(X, 1)$.

There exists a short exact sequence,

$$0 \longrightarrow \zeta^*T_{\text{ev},X} \longrightarrow \zeta^*T_{\overline{\mathcal{M}}_{0,1}(X,1)} \longrightarrow \zeta^*\text{ev}^*T_X \longrightarrow 0$$

Because ζ is very twisting, the first term is positive. Because $\text{ev} \circ \zeta$ is free, the third term is semipositive. Therefore $\zeta^*T_{\overline{\mathcal{M}}_{0,1}(X,1)}$ is semipositive. Because $\zeta^*\text{pr}^*T_{\overline{\mathcal{M}}_{0,0}(X,1)}$ is a quotient of $\zeta^*T_{\overline{\mathcal{M}}_{0,1}(X,1)}$, it is also semipositive. The derivative $d(\text{pr} \circ \zeta) : \mathcal{O}_{\mathbb{P}^1}(2) \to \zeta^*\text{pr}^*T_{\overline{\mathcal{M}}_{0,0}(X,1)}$ is zero iff $\text{pr} \circ \zeta$ is constant. Therefore either $C_1(\zeta^*T_{\overline{\mathcal{M}}_{0,1}(X,1)})$ has positive degree or $\text{pr} \circ \zeta$ is constant.

The scheme $\overline{\mathcal{M}}_{0,1}(X, 1)$ is simply the Fano scheme of lines on X. The canonical bundle of the Fano scheme is straightforward to compute, giving,

$$\text{pr}^*C_1(T_{\overline{\mathcal{M}}_{0,0}(X,1)}) = (n + 1 - \frac{d(d+1)}{2})x.$$

So either $n + 1 > \frac{d(d+1)}{2}$, or else $\text{pr} \circ \zeta$ is constant. In the second case $\deg(\zeta^*x) = 0$, so that,

$$\deg(C_1(\zeta^*T_{\text{ev},X})) = -(n-d-1)\deg(\zeta^*h).$$

Since $d \leq n-1$ this is nonpositive, contradicting that $\zeta^* T_{\mathrm{ev},X}$ is ample. Therefore $n+1 > \frac{d(d+1)}{2}$, i.e. $n - \frac{d(d+1)}{2} \geq 0$.

Because ζ is very twisting, $\sigma^*\mathcal{O}_\Sigma(\sigma(\mathbb{P}^1))$ has nonnegative degree, i.e., $\deg(\zeta^*\psi) \leq 0$. So $(n - \frac{d(d+1)}{2})\deg(\zeta^*\psi) \leq 0$. Therefore,

$$\deg(C_1(\zeta^* T_{\mathrm{ev},X})) \leq (n+1-d^2)\deg(\zeta^* h).$$

Because $\zeta^* T_{\mathrm{ev},X}$ is ample, $(n+1-d^2)\deg(\zeta^*h) > 0$. And $\deg(\zeta^*h) \geq 0$. Therefore $n+1-d^2 > 0$, i.e., $n \geq d^2$. $\square$

6. Notation

In this section, the notation for the computation is introduced. A specific homogeneous polynomial G of degree d is given, and the hypersurface X is $V(G) \subset \mathbb{P}^n$. A minimal twisting family is specified,

$$\zeta_0 = (\pi : \Sigma \to B, \sigma : B \to \Sigma, f_0 : \Sigma \to X).$$

In a later section it is proved that ζ_0 is a smooth point of $\mathrm{Hom}(\mathbb{P}^1, \overline{\mathcal{M}}_{0,1}(X,1))$. An Artin local scheme D is specified that is a closed subscheme of the first-order neighborhood of $[\zeta_0]$ in $\mathrm{Hom}(\mathbb{P}^1, \overline{\mathcal{M}}_{0,1}(X,1))$. The family ζ_0 is "thickened" to a family,

$$\zeta = ((1,\pi) : D \times \Sigma \to D \times B, (1,\sigma) : D \times B \to D \times \Sigma, f : D \times \Sigma \to X).$$

For each closed point $\mathsf{s} \in B$, the pointed line $\zeta_0(\mathsf{s})$ is denoted by $[L_{\mathsf{s}}, x_{\mathsf{s}}]$, and the scheme $M_{\mathsf{s},[L_{\mathsf{s}},x_{\mathsf{s}}]}$ from Notation 5.4 is denoted by M_{s}. Similarly, the morphism $q_{\mathsf{s},[L_{\mathsf{s}},x_{\mathsf{s}}]}$ from Notation 5.5 is denote by q_{s}. The subscheme $D \cap M_{\mathsf{s}} \subset D$ is denoted by D_{s}. It turns out that to prove $dq_{\mathsf{s}}|_{\zeta_0}$ is surjective, it suffices to restrict to the Zariski tangent space $T_0 D_{\mathsf{s}} \subset T_{[\zeta_0]} M_{\mathsf{s}}$. This restriction is denoted $d'q_{\mathsf{s}}$.

Let K be an algebraically closed field with $\mathrm{char}(K) \neq 2$. Let $d \geq 3$ be an integer and denote $n = d^2$. Denote by I_d the set with $d^2 - 4$ elements,

$$\{(i,j) \in \mathbb{Z}^2 | 0 \leq i,j \leq d-1, (i,j) \neq (0,0), (0,1), (1,0), (1,1)\}.$$

Denote by V the $(n+1)$-dimensional K-vector space with ordered basis,

$$(a_0, a_1, a_2, a_3) \cup (b_{(i,j)} | (i,j) \in I_d) \cup (c).$$

Denote the dual ordered basis of $V^\vee$ by,

$$(X_0, X_1, X_2, X_3) \cup (Y_{(i,j)} | (i,j) \in I_d) \cup (Z).$$

Denote by V_a the subspace of V generated by $a_0, \dots, a_3$.

Given a pair of nonnegative integers, i and j, denote by $k = k(i,j)$ the minimum. Denote by $G \in \mathrm{Sym}^d(V^)$ the homogeneous polynomial,

$$G = (X_0X_3 - X_1X_2)X_3^{d-2} + \sum_{(i,j)\in I_d} X_0^k X_1^{i-k} X_2^{j-k} X_3^{d-1-i-j+k} Y_{(i,j)} + \sum_{(i,j)\in I_d} Y_{(i,j)} Z \gamma_{(i,j)} + Z^2 \gamma_z,$$

where the elements $\gamma_{(i,j)}$ and γ_z are homogeneous polynomials of degree $d-2$ in the variables X_0, X_1, X_2, X_3. Denote by $X \subset \mathbb{P}V$ the hypersurface defined by G.

Denote by Σ the surface,

$$\mathbb{P}^1_s \times \mathbb{P}^1_t = \{([S_0 : S_1], [T_0 : T_1]) | [S_0 : S_1], [T_0 : T_1] \in \mathbb{P}^1\}.$$

For each pair of integers (i,j), denote by $\mathcal{O}_\Sigma(i,j)$ the invertible sheaf $\mathrm{pr}^*_{\mathbb{P}^1_s}\mathcal{O}_{\mathbb{P}^1_s}(i) \otimes \mathrm{pr}^*_{\mathbb{P}^1_t}\mathcal{O}_{\mathbb{P}^1_t}(j)$.

Denote by B the curve $\mathbb{P}^1_s$. Denote by $\pi : \Sigma \to B$ projection onto the first factor. Denote by $\sigma : B \to \Sigma$ the section of π,

$$\sigma([S_0 : S_1]) = ([S_0 : S_1], [1 : 0]).$$

Denote by E the K-vector space with ordered basis,

$$(u^0_{(i,j)}, u^1_{(i,j)}, u^2_{(i,j)}, u^3_{(i,j)} | (i,j) \in I_d) \cup (v^0, v^1, v^2, v^3).$$

Denote by D' the local, Artin K-scheme,

$$D' = \mathrm{Spec}\ \mathrm{Sym}^\bullet(E)/\mathrm{Sym}^2(E) \cdot \mathrm{Sym}^\bullet(E).$$

There is a sheaf homomorphism, $V^\vee \otimes_K \mathcal{O}_{D'} \to V_a^\vee \otimes_K \mathcal{O}_{D'}$ by,

$$\begin{cases} X_l & \mapsto & X_l, & l = 0,1,2,3 \\ Y_{(i,j)} & \mapsto & \sum_{l=0}^3 u^l_{(i,j)} X_l, & (i,j) \in I_d \\ Z & \mapsto & \sum_l v^l X_l \end{cases}$$

This induces a morphism of schemes, $\phi' : D' \times V_a \to D' \times V$. There is a unique morphism of schemes $\psi' : D' \times \Sigma \to D' \times \mathbb{P}V_a$ such that $(\psi')^*\mathcal{O}_{\mathbb{P}V_a}(1) = \mathcal{O}_\Sigma(1,1)$ and such that,

$$X_0 \mapsto S_0T_0,\ X_1 \mapsto S_0T_1,\ X_2 \mapsto S_1T_0,\ X_3 \mapsto S_1T_1.$$

Define $f' : D' \times \Sigma \to D' \times \mathbb{P}V$ to be the composition $f' = \phi' \circ \psi'$.

If (i,j) is a pair that is not in I_d, define $u^l_{(i,j)}$ to be 0. Denote by $D \subset D'$ the closed subscheme whose ideal sheaf is,

$$I(D) = \langle u^0_{(i-1,j-1)} + u^1_{(i-1,j)} + u^2_{(i,j-1)} + u^3_{(i,j)} | 0 \leq i,j \leq d \rangle.$$

Define ϕ to be the restriction of ϕ' to $D \times V$, define ψ to be the restriction of ψ' to $D \times \Sigma$, and define f to be the composition $f = \phi \circ \psi$. Define ϕ_0, ψ_0 and f_0 to be the restrictions to the closed point $0 \in D$.

Lemma 6.1. *The preimage f^*G in $H^0(D \times \Sigma, f^*\mathcal{O}_{\mathbb{P}V}(d))$ equals 0. Therefore the morphism f factors through $D \times X \subset D \times \mathbb{P}V$.*

Proof. For every $(i,j) \in I_d$, $f^*Y_{(i,j)} \in \mathfrak{m} \cdot H^0(D \times \Sigma, f^*\mathcal{O}_{\mathbb{P}V}(d))$. Similarly for f^*Z. Because $\mathfrak{m}^2 = 0$, this implies that every term that involves $Y_{(i,j)}Z$ or Z^2 pulls back to 0. So these terms are ignored. Without these terms the preimage of f^*G is,

$$\begin{gathered}(S_0T_0 \cdot S_1T_1 - S_0T_1 \cdot S_1T_0)(S_1T_1)^{d-2} + \\ \textstyle\sum_{(i,j)\in I_d} S_0^i S_1^{d-1-i} T_0^j T_1^{d-1-j} (u^0_{(i,j)} S_0T_0 + u^1_{(i,j)} S_0T_1 + u^2_{(i,j)} S_1T_0 + u^3_{(i,j)} S_1T_1).\end{gathered}$$

The first term is 0 because $S_0T_0 \cdot S_1T_1 = S_0T_1 \cdot S_1T_0$. Gathering like monomials $S_0^i S_1^{d-i} T_0^j T_1^{d-j}$, and using the notation that $u^l_{(i,j)} = 0$ if $(i,j) \notin I_d$, the remaining terms sum to,

$$\sum_{0\le i,j\le d} (u^0_{(i-1,j-1)} + u^1_{(i-1,j)} + u^2_{(i,j-1)} + u^3_{(i,j)}) S_0^i S_1^{d-i} T_0^j T_1^{d-j}.$$

By definition, each element $u^0_{(i-1,j-1)} + u^1_{(i-1,j)} + u^2_{(i,j-1)} + u^3_{(i,j)}$ equals 0 in $\mathcal{O}_D$. Therefore f^*G equals 0. □

Denote,

$$\begin{gathered}\zeta_{\mathbb{P}V} = ((1,\pi) : D \times \Sigma \to D \times B, (1,\sigma) : D \times B \to D \times \Sigma, f : \Sigma \to \mathbb{P}V), \\ \zeta_X = ((1,\pi) : D \times \Sigma \to D \times B, (1,\sigma) : D \times B \to D \times \Sigma, f : \Sigma \to X), \\ \zeta_{\mathbb{P}V,0} = (\pi : \Sigma \to B, \sigma : B \to \Sigma, f_0 : \Sigma \to \mathbb{P}V), \\ \zeta_{X,0} = (\pi : \Sigma \to B, \sigma : B \to \Sigma, f_0 : \Sigma \to X)\end{gathered}$$

For each $\mathsf{s} = [\mathsf{s}_0 : \mathsf{s}_1] \in \mathbb{P}^1_s$, denote by $D_\mathsf{s} \subset D$ the closed subscheme whose ideal is,

$$I(D_\mathsf{s}) = \langle \mathsf{s}_0 u^0_{(i,j)} + \mathsf{s}_1 u^2_{(i,j)}, \mathsf{s}_0 u^1_{(i,j)} + \mathsf{s}_1 u^3_{(i,j)} | (i,j) \in I_d \rangle + \langle \mathsf{s}_0 v^0 + \mathsf{s}_1 v^2, \mathsf{s}_0 v^1 + \mathsf{s}_1 v^3 \rangle.$$

Denote by $\Lambda_\mathsf{s} \subset V_a \subset V$ the subspace,

$$\text{span}\{\mathsf{s}_0 a_0 + \mathsf{s}_1 a_2, \mathsf{s}_0 a_1 + \mathsf{s}_1 a_3\}.$$

Denote $L_\mathsf{s} = \mathbb{P}\Lambda_\mathsf{s} \subset \mathbb{P}V$, and denote $x_\mathsf{s} \in L_\mathsf{s}$ the point $[\mathsf{s}_0 a_0 + \mathsf{s}_1 a_2] \in L_\mathsf{s}$.

Lemma 6.2. *The closed subscheme $D_\mathsf{s} \subset D$ is the maximal subscheme over which $f : D_\mathsf{s} \times \pi^{-1}(\mathsf{s}) \to D_\mathsf{s} \times \mathbb{P}V$ factors through $D_\mathsf{s} \times L_\mathsf{s} \subset D_\mathsf{s} \times \mathbb{P}V$.*

Proof. The image of $D \times \pi^{-1}(\mathsf{s})$ under ϕ is V_s, where $V_\mathsf{s} \subset D \times V$ is the vector subbundle with generators,

$$\begin{gathered}\mathsf{s}_0 a_0 + \mathsf{s}_1 a_2 + \textstyle\sum_{(i,j)\in I_d} (\mathsf{s}_0 u^0_{(i,j)} + \mathsf{s}_1 u^2_{(i,j)}) b_{(i,j)} + (\mathsf{s}_0 v^0 + \mathsf{s}_1 v^2) c, \\ \mathsf{s}_0 a_1 + \mathsf{s}_1 a_3 + \textstyle\sum_{(i,j)\in I_d} (\mathsf{s}_0 u^1_{(i,j)} + \mathsf{s}_1 u^3_{(i,j)}) b_{(i,j)} + (\mathsf{s}_0 v^1 + \mathsf{s}_1 v^3) c\end{gathered}$$

Therefore the maximal closed subscheme D_s such that $D_\mathsf{s} \times_D V_\mathsf{s}$ is contained in $D_\mathsf{s} \times \Lambda_\mathsf{s}$ is the closed subscheme where the coefficients of $b_{(i,j)}$ and c in the generators equal 0, i.e., the closed subscheme D_s defined above. □

7. Description of some coherent sheaves

Denote by $\mathrm{ev} : \overline{\mathcal{M}}_{0,1}(\mathbb{P}V, 1) \to \mathbb{P}V$ the evaluation morphism. Denote by $T_{\mathrm{ev},\mathbb{P}V}$ the locally free sheaf on $\overline{\mathcal{M}}_{0,1}(\mathbb{P}V, 1)$ that is the dual of the sheaf of relative differentials of ev. Denote by $\mathrm{ev} : \overline{\mathcal{M}}_{0,1}(X, 1) \to X$ the evaluation morphism and denote by $\overline{\mathcal{M}}_{0,1}(X, 1)_{\mathrm{ev}} \subset \overline{\mathcal{M}}_{0,1}(X, 1)$ the open subscheme where ev is smooth. Denote by $T_{\mathrm{ev},X}$ the locally free sheaf on $\overline{\mathcal{M}}_{0,1}(X, 1)_{\mathrm{ev}}$ that is the dual of the sheaf of relative differentials of ev.

The families $\zeta_{\mathbb{P}V}$ and ζ_X in the last section define morphisms $\zeta_{\mathbb{P}V} : D \times B \to \overline{\mathcal{M}}_{0,1}(\mathbb{P}V, 1)$ and $\zeta_X : D \times B \to \overline{\mathcal{M}}_{0,1}(X, 1)$. In this section it is proved that the image of ζ_X is contained in $\overline{\mathcal{M}}_{0,1}(X, 1)_{\mathrm{ev}}$, and explicit descriptions are given for $\zeta_{\mathbb{P}V}^* T_{\mathrm{ev},\mathbb{P}V}$ and $\zeta_X^* T_{\mathrm{ev},X}$. This description will be the basis for the computation of dq_{s} in the next section.

Denote by $\widetilde{\psi} : D \times \Sigma \to D \times B \times \mathbb{P}V_a$ the unique morphism such that $\mathrm{pr}_{1,3} \circ \widetilde{\psi} = \psi$ and $\mathrm{pr}_2 \circ \widetilde{\psi} = \pi \circ \mathrm{pr}_\Sigma$. Similarly, denote by $\widetilde{f} : D \times \Sigma \to D \times B \times \mathbb{P}V$ the unique morphism such that $\mathrm{pr}_{1,3} \circ \widetilde{f} = f$ and $\mathrm{pr}_2 \circ \widetilde{f} = \pi \circ \mathrm{pr}_\Sigma$. Denote by $\widetilde{\psi} : D \times B \times \mathbb{P}V_a \to D \times B \times \mathbb{P}V$ the unique morphism such that $\mathrm{pr}_{1,3} \circ \widetilde{\psi} = \psi \circ \mathrm{pr}_{1,3}$ and $\mathrm{pr}_2 \circ \widetilde{\psi} = \mathrm{pr}_2$.

Denote by $N_{\widetilde{\phi}}$ the normal bundle of the closed immersion $\widetilde{\phi}$. Denote by $N_{\widetilde{f}}$ the normal bundle of the closed immersion $\widetilde{f}$. There is a short exact sequence of locally free sheaves on $D \times \Sigma$,

$$0 \to N_{\widetilde{\phi}} \xrightarrow{d\widetilde{\psi}} N_{\widetilde{f}} \to V \otimes_K f^*\mathcal{O}_{\mathbb{P}V}(1)/\psi(V_a) \cdot f^*\mathcal{O}_{\mathbb{P}V}(1) \to 0.$$

Using the Euler exact sequence for the tangent bundle of $\mathbb{P}V$ and the tangent bundle of $\mathbb{P}^1_t$, the locally free sheaf $N_{\widetilde{\phi}}$ is the cokernel of the sheaf homomorphism,

$$\mathrm{pr}_\Sigma^* \mathcal{O}_\Sigma(0, 1)\{t_0^\vee, t_1^\vee\} \to \mathrm{pr}_\Sigma^* \mathcal{O}_\Sigma(1, 1)\{a_0, a_1, a_2, a_3\},$$

defined by,

$$t_0^\vee \mapsto s_0 a_0 + s_1 a_2,\ t_1^\vee \mapsto s_0 a_1 + s_1 a_3.$$

The cokernel is the sheaf homomorphism,

$$\mathrm{pr}_\Sigma^* \mathcal{O}_\Sigma(1, 1)\{a_0, a_1, a_2, a_3\} \to \mathrm{pr}_\Sigma^* \mathcal{O}_\Sigma(2, 1)\{e_0, e_1\},$$

defined by,

$$a_0 \mapsto s_1 e_0,\ a_1 \mapsto s_1 e_1,\ a_2 \mapsto -s_0 e_0,\ a_3 \mapsto -s_0 e_1.$$

This cokernel is identifed with $N_{\widetilde{\phi}}$.

There is a canonical surjective sheaf homomorphism $f^* T_{\mathbb{P}V} \to N_{\widetilde{f}}$. Moreover, by the Euler exact sequence for $\mathbb{P}V$, there is a canonical sheaf homomorphism $V \otimes_K \mathcal{O}_{\mathbb{P}V}(1) \to T_{\mathbb{P}V}$. Together, these give a canonical sheaf homomorphism $V \otimes_K \mathrm{pr}_\Sigma \mathcal{O}_\Sigma(1, 1) \to N_{\widetilde{f}}$. For each element v of V, also

Element	∂G_0	$\partial G_{\mathfrak{m}}$
c	0	$\sum_l (2v^l \gamma_z + \sum_{(i,j)} u^l_{(i,j)} \gamma_{(i,j)}) f^* X_l$
$b_{(i,j)}$	$S_0^i S_1^{d-1-i} \cdot T_0^j T_1^{d-1-j}$	$-$
a_0	$S_1^{d-1} \cdot T_1^{d-1}$	$-$
a_1	$-S_1^{d-1} \cdot T_0 T_1^{d-2}$	$-$
a_2	$-S_0 S_1^{d-2} \cdot T_1^{d-1}$	$-$
a_3	$S_0 S_1^{d-2} \cdot T_0 T_1^{d-2}$	$-$
e_0	$S_1^{d-2} \cdot T_1^{d-2}$	$-$
e_1	$-S_1^{d-2} \cdot T_0 T_1^{d-2}$	$-$

FIGURE 1. The homomorphism ∂G

denote by v the induced sheaf homomorphism $\text{pr}_\Sigma^* \mathcal{O}_\Sigma(1,1) \to N_{\widetilde{f}}$, and the induced global section of $N_{\widetilde{f}} \otimes \text{pr}_\Sigma^* \mathcal{O}_\Sigma(-1,-1)$.

In particular there are global sections of $N_{\widetilde{f}} \otimes \text{pr}_\Sigma^* \mathcal{O}_\Sigma(-1,-1)$: c and $b_{(i,j)}$, for $(i,j) \in I_d$. The images of these global sections in $V \otimes_K \mathcal{O}_{D\times\Sigma}/\widetilde{f}(V_a)$ give a basis as a free $\mathcal{O}_{D\times\Sigma}$-module. With the isomorphism from the last paragraph this defines an isomorphism,

$$N_{\widetilde{f}} \cong \text{pr}_\Sigma^* \mathcal{O}_\Sigma(2,1)\{e_0, e_2\} \oplus \text{pr}_\Sigma^* \mathcal{O}_\Sigma(1,1)\{b_{(i,j)} | (i,j) \in I_d\} \oplus \text{pr}_\Sigma^* \mathcal{O}_\Sigma(1,1)\{c\}.$$

The locally free sheaf $\zeta_{\mathbb{P}V}^* T_{\text{ev},\mathbb{P}V}$ is canonically isomorphic to $(1\times\pi)_* N_{\widetilde{f}}(-D\times \sigma(B))$, cf. [HS05, Sec. 3]. Therefore,

$$\zeta_{\mathbb{P}V}^* T_{\text{ev},\mathbb{P}V} \cong \text{pr}_B^* \mathcal{O}_B(2)\{e_0, e_2\} \oplus \text{pr}_B^* \mathcal{O}_B(1)\{b_{(i,j)} | (i,j) \in I_d\} \oplus \text{pr}_B^* \mathcal{O}_B(1)\{c\}.$$

The morphism $T_{\mathbb{P}_V}|_X \to N_{X/\mathbb{P}V}$ induces a sheaf homomorphism,

$$N_f(-D \times \sigma(B)) \to f^*(\mathcal{O}_{\mathbb{P}V}(d))(-D \times \sigma(B)).$$

Applying $(1\times\pi)_*$, this leads to a sheaf homomorphism,

$$\partial G : \zeta_{\mathbb{P}V}^* T_{\text{ev},\mathbb{P}V} \to \text{pr}_B^* \mathcal{O}_B(d)\{t_0^j t_1^{d-1-j} | 0 \le j \le d-1\}.$$

By the deformation theory of stable maps, this 2-term complex is quasi-isomorphic to the good $[0,1]$ truncation of $\mathbb{R}\mathit{Hom}$ of the pullback by ζ_X of the relative cotangent complex of $\text{ev} : \overline{\mathcal{M}}_{0,1}(X,1) \to X$. In particular, the image of ζ_X is contained in $\overline{\mathcal{M}}_{0,1}(X,1)_{\text{ev}}$ iff ∂G is a surjective sheaf homomorphism. And in this case, $\zeta_X^* T_{\text{ev},X}$ is isomorphic to the kernel of ∂G, cf. [HS05, Rmk. 4.4].

There is a canonical splitting $\mathcal{O}_D \cong K \oplus \mathfrak{m}\mathcal{O}_D$. The domain and target of ∂G are pullbacks under pr_B of locally free sheaves on B. Therefore ∂G has a canonical splitting $\partial G = \partial G_0 + \partial G_{\mathfrak{m}}$. By Nakayama's lemma, to prove that ∂G is surjective, it suffices to prove that ∂G_0 is surjective after reducing modulo $\mathfrak{m}$. The homomorphism ∂G is computed in Figure 1.

Some explanation of Figure 1 is in order. The map being described in the first 6 rows is the map on global sections induced by the sheaf homomorphism $V \otimes_K \mathcal{O}_{D\times B} \to \text{pr}_B^*\mathcal{O}_B(d-1)\{T_0^jT_1^{d-1-j}|0 \le j \le d-1\}$ obtained by composing ∂G with the canonical surjection $V \otimes_K \text{pr}_B^*\mathcal{O}_B(1) \to \zeta_{\mathbb{P}V}^* T_{\text{ev},\mathbb{P}V}$ described above, and then twisting by $\mathcal{O}_B(-1)$. Each of these rows is obtained by taking the partial derivative of G with respect to the corresponding dual coordinate, and the pulling back by f^*.

The last two rows are the images of e_0 and e_1 under the map on global section obtained from ∂G by twisting by $\mathcal{O}_B(-2)$. The images are uniquely determined by the images of the a_ls together with the relation between e_0, e_1 and the a_ls.

The images of $b_{(i,j)}$ and e_0, e_1 under $\partial G_{\mathfrak{m}}$ are not given because they will not be used. The image of c will be used.

Proposition 7.1. (i) *The sheaf homomorphism ∂G is surjective. Moreover, the sheaf homomorphism obtained by twisting by $pr_B^*\mathcal{O}_B(-1)$ is surjective on global sections.*

(ii) *The image of ζ_X is contained in $\overline{\mathcal{M}}_{0,1}(X,1)_{ev}$, and $\zeta_X^* T_{ev,X}$ is generated by global sections.*

(iii) *The morphism $\zeta_X : D \times B \to \overline{\mathcal{M}}_{0,1}(X,1)$ is a minimal twisting family parametrized by D.*

(iv) *For every $\mathsf{s} \in B$, the restriction of ζ to D_{s} defines a morphism from D_{s} to M_{s}.*

Proof. **(i):** Because $\text{pr}_B^*\mathcal{O}_B(d)\{T_0^jT_1^{d-1-j}|0 \le j \le d-1\}$ is generated by global sections even after twisting by $\text{pr}_B^*\mathcal{O}_B(-1)$, it suffices to prove the sheaf homomorphism obtained from ∂G after twisting by $\text{pr}_B^*\mathcal{O}_B(-1)$ is surjective on global sections. Every monomial $S_0^iS_1^{d-1-i}T_0^jT_1^{d-1-j}$ in $H^0(B,\mathcal{O}_B(d-1))\{T_0^jT_1^{d-1-j}|0 \le j \le d-1\}$ occurs as the image under ∂G_0 of either $b_{(i,j)}$, if $(i,j) \in I_d$, or of S_0e_0, S_1e_0, S_0e_1 or S_1e_1.

(ii): As discussed above, because ∂G is surjective the image of ζ_X is contained in $\overline{\mathcal{M}}_{0,1}(X,1)_{\text{ev}}$ and $\zeta_X^*T_{\text{ev},X}$ is the kernel of ∂G. Because ∂G is surjective on global sections after twisting by $\text{pr}_B^*\mathcal{O}_B(-1)$, $H^1(D\times B, \zeta_X^*T_{\text{ev},X})$ is a subspace of $H^1(D\times B, \zeta_{\mathbb{P}V}^*T_{\text{ev},\mathbb{P}V})$. But this is zero. Therefore $\zeta_X^*T_{\text{ev},X}$ is generated by global sections.

(iii): By (ii), the image of ζ_X is contained in $\overline{\mathcal{M}}_{0,1}(X,1)_{\text{ev}}$, and the pullback of $T_{\text{ev},X}$ is generated by global sections. By construction, $f : D\times\Sigma \to D\times X$ is a closed immersion whose image is a family of smooth quadric surfaces. Also $f\circ(1,\sigma) : D\times B \to D\times X$ is a closed immersion whose image is a family of lines.

The only thing left to check is that the image of $f\circ(1,\sigma) : D\times B \to D\times X$ is a family of *free* lines. The computation so far is symmetric with respect to the involution that permutes $\mathbb{P}_s^1$ and $\mathbb{P}_t^1$. By (ii), every fiber of $\text{pr}_{\mathbb{P}_s^1}$:

$\mathbb{P}^1_s \times \mathbb{P}^1_t \to \mathbb{P}^1_s$ is mapped under f to a free line. So also every fiber of $\mathrm{pr}_{\mathbb{P}^1_t}$ is mapped to a free line. Since the image of σ is the fiber over $[1:0] \in \mathbb{P}^1_t$, $f \circ (1,\sigma) : D \times \mathbb{P}^1_s \to D \times X$ maps to a family of free lines.

(iv): This follows from (iii) and Lemma 6.2. □

Notation 7.2. A kernel of ∂G is,

$\iota : \mathrm{pr}_B^* \mathcal{O}_B\{f_{(i,j)} | (i,j) \in I_d, i \leq d-2\} \oplus \mathrm{pr}_B^* \mathcal{O}_B\{g_0, g_1\} \oplus \mathrm{pr}_B^* \mathcal{O}_B(1)\{h\} \to \zeta_{\mathbb{P}V}^* T_{\mathrm{ev},\mathbb{P}V}$,

defined by,

$$\left\{ \begin{array}{ccccc} f_{(i,j)} & \mapsto & S_0 b_{(i,j)} - S_1 b_{(i+1,j)} & + & f_{(i,j),\mathfrak{m}} \\ g_0 & \mapsto & S_0^2 e_0 - S_1 b_{(2,0)} & + & g_{0,\mathfrak{m}} \\ g_1 & \mapsto & S_0^2 e_1 + S_1 b_{(2,1)} & + & g_{1,\mathfrak{m}} \\ h & \mapsto & c & + & h_{\mathfrak{m}} \end{array} \right. ,$$

for some choice of

$$f_{(i,j),\mathfrak{m}}, g_{l,\mathfrak{m}} \in \mathfrak{m} \cdot H^0(D \times B, \zeta_{\mathbb{P}V}^* T_{\mathrm{ev},\mathbb{P}V}), \text{ and } h_{\mathfrak{m}} \in \mathfrak{m} \cdot H^0(D \times B, \zeta_{\mathbb{P}V}^* T_{\mathrm{ev},\mathbb{P}V} \otimes \mathrm{pr}_B^* \mathcal{O}_B(-1)).$$

There are many choices of $f_{(i,j),\mathfrak{m}}$ and $g_{l,\mathfrak{m}}$ such that ι is an isomorphism to the kernel of ∂G. The element $h_{\mathfrak{m}}$ is more canonical, and this is crucial to the computation. The space of global sections,

$$H^0(D \times B, \zeta_X^* T_{\mathrm{ev},X} \otimes \mathrm{pr}_B^* \mathcal{O}_B(-1)),$$

is a free $\mathcal{O}_D$-module of rank 1. So h is unique up to multiplication by an element in $1 + \mathfrak{m}$. Therefore $h_{\mathfrak{m}}$ is a well-defined global section of,

$$\mathfrak{m} \otimes_K \left(\zeta_{\mathbb{P}V,0}^* T_{\mathrm{ev},\mathbb{P}V} \otimes \mathcal{O}_B(-1) / \mathcal{O}_B\{c\} \right) = \mathfrak{m} \otimes_K \mathit{Hom}_{\mathcal{O}_B}(\mathcal{O}_B(1)\{c\}, \zeta_{\mathbb{P}V,0}^* T_{\mathrm{ev},\mathbb{P}V} / \mathcal{O}_B(1)\{c\}).$$

A lift of $h_{\mathfrak{m}}$ to $\mathfrak{m} \otimes_K \zeta_{\mathbb{P}V,0}^* T_{\mathrm{ev},\mathbb{P}V} \otimes \mathcal{O}_B(-1)$ is any global section,

$$\sum_{(i,j) \in I_d} h_{\mathfrak{m},(i,j)} b_{(i,j)} + h_{\mathfrak{m},0} e_0 + h_{\mathfrak{m},1} e_1,$$

whose image under ∂G_0 equals $-\partial G_{\mathfrak{m}}(c)$.

8. The derivative map

In the last section, a canonical global section $h_{\mathfrak{m}}$ was identified in,

$$\mathfrak{m}\mathcal{O}_D \otimes_K \mathit{Hom}_{\mathcal{O}_B}(\mathcal{O}_B(1)\{c\}, \zeta_{\mathbb{P}V,0}^* T_{\mathrm{ev},\mathbb{P}V} / \mathcal{O}_B(1)\{c\}).$$

In this section, $h_{\mathfrak{m}}$ is restricted to $\mathfrak{m}\mathcal{O}_{D_{\mathsf{s}}}$ and it is shown that this gives rise to the restriction to $T_0 D_{\mathsf{s}}$ of the derivative map $dq_{\mathsf{s}}|_{\zeta_0}$; this restriction is denoted $d' q_{\mathsf{s}}$.

Fix $\mathsf{s} \in B$. Restrict from D to the closed subscheme D_{s}. By Lemma 6.2, $\zeta_X(D_{\mathsf{s}} \times \{\mathsf{s}\})$ equals $\zeta_0(\mathsf{s})$ equals $[(L_{\mathsf{s}}, x_{\mathsf{s}})]$. Therefore, the map $\iota|_{D_{\mathsf{s}}}$ gives a map of $\mathcal{O}_{D_{\mathsf{s}}}$-modules,

$$\mathrm{pr}_B^* \mathcal{O}_B(1)\{h\} \otimes_{\mathcal{O}_B} \kappa(\mathsf{s}) \to T_{L_{\mathsf{s}}, x_{\mathsf{s}}} = H^0(L_{\mathsf{s}}, N_{L_{\mathsf{s}}/X}(-x_{\mathsf{s}})).$$

This in turn induces a K-linear map,

$$d'q_{\mathsf{s}} : T_0 D_{\mathsf{s}} = \mathrm{Hom}_K(\mathfrak{m}\mathcal{O}_{D_{\mathsf{s}}}, K) \to \mathrm{Hom}_K(\mathcal{O}_B(1)\{h\}\otimes\kappa(\mathsf{s}), T_{L_{\mathsf{s}},x_{\mathsf{s}}}/\mathcal{O}_B(1)\{h\}\otimes\kappa(\mathsf{s})).$$

The map $d'q_{\mathsf{s}}$ has another description. The element $h_{\mathfrak{m}}$ from the last section gives a global section $h_{\mathfrak{m},\mathsf{s}}$ of,

$$\mathfrak{m}\mathcal{O}_{D_{\mathsf{s}}} \otimes_K \mathit{Hom}_{\mathcal{O}_B}(\mathcal{O}_B(1)\{c\}, \zeta^*_{\mathbb{P}V,0}T_{\mathrm{ev},\mathbb{P}V}/\mathcal{O}_B(1)\{c\}).$$

This global section restricts to give an element in,

$$\mathfrak{m}\mathcal{O}_{D_{\mathsf{s}}} \otimes_K \mathit{Hom}_{\mathcal{O}_B}(\mathcal{O}_B(1)\{c\}, \zeta^*_{\mathbb{P}V,0}T_{\mathrm{ev},\mathbb{P}V}/\mathcal{O}_B(1)\{c\}) \otimes_{\mathcal{O}_B} \kappa(\mathsf{s}).$$

The image of this element in,

$$\mathfrak{m}\mathcal{O}_{D_{\mathsf{s}}} \otimes_K \mathcal{O}_B(d-1)\{T_0^j T_1^{d-1-j} | 0 \leq j \leq d-1\} \otimes_{\mathcal{O}_B} \kappa(\mathsf{s}),$$

is zero. Therefore this is an element in,

$$\mathfrak{m}\mathcal{O}_{D_{\mathsf{s}}} \otimes_K \mathit{Hom}_{\mathcal{O}_B}(\mathcal{O}_B(1)\{h\}, \zeta^*_{X,0}T_{\mathrm{ev},X}/\mathcal{O}_B(1)\{h\}) \otimes_{\mathcal{O}_B} \kappa(\mathsf{s}).$$

By adjointness of $\otimes$ and Hom, this induces the homomorphism $d'q_{\mathsf{s}}$ above.

The map $d'q_{\mathsf{s}}$ can be made more explicit. Let θ be an element in $T_0 D_{\mathsf{s}} = \mathrm{Hom}_K(\mathfrak{m}\mathcal{O}_{D_{\mathsf{s}}}, K)$. For each $(i,j) \in I_d$, the element $\langle h_{\mathfrak{m},(i,j)}, \theta\rangle$ is just an element in K. For $l = 0, 1$, the element $\langle h_{\mathfrak{m},l}, \theta\rangle$ is an element of $H^0(B, \mathcal{O}_B(1))$. The image in $H^0(B, \mathcal{O}_B(1) \otimes \mathcal{O}_B(-\mathsf{s}))$ is just an element in K. So $d'q_{\mathsf{s}}(\theta)$ is really just a vector of elements in K. All of these elements of K are uniquely determined by,

$$-\langle \partial G_{\mathfrak{m}}(c), \theta\rangle \in H^0(B, \mathcal{O}_B(d-1)\{T_0^j T_1^{d-1-1} | 0 \leq j \leq d-1\}).$$

The algorithm for computing $d'q_{\mathsf{s}}$ is the following.

Step 1 Choose an ordered basis for $\mathfrak{m}\mathcal{O}_{D_{\mathsf{s}}}$ and a dual basis for $T_0 D_{\mathsf{s}}$.

Step 2 For each basis element w of $\mathfrak{m}\mathcal{O}_{D_{\mathsf{s}}}$ with dual basis element θ, compute $\langle \partial G_{\mathfrak{m}}(c), \theta\rangle$, i.e., compute the coefficient of w in the expression $\partial G_{\mathfrak{m}}(c)$ in the vector space, $\mathfrak{m}\mathcal{O}_{D_{\mathsf{s}}} \otimes H^0(B, \mathcal{O}_B(d-1)\{T_0^j T_1^{d-1-j} | 0 \leq j \leq d-1\})$.

Step 3 Find the unique element in $H^0(B, \zeta^*_{\mathbb{P}V,0}T_{\mathrm{ev},\mathbb{P}V} \otimes \mathcal{O}_B(-1))$ that is mapped by ∂G_0 to $-\langle \partial G_{\mathfrak{m}}(c), \theta\rangle$, i.e., compute all the coefficients of $b_{(i,j)}$, and of $S_0 e_0$, $S_1 e_0$, $S_0 e_1$ and $S_1 e_1$.

Step 4 Compute the image of this element in $\zeta^*_{\mathbb{P}V,0}T_{\mathrm{ev},\mathbb{P}V} \otimes \mathcal{O}_B(-1) \otimes_{\mathcal{O}_B} \kappa(\mathsf{s})$. In essence, consider the element modulo $H^0(B, \mathcal{O}_B(1)\{e_0, e_1\} \otimes \mathcal{O}_B(-\mathsf{s}))$.

Step 5 Compute the element of $\zeta^*_{X,0}T_{\mathrm{ev},X} \otimes \mathcal{O}_B(-1)/\mathcal{O}_B\{h\} \otimes_{\mathcal{O}_B} \kappa(\mathsf{s})$ that is mapped by $\iota \otimes \kappa(\mathsf{s})$ to this element, i.e., compute all the coefficients of $f_{(i,j)}$, g_0 and g_1.

Proposition 8.1. *There exist polynomials γ_z and $\gamma_{(i,j)}$, for $(i,j) \in I_d$, and there exists $\mathsf{s} \in \mathbb{P}^1_s$ such that q_{s} is smooth at $\zeta_X|_0$.*

Index	Polynomial	f^*Polynomial
$\gamma_{(i,j)}$, $(i,j) \in I_d, i,j \le d-2$	$C_{(i,j)} X_0^k X_1^{i-k} X_2^{j-k} X_3^{d-2-i-j+k}$	$C_{(i,j)} S_0^i S_1^{d-2-i} \cdot T_0^j T_1^{d-2-j}$
$\gamma_{(d-1,j)}$, $2 \le j \le d-1$	$C_j X_1^{j-1} X_3^{d-1-j}$	$C_j S_0^{j-1} S_1^{d-1-j} T_1^{d-2}$
$\gamma_{(d-1,1)}$	$C_{1,a} X_1 X_3^{d-3} + C_{1,b} X_2 X_3^{d-3}$	$C_{1,a} S_0 S_1^{d-3} \cdot T_1^{d-2} + C_{1,b} S_1^{d-2} \cdot T_0 T_1^{d-3}$
γ_z	$C_{z,a} X_0 X_3^{d-3} + C_{z,b} X_1 X_3^{d-3}$	$C_{z,a} S_0 S_1^{d-3} \cdot T_0 T_1^{d-3} + C_{z,b} S_0 S_1^{d-3} \cdot T_1^{d-2}$
$\gamma_{(i,d-1)}$, $0 \le i \le d-2$	0	0
$\gamma_{(d-2,d-1)}$	0	0
$\gamma_{(d-1,0)}$	0	0

FIGURE 2. The polynomials γ

By Lemma 5.6, the domain and target of q_{s} are smooth. So, by the Jacobian criterion, q_{s} is smooth at ζ_0 iff $dq_{\mathsf{s}}|_{\zeta_0}$ is surjective. To prove this is surjective, it suffices to prove the restriction $d'q_{\mathsf{s}}$ is surjective.

9. Proof of the proposition

The point s is $[1:0]$. The polynomials $\gamma_{(i,j)}$ and γ_z are given in Figure 2. The terms $C_{(i,j)}$, C_j, $C_{1,a}$, $C_{1,b}$, $C_{z,a}$ and $C_{z,b}$ are elements of K that are generic (this is made more precise later). For any index (i,j) not appearing above, $C_{(i,j)}$ is defined to be 0. The remainder of this section carries out the algorithm of the previous section.

Step 1: Define $J_d = \{(i,j) \in I_d | (i,j) \neq (i,0), (0,2), (1,2)\}$. There is an isomorphism of K-vector spaces,

$$\mathfrak{m}\mathcal{O}_{D_{\mathsf{s}}} \to K\{v^2, v^3\} \cup \{w_{(i,j)} | (i,j) \in J_d\},$$

given in Figure 3.

Step 2: With respect to the isomorphism above, $h_{\mathfrak{m}}$ equals,

$$h_{\mathfrak{m}} = \sum_{(i,j)\in J_d} w_{(i,j)} \cdot \langle h_{\mathfrak{m}}, w_{(i,j)}^\vee \rangle + v^1 \cdot \langle h_{\mathfrak{m}}, (v^1)^\vee \rangle + v^2 \cdot \langle h_{\mathfrak{m}}, (v^2)^\vee \rangle$$

given in Figure 4. Each computation in Figure 4 is straightforward, and is left to the reader.

Step 3: For each w in the basis of $\mathfrak{m}\mathcal{O}_{D_{\mathsf{s}}}$, an element in $H^0(B, \zeta_{\mathbb{P}V,0}^* T_{\mathrm{ev},\mathbb{P}V} \otimes \mathcal{O}_B(-1))$ that maps under ∂G_0 to $-\langle h_{\mathfrak{m}}, w^\vee \rangle$ is given in Figure 5

Step 4: The images of these elements in $\zeta\mathbb{P}V, 0^* T_{\mathrm{ev},\mathbb{P}V} \otimes \mathcal{O}_B(-1) \otimes_{\mathcal{O}_B} \kappa(\mathsf{s})$ are obtained by setting S_1 equal to 0. This only effects the term for $w_{(d-1,1)}$.

Variable	Image	Range
$u^0_{(i,j)}$	0	$-$
$u^1_{(i,j)}$	0	$-$
$u^2_{(i,j)}$	$-w_{(i,j+1)}$	$j \leq d-2$
$u^2_{(i,d-1)}$	0	$-$
$u^3_{(i,j)}$	$w_{(i,j)}$	$(i,j) \in J_d$
$u^3_{(i,j)}$	0	$(i,j) = (i,0), (0,2), (1,2)$
v^0	0	$-$
v^1	0	$-$
v^2	v^2	$-$
v^3	v^3	$-$

FIGURE 3. Basis for $\mathfrak{m}\mathcal{O}_{D_s}$

Element w	$\langle h_{\mathfrak{m}}, w^\vee \rangle$
$w_{(i,j)}$, $(i,j) \in J_d, i \leq d-2$	$(C_{(i,j)} - C_{(i,j-1)}) S_0^i S_1^{d-1-i} T_0^j T_1^{d-1-j}$
$w_{(d-1,j)}$, $3 \leq j \leq d-1$	$C_j S_0^{j-1} S_1^{d-j} T_1^{d-1} - C_{j-1} S_0^{j-2} S_1^{d+1-j} T_0 T_1^{d-2}$
$w_{(d-1,2)}$	$-C_{1,a} S_0 S_1^{d-2} T_0 T_1^{d-2} - C_{1,b} S_1^{d-1} T_0^2 T_1^{d-3} + C_2 S_0 S_1^{d-2} T_1^{d-1}$
$w_{(d-1,1)}$	$C_{1,a} S_0 S_1^{d-2} T_1^{d-1} + C_{1,b} S_1^{d-1} T_0 T_1^{d-2}$
v^2	$C_{z,a} S_0 S_1^{d-2} T_0^2 T_1^{d-3} + C_{z,b} S_0 S_1^{d-2} T_0 T_1^{d-2}$
v^3	$C_{z,a} S_0 S_1^{d-2} T_0 T_1^{d-2} + C_{z,b} S_0 S_1^{d-2} T_1^{d-1}$

FIGURE 4. Computation of $\langle h_{\mathfrak{m}}, w^\vee \rangle$

Step 5: The vector space map,

$\zeta^*_{X,0} T_{\mathrm{ev},X} \otimes \mathcal{O}_B(-1)/\mathcal{O}_B\{h\} \otimes_{\mathcal{O}_B} \kappa(\mathsf{s}) \to \zeta^*_{\mathbb{P}V,0} T_{\mathrm{ev},\mathbb{P}V} \otimes \mathcal{O}_B(-1)/\mathcal{O}_B\{c\} \otimes_{\mathcal{O}_B} \kappa(\mathsf{s})$,

is given by,

$$\begin{cases} f_{(i,j)} & \mapsto & S_0 b_{(i,j)} \\ g_0 & \mapsto & S_0^2 e_0 \\ g_1 & \mapsto & S_0^2 e_1 \end{cases}$$

For each basis element $w \in \mathfrak{m}\mathcal{O}_{D_\mathsf{s}}$, $d' q_\mathsf{s}(w^\vee)$ is the preimage of the element in the previous table. This is given in Figure 6.

Element w	$-\partial G_0^{-1}\langle h_{\mathfrak{m}}, w^\vee\rangle$
$w_{(i,j)}$, $(i,j) \in J_d, i \leq d-2$	$(-C_{(i,j)} + C_{(i,j-1)})b_{(i,j)}$
$w_{(d-1,j)}$, $4 \leq j \leq d-1$	$-C_j b_{(j-1,0)} + C_{j-1} b_{(j-2,1)}$
$w_{(d-1,3)}$	$-C_3 b_{(2,0)} - C_2 S_0 e_1$
$w_{(d-1,2)}$	$-C_{1,a} S_0 e_1 + C_{1,b} b_{(0,2)} - C_2 S_0 e_0$
$w_{(d-1,1)}$	$-C_{1,a} S_0 e_0 + C_{1,b} S_1 e_1$
v^2	$-C_{z,a} b_{(1,2)} + C_{z,b} S_0 e_1$
v^3	$C_{z,a} S_0 e_1 - C_{z,b} S_0 e_0$

FIGURE 5. Computation of $-\partial G_0^{-1}\langle h_{\mathfrak{m}}, w^\vee\rangle$

Element w	$d'q_{\mathsf{s}}(w^\vee)$
$w_{(i,j)}$, $(i,j) \in J_d, i \leq d-2$	$(-C_{(i,j)} + C_{(i,j-1)})\frac{1}{S_0} f_{(i,j)}$
$w_{(d-1,j)}$, $4 \leq j \leq d-1$	$-C_j \frac{1}{S_0} f_{(j-1,0)} + C_{j-1} \frac{1}{S_0} f_{(j-2,1)}$
$w_{(d-1,3)}$	$-C_3 \frac{1}{S_0} f_{(2,0)} - C_2 \frac{1}{S_0} g_1$
$w_{(d-1,2)}$	$-C_{1,a} \frac{1}{S_0} g_1 + C_{1,b} \frac{1}{S_0} f_{(0,2)} - C_2 \frac{1}{S_0} g_0$
$w_{(d-1,1)}$	$-C_{1,a} \frac{1}{S_0} g_0$
v^2	$-C_{z,a} \frac{1}{S_0} f_{(1,2)} + C_{z,b} \frac{1}{S_0} g_1$
v^3	$C_{z,a} \frac{1}{S_0} g_1 - C_{z,b} \frac{1}{S_0} g_0$

FIGURE 6. Computation of $d'q_{\mathsf{s}}$

From the first row, the image of $d'q_{\mathsf{s}}$ contains $\frac{1}{S_0} f_{(i,j)}$ for all $(i,j) \in J_d$ and $i \leq d-2$. In particular, $(j-2,1)$ is such an index for $4 \leq j \leq d-1$. So, from the second row, the term $-C_{j-1}\frac{1}{S_0} f_{(j-2,1)}$ is already in the image of $d'q_{\mathsf{s}}$. Therefore $\frac{1}{S_0} f_{(j-1,0)}$ is in the image of $d'q_{\mathsf{s}}$ for $4 \leq j \leq d-1$.

Among the terms $f_{(i,j)}$, the only ones not covered by the first 2 rows are $(i,j) = (0,2), (1,2)$ and $(2,0)$.

From the fifth row, $\frac{1}{S_0} g_0$ is in the image of $d'q_{\mathsf{s}}$. Together with the last row, then $\frac{1}{S_0} g_1$ is in the image of $d'q_{\mathsf{s}}$. With the fourth row, $\frac{1}{S_0} f_{(0,2)}$ is in the image. With the sixth row, $\frac{1}{S_0} f_{(1,2)}$ is in the image. And with the third row, $\frac{1}{S_0} f_{(2,0)}$ is in the image. Therefore $d'q_{\mathsf{s}}$ is surjective.

10. A spanning minimal twisting family

If $d = 1$ or 2, any minimal twisting family is already very twisting.

Proposition 10.1. *Let $X \subset \mathbb{P}^n$ be a general hypersurface of degree $d \geq 3$. If $n \geq d^2$, there exists a point $\mathsf{s} \in \mathbb{P}^1$, a point $[L,x] \in \overline{\mathcal{M}}_{0,1}(X,1)$ and a minimal twisting morphism $\zeta_{X,0} : (\mathbb{P}^1, \mathsf{s}) \to (\overline{\mathcal{M}}_{0,1}(X,1), [L,x])$ such that $q_{\mathsf{s},[L,x]}$ is spanning at $\zeta_{X,0}$.*

Let $d \geq 3$ and let $n' \geq d^2$. Denote $n = d^2$. Choose a linear subspace $\mathbb{P}^n \subset \mathbb{P}^{n'}$. Let $X \subset \mathbb{P}^n$ be the hypersurface from Section 9. Let $X' \subset \mathbb{P}^n$ be a cone over X. Let $\mathsf{s} = [1:0]$ and let $\zeta_X : D_{\mathsf{s}} \times B \to \overline{\mathcal{M}}_{0,1}(X,1)$ be the minimal twisting family from Section 6. This is also a minimal twisting family for X'; denote by $\zeta_{X'} : D_{\mathsf{s}} \times B \to \overline{\mathcal{M}}_{0,1}(X',1)$ the corresponding morphism. There is a short exact sequence,

$$0 \longrightarrow \zeta_X^* T_{\mathrm{ev},X} \longrightarrow \zeta_{X'}^* T_{\mathrm{ev},X'} \longrightarrow \mathrm{pr}_B^* \mathcal{O}_B(1)^{n'-n} \longrightarrow 0.$$

Therefore, the quotient of $T_{\mathrm{ev},X'} \otimes \kappa([L_{\mathsf{s}}, x_{\mathsf{s}}])$ by $\mathrm{pr}_B^* \mathcal{O}_B(1)^{n'+1-d^2}$ is canonically isomorphic to the quotient of $T_{\mathrm{ev},X} \otimes \kappa([L_{\mathsf{s}}, x_{\mathsf{s}}])$ by $\mathrm{pr}_B^* \mathcal{O}_B(1)^{n+1-d^2}$. By Proposition 8.1, $d'q_{\mathsf{s}}$ surjects to this quotient. Therefore the morphism q_{s} for X' is smooth at ζ_0; i.e., q_{s} is spanning. By Proposition 5.7, for a general hypersurface of degree d in $\mathbb{P}^{n'}$, there exists a very twisting family of lines.

11. Can we do better?

What is the weakest inequality such that the spaces $\overline{\mathcal{M}}_{0,0}(X,e)$ are rationally connected? What is the weakest inequality such that there exists a very twisting 1-morphism $\zeta : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,1}(X,1)$? The weakest inequality such that there exists a very twisting morphism is $n \geq d^2$, cf. Proposition 5.8. But rational connectedness holds under slightly weaker hypotheses. The basic idea is contained in the following result.

Theorem 11.1 (de Jong). *Let K be an algebraically closed field with char$(K) = 0$. Let (d,n) be positive integers such that either $d = 1$ and $n \geq 2$ or $d \geq 2$ and $n+1 \geq d^2$. Let $X \subset \mathbb{P}^n$ be a general hypersurface of degree d. For every $e \geq 1$, every irreducible component of $\overline{\mathcal{M}}_{0,0}(X,e)$ is uniruled (or is a point if $(d,n,e) = (1,2,1)$).*

Proof. If $d = 1$ or $d = 2$, this is easy and follows from a stronger result of Kim and Pandharipande, cf. [KP01]. In those cases there is an action of SL_n, resp. SO_{n+1}, on $\overline{\mathcal{M}}_{0,0}(X, e)$. With the one exception of $(d, n, e) = (1, 2, 1)$, the stabilizer of a general point is a subgroup of positive codimension. Because SL_n and SO_{n+1} are rational, the orbit of a general point is a positive dimensional, unirational variety; in particular it is uniruled. Because the orbit of a general point is uniruled, $\overline{\mathcal{M}}_{0,0}(X, e)$ is uniruled.

Let $d \geq 3$. By [HS05, Prop. 6.5], there exists a twisting family $\zeta : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,1}(X, 1)$ such that $\mathrm{ev} \circ \zeta : \mathbb{P}^1 \to X$ is a line. Let $g : \mathbb{P}^1 \to \mathbb{P}^1$ be a finite morphism of degree e. Then $\zeta \circ g : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,1}(X, 1)$ is also twisting; i.e., there exists a degree e stable map $\mathrm{ev} \circ \zeta \circ g : \mathbb{P}^1 \to X$ that is *twistable*, cf. [HS05, Def. 4.7]. By [HS05, Prop. 4.8], there is a nonempty open subset of $\overline{\mathcal{M}}_{0,0}(X, e)$ of twistable stable maps. By [HRS04], $\overline{\mathcal{M}}_{0,0}(X, e)$ is irreducible, therefore this open set is dense. If $h : \mathbb{P}^1 \to X$ is in this open set, then there exists a twisting morphism $\xi : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,1}(X, 1)$ such that $\mathrm{ev} \circ \xi : \mathbb{P}^1 \to X$ equals h. Let $(\pi : \Sigma \to \mathbb{P}^1, \sigma : \mathbb{P}^1 \to \Sigma, g : \Sigma \to X)$ be the family inducing ζ. Then $\sigma(B) \subset \Sigma$ deforms in its linear equivalence class. This deformation defines a non-constant, rational transformation $\mathbb{P}^1 \dashrightarrow \overline{\mathcal{M}}_{0,0}(X, 1)$ whose image contains $[h]$. Therefore $\overline{\mathcal{M}}_{0,0}(X, 1)$ is uniruled. □

A. J. de Jong's proof is different, and proves a stronger result: for *every* smooth hypersurface, *every* irreducible component of $\overline{\mathcal{M}}_{0,0}(X, e)$ is uniruled. A. J. de Jong's theorem motivated a re-investigation of [HS05, Lem. 7.4] and the proof of Theorem 1.1.

In a forthcoming paper, by elaborating on the proof of Theorem 11.1, de Jong and I prove that if $n \geq d^2 + 1$, then the spaces of curves are rationally connected. However, that proof does not give the existence of very twisting families of lines.

This may seem inconsequential, but it is actually important for another purpose: de Jong and I have a method of generalizing the Tsen-Lang theorem. In that method, existence of a very twisting family of lines plays an important role. As the Tsen-Lang theorem holds for $n \geq d^2$ but *does not* hold for $n = d^2 + 1$, non-existence of a very twisting family of lines appears to be signficant.

In the other direction, there is reason to expect that $\overline{\mathcal{M}}_{0,0}(X, e)$ is not uniruled if $n + 2 \leq d^2$.

Proposition 11.2. [Sta03] *Let K be an algebraically closed field with char$(K) = 0$. Let (d, n) be positive integers such that $d < \min(n - 3, \frac{n+1}{2})$ and $n + 2 \leq d^2$ (if $n \geq 6$, the conditions are $d \leq \frac{n}{2}$ and $n + 2 \leq d^2$). Let $X \subset \mathbb{P}^n$ be a general hypersurface of degree d. For every $e \gg 0$ the canonical divisor class on $\overline{\mathcal{M}}_{0,0}(X, e)$ is* big. *If $e \leq n - d$, then the coarse moduli space $\overline{M}_{0,0}(X, e)$ is of general type (the singularities are* canonical*).*

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