# VERY TWISTING FAMILIES OF POINTED LINES ON GRASSMANNIANS

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Abstract. This excerpt is a section from an article in progress. This section proves that for the *Grassmannians*, i.e., the homogeneous spaces of Picard number one for the classical simple algebraic groups, there exists a very twisting family of pointed lines.

## 1. Very twisting lines on Grassmannians

This is an extract from an article in progress relating rational connectedness of spaces of rational curves to existence of sections of families over surfaces. Two other articles, [dJS05b] and [dJS05a], also deal with aspects of this work.

This extract is concerned with a very limited problem: extending the basic arguments from [HS05] to Grassmannians and isotropic Grassmannians. This might seem superfluous since Kim and Pandharipande prove rationality of the spaces of rational curves on every projective homogeneous spaces, [KP01]. However, for sections of families over surfaces, one needs also the existence of very twisting families of lines, which is what this note proves.

Let $\kappa$ be an algebraically closed field. Let $(X, \mathcal{O}_X(1))$ be a quasi-projective $\kappa$-variety together with an ample invertible sheaf. Denote by $X_{\text{sm}}$ the smooth locus of $X$. The scheme $\overline{\mathcal{M}}_{0,1}(X_{\text{sm}}, 1)$ represents the functor of pointed lines in $X$. On $\overline{\mathcal{M}}_{0,1}(X_{\text{sm}}, 1)$, there is a rank 2 locally free sheaf $E$, a rank 1 locally direct summand $L$ of $E$, and a morphism $g : \mathbb{P}(E) \to X_{\text{sm}}$ pulling back $\mathcal{O}_{X_{\text{sm}}}(-1)$ to the universal rank 1 locally direct summand $\mathcal{O}_{\mathbb{P}(E)}(-1)$ of $\pi^*E$. The universal line is the $\mathbb{P}^1$-bundle $\pi : \mathbb{P}(E) \to \overline{\mathcal{M}}_{0,1}(X_{\text{sm}}, 1)$, the universal section of $\pi$, $\sigma : \overline{\mathcal{M}}_{0,1}(X_{\text{sm}}, 1) \to \mathbb{P}(E)$, pulls back $\mathcal{O}_{\mathbb{P}(E)}(-1)$ to the locally direct summand $L$, and the universal map is $g$.

On $\overline{\mathcal{M}}_{0,1}(X_{\text{sm}}, 1)$ there is an important invertible sheaf $\psi^\vee$, defined as $\sigma^*\mathcal{O}_{\mathbb{P}(E)}(\text{Image}(\sigma)) = \sigma^*T_\pi$, the pullback of the normal bundle of $\sigma$, which is also the pullback of the vertical tangent bundle of $\pi$. Equivalently, $\psi^\vee$ satisfies a canonical isomorphism,

$$\psi^\vee \cong \det(E) \otimes (L^\vee)^{\otimes 2}.$$

The *evaluation morphism*, denoted $\text{ev} : \overline{\mathcal{M}}_{0,1}(X_{\text{sm}}, 1) \to X_{\text{sm}}$, is $g \circ \sigma$. The open subset where ev is smooth is denoted $U$. On this open set, an important locally free sheaf is the vertical tangent bundle $T_{\text{ev}}$ of ev, i.e., the dual of the sheaf of relative differentials of ev. The morphism $(\pi, g) : \mathbb{P}(E) \to \overline{\mathcal{M}}_{0,1}(X_{\text{sm}}, 1) \times X_{\text{sm}}$ is a regular embedding, and thus has a locally free normal sheaf $N$. The open subset $U$ is the

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maximal open set over which $N$ is $\pi$-relatively globally generated, and $T_{\mathrm{ev}}$ has an equivalent definition,

$$T_{\mathrm{ev}} \cong \pi_*(N(-\mathrm{Image}(\sigma))).$$

**Definition 1.1.** [HS05] A *very twisting family of pointed lines on* $X$ is a morphism $\zeta : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,1}(X_{\mathrm{sm}}, 1)$ such that,

(i) $U$ contains $\mathrm{Image}(\zeta)$,
(ii) $\zeta^* T_{\mathrm{ev}}$ is ample, and
(iii) the degree of $\zeta^*\psi^\vee$ is nonnegative.

Using the canonical isomorphisms, (i)–(iii) are equivalent to,

(i') the restriction of $N$ to the fiber of $\pi$ over each point of $\mathrm{Image}(\zeta)$ is globally generated,
(ii') $\zeta^*\pi_*(N(-\sigma(\mathbb{P}^1)))$ is ample, and
(iii') $\zeta^*\det(E) \otimes (L^\vee)^{\otimes 2}$ has nonnegative degree.

The open subset $U$ intersects each irreducible component of $\overline{\mathcal{M}}_{0,1}(X_{\mathrm{sm}}, 1)$ whose lines cover a dense subset of $X_{\mathrm{sm}}$, cf. [KMM92, 1.1].

**Lemma 1.2.** *If $T_{X_{sm}}$ is globally generated, then $U$ equals $\overline{\mathcal{M}}_{0,1}(X_{sm}, 1)$. In particular, if $X$ is a homogeneous space $G/P$, then $U$ equals $\overline{\mathcal{M}}_{0,1}(X, 1)$.*

*Proof.* Since $T_{X_{\mathrm{sm}}}$ is globally generated, $g^*T_{X_{\mathrm{sm}}}$ is globally generated, and thus the quotient $N$ is globally generated. For a homogeneous space $G/P$, $T_X$ is globally generated by $T_eG \otimes_\kappa \mathcal{O}_X$. □

Unfortunately, there typically exist rational curves in $\overline{\mathcal{M}}_{0,1}(G/P, 1)$ on which $\psi^\vee$ or $T_{\mathrm{ev}}$ has negative degree. Thus, if there exists a very twisting family, it is a special rational curve. There are some obvious special rational curves in $\overline{\mathcal{M}}_{0,1}(X, 1)$; in some cases these give very twisting families.

**Definition 1.3.** Let $\lambda : \mathbf{G}_m \to G$ be a 1-parameter subgroup. Let $s : G \times \overline{\mathcal{M}}_{0,1}(G/P, 1) \to \overline{\mathcal{M}}_{0,1}(G/P, 1)$ be the canonical action. Let $p \in \overline{\mathcal{M}}_{0,1}(G/P, 1)$ be a point. There is an induced morphism $\zeta^o : \mathbf{G}_m \to \overline{\mathcal{M}}_{0,1}(G/P, 1)$ by $\zeta(t) = s(\lambda(t), p)$. This extends uniquely to a morphism $\zeta : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,1}(G/P, 1)$, by the valuative criterion of properness. An *orbit curve* is a morphism $\zeta$ thus obtained.

**Question 1.4.** Does there exist a very twisting family $\zeta : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,1}(G/P, 1)$?

If the answer is affirmative, a second question is whether there exists a very twisting orbit curve.

We answer Question 1.4 when $X$ is the Grassmannian $\mathrm{Flag}(k, V)$ of rank $k$ subspaces of an $n$-dimensional vector space $V$ and when $X$ is the Grassmannian $\mathrm{Flag}_k(V, \beta)$ of rank $k$ isotropic subspaces of an $n$-dimensional vector space $V$ with a symmetric or skew-symmetric bilinear pairing $\beta$. Thus we answer the question when $G$ is one of the classical simple groups $\mathbf{SL}_n$, $\mathbf{SO}_n$, $\mathbf{Sp}_{2n}$ and $P$ is a maximal parabolic group.

**The exceptional cases.** There are some exceptional cases: there does not exist a very twisting family of pointed lines if $X$ equals a finite set, $\mathbb{P}^1$, or $\mathbb{P}^1 \times \mathbb{P}^1$. In these case ev is finite, and thus $T_{\mathrm{ev}}$ is the zero sheaf.

(i) For the classical Grassmannian, the single exceptional case is $(n, k) = (2, 1)$.

(ii) In the skew-symmetric case, the single exceptional case is $(n,k)=(2,1)$.

(iii) In the symmetric case, the exceptional cases are $(n,k)=(2,1),(3,1),(4,1)$, and $(4,2)$.

**Theorem 1.5.** *For every pair of positive integers $(n,k)$ satisfying $n \geq 2k$ and not on the exceptional list above, there is a very twisting family of pointed lines to $X$. In many cases, there is a very twisting orbit curve.*

## 2. Pointed lines on classical Grassmannians

Let $V$ be a rank $n$ $\kappa$-vector space, and let $k$ be an integer, $0<k<n$. Denote by $X$ the Grassmannian $\text{Flag}(k,V)$, and denote by $S(k,V)$ the universal rank $k$ locally direct summand of $V \otimes_\kappa \mathcal{O}_X$. Denote by $\mathcal{O}_X(1)$ the ample invertible sheaf giving the Plücker embedding, i.e., the ample generator of the Picard group of $X$. Denote by $\overline{\mathcal{M}}$ the scheme $\overline{\mathcal{M}}_{0,1}(X,1)$. Denote by $M$ the flag variety $\text{Flag}(k-1,k,k+1;V)$, and denote by $S_{k-1} \subset S_k \subset S_{k+1} \subset V \otimes \mathcal{O}_M$ the universal $(k-1,k,k+1)$-flag of locally direct summands of $V$.

**Lemma 2.1.** *On $\mathbb{P}(E)$, the locally free sheaf $g^*S(k,V)^\vee$ is globally generated. On $\overline{\mathcal{M}}$, $\pi_*[g^*S(k,V)^\vee \otimes \mathcal{O}_{\mathbb{P}(E)}(-1)]$ is an invertible sheaf, and $\pi_*[g^*S(k,V)^\vee]$ is locally free of rank $k+1$. The tautological map $V^\vee \otimes \mathcal{O}_{\overline{\mathcal{M}}} \to \pi_*[g^*S(k,V)^\vee]$ is surjective.*

*Proof.* Since $V^\vee \otimes \mathcal{O}_X$ generates $S(k,V)^\vee$, $V^\vee \otimes \mathcal{O}_{\mathbb{P}(E)}$ generates $g^*S(k,V)^\vee$. So the restriction of $g^*S(k,V)^\vee$ to every fiber of $\pi$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_k)$ for integers $0 \leq a_1 \leq \cdots \leq a_k$. By definition, $\mathcal{O}_X(1) = \bigwedge^k S(k,V)^\vee$ has degree 1 on every fiber of $\pi$. Thus the restriction of $g^*S(k,V)^\vee$ to every fiber of $\pi$ is isomorphic to $F := \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus(k-1)}$. So, firstly, $\pi_*[g^*S(k,V)^\vee]$ is locally free of rank $h^0(\mathbb{P}^1,F)=k+1$, and $\pi^*[g^*S(k,V)^\vee \otimes \mathcal{O}_{\mathbb{P}(E)}(-1)]$ is locally free of rank $h^0(\mathbb{P}^1,F(-1))=1$. Since the only subspace of $H^0(\mathbb{P}^1,F)$ generating $F$ is all of $H^0(\mathbb{P}^1,F)$, and since $V^\vee \otimes \mathcal{O}_{\mathbb{P}(E)}$ generates $g^*S(k,V)^\vee$, also $V^\vee \otimes \mathcal{O}_{\overline{\mathcal{M}}} \to \pi_*[g^*S(k,V)^\vee]$ is surjective. ☐

There is a $(1,2)$-flag of locally direct summands of $\pi_*[g^*S(k,V)^\vee]$,

$$\pi_*[g^*S(k,V)^\vee \otimes \mathcal{O}_{\mathbb{P}(E)}(-1)] \otimes (E/L)^\vee \subset \pi_*[g^*S(k,V)^\vee \otimes \mathcal{O}_{\mathbb{P}(E)}(-1)] \otimes E^\vee \subset \pi_*[g^*S(k,V)^\vee].$$

Dually, there is a $(k-1,k)$-flag of locally direct summands of $(\pi_*[g^*S(k,V)^\vee])^\vee$,

$$(\pi_*[g^*S(k,V)^\vee \otimes \mathcal{O}_{\mathbb{P}(E)}(-1)] \otimes E^\vee)^\perp \subset (\pi_*[g^*S(k,V)^\vee \otimes \mathcal{O}_{\mathbb{P}(E)}(-1)] \otimes (E/L)^\vee)^\perp \subset (\pi_*[g^*S(k,V)^\vee])^\vee.$$

Because $V^\vee \otimes \mathcal{O}_M \to \pi_*[g^*S(k,V)^\vee]$ is surjective, $(\pi_*[g^*S(k,V)^\vee])^\vee$ is canonically a rank $k+1$ locally direct summand of $V \otimes \mathcal{O}_{\overline{\mathcal{M}}}$. This defines a $(k-1,k,k+1)$-flag of locally direct summands of $V \otimes \mathcal{O}_{\overline{\mathcal{M}}}$, denoted $E_{k-1} \subset E_k \subset E_{k+1} \subset V \otimes \mathcal{O}_{\overline{\mathcal{M}}}$,

$$\begin{array}{rcc} E_{k-1} & := & (\pi_*[g^*S(k,V)^\vee \otimes \mathcal{O}_{\mathbb{P}(E)}(-1)] \otimes E^\vee)^\perp, \\ E_k & := & (\pi_*[g^*S(k,V)^\vee \otimes \mathcal{O}_{\mathbb{P}(E)}(-1)] \otimes (E/L)^\vee)^\perp, \\ E_{k+1} & := & (\pi_*[g^*S(k,V)^\vee])^\vee. \end{array}$$

By the universal property of the flag variety, there exists a unique morphism $\iota' : \overline{\mathcal{M}} \to M$ pulling back $S_{k-1} \subset S_k \subset S_{k+1} \subset V \otimes \mathcal{O}_M$ to $E_{k-1} \subset E_k \subset E_{k+1} \subset V \otimes \mathcal{O}_{\overline{\mathcal{M}}}$.

**Proposition 2.2.** *The morphism $\iota'$ is an isomorphism. Moreover, $ev : \overline{\mathcal{M}} \to Flag(k,V)$ is the composition of $\iota'$ with the tautological projection $Flag(k-1,k,k+1;V) \to Flag(k,V)$.*

*Proof.* On $M$, denote by $E'$ the rank 2 locally free sheaf $S_{k+1}/S_{k-1}$. Denote by $L'$ the rank 1 locally direct summand $S_k/S_{k-1}$. Denote by $\pi' : \mathbb{P}(E') \to \text{Flag}(k-1,k,k+1;V)$ the associated $\mathbb{P}^1$-bundle. There is a unique section $\sigma' : \text{Flag}(k-1,k,k+1;V) \to \mathbb{P}(E')$ pulling back the universal rank 1 locally direct summand $\mathcal{O}_{\mathbb{P}(E')}(-1)$ of $(\pi')^*E'$ to $L'$. On $\mathbb{P}(E')$, there is a rank $k$ locally direct summand $S'_k$ of $V \otimes \mathcal{O}_{\mathbb{P}(E')}$ defined to be the preimage in $(\pi')^*S_{k+1}$ of the universal rank 1 locally direct summand $\mathcal{O}_{\mathbb{P}(E')}(-1) \subset (\pi')^*(S_{k+1}/S_{k-1})$. By the universal property of $X$, there exists a unique morphism $g' : \mathbb{P}(E') \to X$ pulling back the locally direct summand $S(k,V)$ to $S'_k$. By definition of $\sigma'$, $(\sigma')^*S'_k$ equals $S_k$ as a locally direct summand of $V \otimes \mathcal{O}_F$. Therefore $g' \circ \sigma'$ is the tautological projection $\text{Flag}(k-1,k,k+1;V) \to \text{Flag}(k,V)$.

By the definition of $\mathcal{O}_X(1)$, $(g')^*\mathcal{O}_X(-1)$ is isomorphic to $\bigwedge^k S'_k$. By definition of $S'_k$, this is isomorphic to $(\pi')^* \bigwedge^{k-1} S_{k-1} \otimes \mathcal{O}_{\mathbb{P}(E')}(-1)$. In particular, the restriction of $(g')^*\mathcal{O}_X(1)$ to every fiber of $\pi'$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(1)$. Thus $(\pi', \sigma', g')$ is a family of pointed lines in $X$. By the universal property of $\overline{\mathcal{M}}_{0,1}(X,1)$, there exists a unique morphism $\iota : M \to \overline{\mathcal{M}}$ pulling back $(\pi,\sigma,g)$ to $(\pi',\sigma',g')$. It follows easily that $\iota' \circ \iota$ is the identity map $\text{Id}_M$.

To prove that $\iota \circ \iota'$ is $\text{Id}_{\overline{\mathcal{M}}}$, it suffices to find an isomorphism $h : \mathbb{P}(E) \to (\iota')^*\mathbb{P}(E')$ such that $(\iota')^*\sigma'$ equals $h \circ \sigma$ and $g$ equals $(\iota')^*g' \circ h$. By construction of $\iota'$, there is a canonical isomorphism of $(\iota')^*E'$ with $E \otimes (\pi_*[g^*S(k,V)^\vee \otimes \mathcal{O}_{\mathbb{P}(E)}(-1)])^\vee$. By the universal property of $\mathbb{P}(E')$, there exists a unique isomorphism $h : \mathbb{P}(E) \to (\iota')^*\mathbb{P}(E')$ pulling back the universal locally direct summand $(\iota')^*\mathcal{O}_{\mathbb{P}(E')}(-1)$ of $(\iota')^*(\pi')^*E'$ to the locally direct summand,

$$\mathcal{O}_{\mathbb{P}(E)}(-1) \otimes (\pi^*\pi_*[g^*S(k,V) \otimes \mathcal{O}_{\mathbb{P}(E)}(-1)])^\vee \subset \pi^*E \otimes (\pi^*\pi_*[g^*S(k,V) \otimes \mathcal{O}_{\mathbb{P}(E)}(-1)])^\vee \cong \pi^*(\iota')^*E',$$

obtained from the universal locally direct summand $\mathcal{O}_{\mathbb{P}(E)}(-1) \subset \pi^*E$. Since $(\iota')^*L'$ equals $L \otimes (\pi_*[g^*S(k,V)^\vee \otimes \mathcal{O}_{\mathbb{P}(E)}(-1)])^\vee$ as locally direct summands of $(\iota')^*E'$, $(\iota')^*\sigma'$ equals $h \circ \sigma$. To prove $(\iota')^*g' \circ h$ equals $g$, it suffices to prove that $(\iota')^*S'_k$ equals $g^*S(k,V)$ as locally direct summands of $V \otimes \mathcal{O}_{\mathbb{P}(E)}$. By definition of $\iota'$, the locally direct summand $(\iota')^*(\pi')^*S_{k-1}$ of $V \otimes \mathcal{O}_{\mathbb{P}(E)}$ equals $\pi^*E_{k-1}$, which is contained in $g^*S(k,V)$. Forming the corresponding quotients, it suffices to prove that $g^*S(k,V)/\pi^*E_{k-1}$ equals $(\iota')^*\mathcal{O}_{\mathbb{P}(E')}(-1)$ as locally direct summands of $(\iota')^*(\pi')^*E'$, i.e., of $\pi^*(E_k/E_{k-1})$. As $g^*S(k,V)/\pi^*E_{k-1} \subset \pi^*(E_k/E_{k-1})$ is isomorphic to,

$$\mathcal{O}_{\mathbb{P}(E)}(-1) \otimes (\pi^*\pi_*[g^*S(k,V) \otimes \mathcal{O}_{\mathbb{P}(E)}(-1)])^\vee \subset \pi^*E \otimes (\pi^*\pi_*[g^*S(k,V) \otimes \mathcal{O}_{\mathbb{P}(E)}(-1)])^\vee,$$

compatibly with the isomorphism to $\pi^*\iota^*E'$, this follows from the definition of $h$. ☐

**Corollary 2.3.** *Denote by $\iota : Flag(k-1,k,k+1;V) \to M$ the inverse morphism of $\iota'$. There are canonical isomorphisms,*

$$\iota^*T_{ev} \cong [(S_{k+1}/S_k)^\vee \otimes ((V \otimes \mathcal{O}_F)/S_{k+1})] \oplus [(S_k/S_{k-1}) \otimes S_{k-1}^\vee],$$

$$\iota^*\psi^\vee \cong (S_{k+1}/S_k) \otimes (S_k/S_{k-1})^\vee.$$

*Proof.* Because $\mathrm{ev} \circ \iota'$ is the tautological projection, $(\iota')^* T_{\mathrm{ev}}$ is the vertical tangent bundle of the projection. This projection is the fiber product of the relative Grassmannian $\mathrm{Flag}(1, (V \otimes \mathcal{O}_F)/S_k)$ and the relative Grassmannian $\mathrm{Flag}(k-1, S_k)$. The first isomorphism follows from the well-known computation of the vertical tangent bundle of a Grassmannian bundle. The second isomorphism follows from the isomorphisms $(\iota')^* L = L' = S_k/S_{k-1}$ and $(\iota')^*(E/L) = E'/L' = S_{k+1}/S_k$. □

## 3. Pointed lines on isotropic Grassmannians

Assume that $\mathrm{char}(\kappa)$ is not 2. This subsection gives the analogues of Proposition 2.2 and Corollary 2.3 in the isotropic case. Let $V$ be an $n$-dimensional vector space with a symmetric or skew-symmetric nondegenerate pairing. Let $X$ be the Grassmannian of isotropic $k$-planes in $V$. The later sections prove existence of a very twisting family of pointed lines. The proof breaks up into several cases.

- I. This is the case when $k$ is odd and $n \geq \max(4, 2k+2)$.
- II. This is the case when $k$ is even and $n \geq 2k+2$.
- III. This is the case when $n = 2k$ and the pairing is symmetric. The case when $n = 2k-1$ and the pairing is symmetric reduces to this case.
- IV. This is the case when $n = 2k$ and the pairing is skew-symmetric.

**Definition 3.1.** Let $B$ be a $\kappa$-scheme. A *symmetric pairing over* $B$, resp. a *skew-symmetric pairing on* $B$, is a triple $(E, \mathcal{L}, \beta)$ of a locally free $\mathcal{O}_B$-module $E$ of finite, constant rank, an invertible $\mathcal{O}_B$-module $\mathcal{L}$, and an isomorphism of $\mathcal{O}_B$-modules, $\beta : E \to E^\vee \otimes \mathcal{L}$ such that $\beta$ equals $\beta^\dagger \otimes \mathrm{Id}_{\mathcal{L}}$, resp. $\beta$ equals $-\beta^\dagger \otimes \mathrm{Id}_{\mathcal{L}}$. If the invertible sheaf $\mathcal{L}$ equals $\mathcal{O}_B$, the pair is written $(E, \beta)$.

**Notation 3.2.** Let $(E, \beta)$ be a symmetric pairing over $B$ or a skew-symmetric pairing over $B$. For every increasing sequence of integers $\underline{k} = (k_1 < \cdots < k_r)$, denote by $\mathrm{Flag}_{\underline{k}}(E, \beta)$ the bundle over $B$ parametrizing $\underline{k}$-flags of isotropic locally direct summands of $E$. Denote by $\pi : \mathrm{Flag}_{\underline{k}}(E, \beta) \to B$ the projection, and denote by $S_{k_1}(E, \beta) \subset \cdots \subset S_{k_r}(E, \beta) \subset \pi^* E$ the universal $\underline{k}$-flag of isotropic locally direct summands.

Let $k$ and $n$ be positive integers with $n \geq 2k$. Let $W$ be a $\kappa$-vector space of dimension $n$. Let $(W, \beta)$ be a symmetric or skew-symmetric pairing. Denote by $X$ the isotropic flag variety,

$$X = \mathrm{Flag}_k(W, \beta).$$

By the universal property of $\mathrm{Flag}(k; W)$, there exists a unique morphism $e : X \to \mathrm{Flag}(k; W)$ pulling back the universal locally direct summand $S(k, W)$ of $W$ to the universal isotropic locally direct summand $S_k(W, \beta)$. The morphism $e$ is a closed immersion. Except when $\beta$ is odd and $n = 2k$ or $2k-1$, the invertible sheaf $\mathcal{O}_X(1)$ is defined to be the pullback by $e$ of the Plücker invertible sheaf $\mathcal{O}(1)$ on $\mathrm{Flag}(k; W)$. In these cases, the induced morphism $\overline{\mathcal{M}}_{0,1}(e, 1) : \overline{\mathcal{M}}_{0,1}(X, 1) \to \overline{\mathcal{M}}_{0,1}(\mathrm{Flag}(k; W), 1)$ is a closed immersion. Thus, using Proposition 2.2, there is a canonical closed immersion of $\overline{\mathcal{M}}_{0,1}(X, 1)$ in $\mathrm{Flag}(k-1, k, k+1; W)$. The cases when $n = 2k$ or $2k-1$ are a bit more complicated.

**Symmetric case,** $n \geq 2k+2$**.** Denote by $\overline{\mathcal{M}}$ the scheme $\overline{\mathcal{M}}_{0,1}(X, 1)$. Denote by $M$ the isotropic flag variety,

$$M = \mathrm{Flag}_{k-1,k,k+1}(W, \beta).$$

Denote by $S_{k-1} \subset S_k \subset S_{k+1} \subset W \otimes_\kappa \mathcal{O}_M$ the universal isotropic flag. By the universal property of $\text{Flag}(k-1,k,k+1;W)$, there is a unique morphism $e' : M \to \text{Flag}(k-1,k,k+1;W)$ pulling back the universal flag to $S_{k-1} \subset S_k \subset S_{k+1} \subset W \otimes_\kappa \mathcal{O}_M$. The morphism $e'$ is a closed immersion.

Define $E'$ to be the rank 2 locally free sheaf, $S_{k+1}/S_{k-1}$, and define $L'$ to be the invertible sheaf, $S_k/S_{k-1}$. Denote by $\pi' : \mathbb{P}(E') \to M$ the associated $\mathbb{P}^1$-bundle. There is a unique section $\sigma' : M \to \mathbb{P}(E')$ such that the pullback of the rank 1 locally direct summand $\mathcal{O}_{\mathbb{P}(E')}(-1)$ of $(\pi')^*E'$ equals $L'$. On $\mathbb{P}(E')$ there is a rank $k$ locally direct summand $S'_k$ of $(\pi')^*S_{k+1}$ defined as the preimage of the rank 1 locally direct summand $\mathcal{O}_{\mathbb{P}(E')}(-1)$ of $(\pi')^*(S_{k+1}/S_{k-1})$. Because $(\pi')^*S_{k+1}$ is a rank $k+1$ locally direct summand of $W \otimes_\kappa \mathcal{O}_{\mathbb{P}(E')}$, $S'_k$ is a rank $k$ locally direct summand of $W \otimes_\kappa \mathcal{O}_{\mathbb{P}(E')}$. Because $(\pi')^*S_{k+1}$ is isotropic, also $S'_k$ is isotropic. By the universal property of $\text{Flag}_k(W,\beta)$, there exists a unique morphism $g' : \mathbb{P}(E') \to \text{Flag}_k(W,\beta)$ such that $(g')^*S_k$ equals $S'_k$ as a locally direct summand of $W \otimes_\kappa \mathcal{O}_{\mathbb{P}(E')}$.

**Proposition 3.3.** *Assume $n$ is at least $2k+2$. The datum $(\pi',\sigma',g')$ is a family of pointed lines in $\mathit{Flag}_k(W,\beta)$ parametrized by $M$. The associated morphism $\iota : M \to \overline{\mathcal{M}}$ is an isomorphism, compatible with the closed immersions into $\mathit{Flag}(k-1,k,k+1;W)$. Moreover, $ev \circ \iota$ equals the tautological projection $\mathit{Flag}_{k-1,k,k+1}(W,\beta) \to \mathit{Flag}_k(W,\beta)$.*

*Proof.* The proof that $(\pi',\sigma',g')$ is a family of pointed lines is identical to the argument in the proof of Proposition 2.2. The morphism $\iota$ is clearly compatible with the closed immersions into $\text{Flag}(k-1,k,k+1;W)$. So $\iota$ is a closed immersion. Since both $M$ and $\overline{\mathcal{M}}$ are smooth, to prove $\iota$ is an isomorphism, it suffices to prove that $\iota$ is surjective. The fact about ev follows from the analogous fact in Proposition 2.2.

Let $[E_{k-1} \subset E_k \subset E_{k+1} \subset W]$ be any flag in $\text{Flag}(k-1,k,k+1;W)$ contained in the image of $\overline{\mathcal{M}}$. Because the pointed line in $\text{Flag}(k;W)$ associated to this flag is contained in $\text{Flag}_k(W,\beta)$, for every rank $k$ subspace $S'_k \subset E_{k+1}$ containing $E_{k-1}$, $S'_k$ is isotropic. In particular, since every vector in $E_{k+1}$ is contained in such a subspace, every vector in $E_{k+1}$ is isotropic. By the polarization identity for symmetric bilinear pairings (which holds because $\text{char}(k)$ is not two!), the subspace $E_{k+1}$ is isotropic. Thus $[E_{k-1} \subset E_k \subset E_{k+1} \subset W]$ is contained in the image of $M$. □

On $M$ there is a rank $n-k-1$ locally direct summand $S_{k+1}^\perp$ of $W \otimes_\kappa \mathcal{O}_M$ defined as the annihilator of $S_{k+1}$ under $\beta$. Since $S_{k+1}$ is isotropic, by definition $S_{k+1}$ is a locally direct summand of $S_{k+1}^\perp$.

**Corollary 3.4.** *Assume $n$ is at least $2k+2$. The pullbacks under $\iota$ of $T_{ev}$ and $\psi^\vee$ admit canonical isomorphisms,*

$$\iota^*T_{ev} \cong [(S_{k+1}/S_k)^\vee \otimes (S_{k+1}^\perp/S_{k+1})] \oplus [(S_k/S_{k-1}) \otimes S_{k-1}^\vee],$$
$$\iota^*\psi^\vee \cong (S_{k+1}/S_k) \otimes (S_k/S_{k-1})^\vee.$$

*Proof.* The projection $\text{Flag}_{k-1,k,k+1}(W,\beta) \to \text{Flag}_k(W,\beta)$ is the fiber product of the relative isotropic Grassmannian, $\text{Flag}_1(S_k^\perp/S_k,\widetilde{\beta})$, and the relative classical Grassmannian, $\text{Flag}(k-1,S_k)$. The proof of the corollary is almost identical the the proof of Corollary 2.3. The one new element is the well-known isomorphism of the vertical tangent bundle of $\text{Flag}_1(E,\beta) \to B$ with $S_1(E,\beta)^\vee \otimes (S_1(E,\beta)^\perp/S_1(E,\beta))$, using Notation 3.2. $\square$

**Remark 3.5.** If $k = 1$, then $S_{k-1}$ is the zero sheaf. If $n = 2k+2$, then $S_{k+1}^\perp$ equals $S_{k+1}$ so that $S_{k+1}^\perp/S_{k+1}$ is the zero sheaf.

**Symmetric case, $n = 2k+1, 2k+2$.** If $n$ equals $2k+1$ or $2k+2$, then the Picard group is $\mathbb{Z}$, but the invertible sheaf giving the Plücker embedding, the *Plücker invertible sheaf*, is not a generator of the Picard group. First consider the case when $n = 2k+1$. Let $(\mathbf{1}, \beta_{\mathbf{1}})$ be a symmetric pairing such that $\dim(\mathbf{1})$ equals 1. Define $(W', \beta')$ to be the orthogonal direct sum of $(W,\beta)$ and $(\mathbf{1}, \beta_{\mathbf{1}})$. This has rank $n' = n+1$. Denote $k+1$ by $k'$. Denote by $M'$ the isotropic flag variety, $M' = \text{Flag}_{k'}(W', \beta')$. Because every isotropic subspace of $W$ has dimension $\leq k$, no $k'$-dimensional isotropic subspace of $W'$ is contained in $W$. Therefore the following morphism of $\mathcal{O}_{M'}$-modules is surjective,

$$S_{k'} \to W' \otimes_\kappa \mathcal{O}_{M'} \xrightarrow{\text{pr}_{\mathbf{1}}} \mathbf{1} \otimes_\kappa \mathcal{O}_{M'}.$$

Denote the kernel by $K$. This is a rank $k$ locally direct summand of $W \otimes_\kappa \mathcal{O}_{M'}$. Because $S_{k'}$ is isotropic for $\beta'$, $K$ is isotropic for $\beta$. By the universal property of $\text{Flag}_k(W,\beta)$, there exists a unique morphism $e : M' \to M$, pulling back the universal isotropic flag $S_k \subset W \otimes_\kappa \mathcal{O}_M$ to $K \subset W \otimes_\kappa \mathcal{O}_{M'}$.

**Lemma 3.6.** *If $n = 2k+1$, the morphism $e : \text{Flag}_{k+1}(W', \beta') \to \text{Flag}_k(W,\beta)$ is an étale, finite morphism of degree 2 identifying $\text{Flag}_{k+1}(W',\beta')$ with a disjoint union of 2 copies of $\text{Flag}_k(W,\beta)$.*

*Proof.* The first part is just the fact that a symmetric, bilinear pairing on a rank 2 vector space has precisely 2 isotropic lines. That this cover is trivial can be checked directly. It also follows from the fact that $\text{Flag}_k(W,\beta)$ is separably rationally connected, in fact separably unirational, together with a corollary of Kollár: a separably rationally connected variety has trivial étale fundamental group, cf. [Deb03, Cor. 3.6]. $\square$

Of course $\dim(W') = n+1 = 2(k+1) = 2k'$. Thus the case $n = 2k+1$ is reduced to the case $n' = 2k'$.

Next consider the case when $n = 2k$. Let $X$ be one of the two connected components of $\text{Flag}_k(V,\beta)$. As above, there is an embedding of $X$ in $\text{Flag}(k,V)$, and thus a Plücker invertible sheaf on $X$. Because $X$ is smooth and rational, there is no torsion in the Picard group of $X$. Therefore there exists a minimal ample invertible sheaf some power of which equals the Plücker invertible sheaf.

**Notation 3.7.** Assume $n$ equals $2k$. Denote by $\mathcal{O}_X(1)$ the unique minimal ample invertible sheaf on $\text{Flag}_k(W,\beta)$ some power of which equals the Plücker invertible sheaf.

**Lemma 3.8.** *The Picard group of $X$ is generated by $\mathcal{O}_X(1)$ and the Plücker invertible sheaf is isomorphic to $\mathcal{O}_X(2)$. In particular, a smooth rational curve in $X$ is a line with respect to $\mathcal{O}_X(1)$ iff it has degree 2 with respect to the Plücker invertible sheaf.*

*Proof.* Write $X$ as $\mathbf{SO}_{2k}/P = \mathbf{Spin}_{2k}/P'$. There is a natural isomorphism of the Picard group of $X$ and the character group of $P'$. Choose an isomorphism $W \cong V \oplus V^\vee$ sending $\beta$ to the canonical symmetric bilinear pairing on $V \oplus V^\vee$. Then the stabilizer group $P$ of the isotropic flag $V \subset V \oplus V^\vee$ is the group of all maps,

$$\left( \begin{array}{c|c} U^{-1} & B \\ \hline 0 & U^\dagger \end{array} \right),$$

where $U$ is in $\mathbf{GL}(V)$ and $B : V^\vee \to V$ is any skew-symmetric map. The group $P'$ is a connected extension of $P$ by $\mu_2$. Because $\text{char}(\kappa)$ is not 2, there is no nontrivial $\mu_2$-extension of the additive group of skew-symmetric matrices. Therefore $P'$ is the basechange by $P \to \mathbf{GL}(V)$ of a connected $\mu_2$-extension $G'$ of $\mathbf{GL}(V)$. There is precisely one such, namely the basechange by $\det : \mathbf{GL}(V) \to \mathbf{G}_m$ of the unique connected $\mu_2$ extension of $\mathbf{G}_m$, $(*)^2 : \mathbf{G}_m \to \mathbf{G}_m$. The character group of $P'$ equals the character group of $G'$, which is a free Abelian group containing the character group of $\mathbf{GL}(V)$ as an index 2 subgroup. The Plücker invertible sheaf corresponds to the character $\det : \mathbf{GL}(V) \to \mathbf{G}_m$. Since this character is twice the generator of the character group of $G'$, the Plücker invertible sheaf is isomorphic to the square of the generator of the Picard group of $X$. $\square$

Denote by $\overline{\mathcal{M}}$ the the space $\overline{\mathcal{M}}_{0,1}(X, 1)$. Denote by $M$ the isotropic flag variety,

$$M = \text{Flag}_{k-2,k}(W, \beta).$$

Denote by $S_{k-2} \subset S_k \subset W \otimes_\kappa \mathcal{O}_M$ the universal isotropic flag. Denote by $S_{k-2}^\perp$ the rank $k+2$ locally direct summand of $W \otimes_\kappa \mathcal{O}_M$ defined as the annihilator of $S_{k-2}$. Because $S_k$, resp. $S_{k-2}$, is isotropic, it is a locally direct summand of $S_{k-2}^\perp$. Denote by $F$ the rank 4 locally free sheaf, $S_{k-2}^\perp / S_{k-2}$. Associated to $\beta$ there is a symmetric pairing $(F, \beta_F)$ on $M$. Denote by $C'$ the isotropic flag variety,

$$C' = \text{Flag}_2(E, \widetilde{\beta}).$$

Denote by $\pi : C' \to M$ the projection. Denote by $G$ the rank 2, locally direct summand $S_k/S_{k-2}$ of $F$. Because $S_k$ is isotropic for $\beta$, $G$ is isotropic for $\beta_F$. By the universal property of the isotropic flag variety, there exists a unique section $\sigma : M \to C'$ such that the pullback of the universal flag $S_2(F, \beta_F) \subset \pi^* F$ equals $G \subset F$. On $C'$ there is a rank $k$ locally direct summand $S'_k$ of $\pi^* S_{k-2}^\perp$ defined as the preimage of the rank 2 locally direct summand $S_2(E, \beta_F)$ of $\pi^* F = \pi^*(S_{k-2}^\perp / S_{k-2})$. Because $S_2(F, \beta_F)$ is isotropic for $\beta_F$, $S'_k$ is isotropic for $\beta$. By the universal property of $\text{Flag}_k(W, \beta)$, there exists a unique morphism $g : C' \to \text{Flag}_k(W, \beta)$ such that $g^* S_k$ equals $S'_k$ as a locally direct summand of $W \otimes_\kappa \mathcal{O}_{C'}$.

**Proposition 3.9.** *Assume $n$ equals $2k$. The morphism $\pi' : C' \to M$ is a proper, smooth morphism, and every fiber is a disjoint union of 2 copies of $\mathbb{P}^1$. There is a unique open and closed subscheme $C$ containing the image of $\sigma$ such that $\pi : C \to M$ is a $\mathbb{P}^1$-bundle. The datum $(\pi : C \to M, \sigma, g)$ is a family of pointed lines in $Flag_k(W, \beta)$ parametrized by $M$. The associated morphism $\iota : M \to \overline{\mathcal{M}}_{0,1}(Flag_k(W, \beta), 1)$ is an isomorphism, and $ev \circ \iota$ is the tautological projection $Flag_{k-2,k}(W, \beta) \to Flag_k(W, \beta)$.*

*Proof.* The first part follows from the fact that for $n = 4$, $\text{Flag}_2(W, \beta)$ is a disjoint union of two copies of $\mathbb{P}^1$. Each $\mathbb{P}^1$ has degree 2 with respect to the Plücker embedding, thus it has degree 1 with respect to $\mathcal{O}_X(1)$. Therefore $(\pi, \sigma, g)$ is a family of pointed lines in $X$. As in the proofs of Proposition 2.2 and Proposition 3.3, to prove $\iota$ is an isomorphism, it suffices to prove it is bijective on points.

Let $C$ be a smooth conic in $\mathrm{Flag}(k,W)$. By [Buc03, Lemma 1], there is a rank $k-2$ subspace $S_{k-2}$ of $W$ such that every point of $C$ parametrizes a subspace $S_k$ containing $S_{k-2}$. If $C$ is contained in $\mathrm{Flag}_k(W,\beta)$, then $S_k$ is isotropic, hence also $S_{k-2}$ is isotropic. Because $S_k$ is isotropic and contains $S_{k-2}$, $S_k$ is contained in the rank $k+2$ annihilator, $S_{k-2}^{\perp}$. Denote by $F$ the quotient space $S_{k-2}^{\perp}/S_{k-2}$. Associated to $\beta$ there is a symmetric bilinear pairing $\beta_F$ on $F$. A rank $k$ subspace $S_k$ of $W$ containing $S_{k-2}$ is isotropic for $\beta$ iff the subspace $S_k/S_{k-2}$ of $F$ is isotropic for $\beta_F$. Therefore $C$ is also a smooth conic in $\mathrm{Flag}_2(F,\beta_F)$. As above, $\mathrm{Flag}_2(F,\beta_F)$ is a disjoint union of two smooth conics, i.e., $C$ is one of the two connected components of $\mathrm{Flag}_2(F,\beta_F)$. Also, since the subspaces of $F$ parametrized by $\mathrm{Flag}_2(F,\beta_F)$ collectively span $F$ and have common intersection $(0)$, $S_{k-2}$ is the common intersection in $W$ of the spaces $S_k$ for every point $[S_k]$ of $C$. Therefore the space of conics in $\mathrm{Flag}_k(W,\beta)$ is the bijective image of the étale double cover $\mathrm{Flag}_2(F,\beta_F)$ of $\mathrm{Flag}_{k-2}(W,\beta)$. Therefore $\iota$ is bijective. □

**Corollary 3.10.** *Assume $n$ equals $2k$. The pullbacks under $\iota$ of $T_{ev}$ and $\psi^\vee$ admit canonical isomorphisms,*

$$\iota^*T_{ev} \cong (S_k/S_{k-2}) \otimes S_{k-2}^\vee,$$
$$\iota^*\psi^\vee \cong \bigwedge^2 (S_k/S_{k-2})^\vee.$$

*Proof.* The projection $\mathrm{Flag}_{k-2,k}(W,\beta) \to \mathrm{Flag}_k(W,\beta)$ is the relative Grassmannian $\mathrm{Flag}(k-2,S_k)$. So the first isomorphism follows from the well-known isomorphism of the vertical tangent bundle of a relative Grassmannian. By construction, $\pi : C \to M$ is one connected component of the relative isotropic Grassmannian $\mathrm{Flag}_2(F,\beta_F)$, where $F := S_{k-2}^{\perp}/S_{k-2}$. The vertical tangent bundle of $\mathrm{Flag}(2,F)$ equals $S(2,F)^\vee \otimes \pi^*F/S(2,F)$. The restriction of this sheaf to $\mathrm{Flag}_2(F,\beta_F)$ is $S_2(F,\beta_F)^\vee \otimes S_2(F,\beta_F)^\vee$. By [HT84, p. 474], the normal bundle of the regular embedding $\mathrm{Flag}_2(F,\beta_F) \to \mathrm{Flag}(2,F)$ equals $\mathrm{Sym}^2(S_2(F,\beta_F))^\vee$. Therefore the vertical tangent bundle of $\mathrm{Flag}_2(F,\beta_F)$ is the kernel of the map $S_2(F,\beta_F)^\vee \otimes S_2(F,\beta_F)^\vee \to \mathrm{Sym}^2(S_2(F,\beta_F))^\vee$, i.e., $\bigwedge^2 S_2(F,\beta_F)^\vee$. The bundle $\psi^\vee$ is the pullback of the vertical tangent bundle by the section $\sigma$. Since $\sigma$ pulls back $S_2(F,\beta_F)$ to $G = S_k/S_{k-2}$, the bundle $\psi^\vee$ is canonically isomorphic to $\bigwedge^2(S_k/S_{k-2})^\vee$. □

**Skew-symmetric case, $n \geq 2k$.** Assume $(W,\beta)$ is a skew-symmetric pairing of dimension $n$. Assume $n$ is at least $2k$. There is a natural embedding of the isotropic Grassmannian in the classical Grassmannian, $\mathrm{Flag}_k(W,\beta) \subset \mathrm{Flag}(k,W)$. Denote by $\mathcal{O}_X(1)$ the pullback of $\mathcal{O}_{\mathbb{P}(\bigwedge^k W)}(1)$ under the Plücker embedding.

Denote by $\overline{\mathcal{M}}$ the space $\overline{\mathcal{M}}_{0,1}(X,1)$. Denote by $M_{\mathrm{pre}}$ the isotropic flag variety,

$$M = \mathrm{Flag}_{k-1,k}(W,\beta).$$

Denote by $S_{k-1,\mathrm{pre}} \subset S_{k,\mathrm{pre}} \subset W \otimes_\kappa \mathcal{O}_{M_{\mathrm{pre}}}$ the universal isotropic flag. On $M_{\mathrm{pre}}$ there is a rank $n-k+1$ locally direct summand $S_{k-1,\mathrm{pre}}^{\perp}$ of $W \otimes_\kappa \mathcal{O}_{M_{\mathrm{pre}}}$ defined as the annihilator of $S_{k-1,\mathrm{pre}}$ under $\beta$. Since $S_{k,\mathrm{pre}}$ and $S_{k-1,\mathrm{pre}}$ are isotropic, each one is a locally direct summand of $S_{k-1,\mathrm{pre}}$. Denote by $G$ the rank $n+1-2k$, locally free sheaf $S_{k-1,\mathrm{pre}}^{\perp}/S_{k,\mathrm{pre}}$.

Denote by $\rho : M \to M_{\text{pre}}$ the projective bundle,

$$M = \text{Flag}(1, G) = \mathbb{P}(G).$$

Denote by $S_{k-1}$, resp. $S_k$, the locally direct summand of $W \otimes_\kappa \mathcal{O}_M$ obtained by pulling back $S_{k-1,\text{pre}}$, resp. $S_{k,\text{pre}}$. Each one is an isotropic locally direct summand of $W \otimes_\kappa \mathcal{O}_M$. Also the annihilator $S_{k-1}^\perp$ of $S_{k-1}$ with respect to $\beta$ equals $\rho^* S_{k-1,\text{pre}}^\perp$ as subsheaves of $W \otimes_\kappa \mathcal{O}_M$. There is a rank $k+1$ locally direct summand $R_{k+1}$ of $S_{k-1}^\perp$ defined as the preimage of the rank 1 locally direct summand $\mathcal{O}_{PP(G)}(-1)$ of $\pi^* G = S_{k-1}^\perp / S_k$.

Altogether, this defines a flag of locally direct summands of $W \otimes_\kappa \mathcal{O}_M$,

$$S_{k-1} \subset S_k \subset R_{k+1} \subset W \otimes_\kappa \mathcal{O}_M.$$

The first two terms are isotropic. The term $R_{k+1}$ is not necessarily isotropic, but it is contained in $S_{k-1}^\perp$. The following lemma is straightforward.

**Lemma 3.11.** *The flag $S_{k-1} \subset S_k \subset R_{k+1} \subset W \otimes_\kappa \mathcal{O}_M$ is the universal $(k-1, k, k+1)$-flag of locally direct summands of $W$ such that $S_k$ is isotropic and $R_{k+1}$ is contained in $S_{k-1}^\perp$.*

**Remark 3.12.** There is a natural action of $\mathbf{Sp}(W, \beta)$ on $M$. There are 2 orbits. The closed orbit is the projective homogeneous space $\text{Flag}_{k-1,k,k+1}(W, \beta)$. The complement of the closed orbit is a non-projective homogeneous space. If $1 < k < n/2$, the automorphism group of $M$ equals $\mathbf{Sp}(W, \beta)$, thus $M$ is not a homogeneous space for any group.

Define $E'$ to be the rank 2 locally free sheaf, $R_{k+1}/S_{k-1}$, and define $L'$ to be the invertible sheaf, $S_k/S_{k-1}$. Denote by $\pi : \mathbb{P}(E') \to M$ the associated $\mathbb{P}^1$-bundle. There is a unique section $\sigma' : M \to \mathbb{P}(E')$ such that the pullback of the rank 1 locally direct summand $\mathcal{O}_{\mathbb{P}(E')}(-1)$ of $\pi^* E'$ equals $L'$.

On $\mathbb{P}(E')$ there is a rank $k$ locally direct summand $S'_k$ of $\pi^* R_{k+1}$ defined as the preimage of the rank 1 locally direct summand $\mathcal{O}_{\mathbb{P}(E')}(-1)$ of $\pi^* E' = \pi^*(R_{k+1}/S_{k-1})$. Because $\pi^* R_{k+1}$ is a rank $k+1$ locally direct summand of $W \otimes_\kappa \mathcal{O}_{\mathbb{P}(E')}$, $S'_k$ is a rank $k$ locally direct summand of $W \otimes_\kappa \mathcal{O}_{\mathbb{P}(E)}$. Denote by $F$ the quotient $S_{k-1}^\perp / S_{k-1}$. Associated to $\beta$ there is a skew-symmetric pairing $\beta_F$ for $F$. The locally direct summand $\mathcal{O}_{\mathbb{P}(E')}(-1)$ of $\pi^* F$ is isotropic for $\beta_F$ because *every* rank 1 locally direct summand of a skew-symmetric pairing is isotropic. Because $S'_k/\pi^* S_{k-1})$ is isotropic for $\beta_F$, $S'_k$ is isotropic for $\beta$. By the universal property of $\text{Flag}_k(W, \beta)$, there exists a unique morphism $g' : \mathbb{P}(E') \to \text{Flag}_k(W, \beta)$ such that $(g')^* S_k$ equals $S'_k$ as a locally direct summand of $W \otimes_\kappa \mathcal{O}_{\mathbb{P}(E')}$.

**Proposition 3.13.** *Assume $n$ is at least $2k$. The datum $(\pi', \sigma', g')$ is a family of pointed lines in $\text{Flag}_k(W, \beta)$ parametrized by $M$. The associated morphism $\iota : M \to \overline{\mathcal{M}}$ is an isomorphism.*

*Proof.* As in the proof of Proposition 3.3, the first statement follows from Proposition 2.2, and the second statement reduces to surjectivity of $\iota$. Let $[S_{k-1} \subset S_k \subset R_k \subset W]$ be a flag parametrizing a line in $\text{Flag}(k, W)$ contained in $\text{Flag}_k(W, \beta)$. Then $S_k$ is isotropic, and hence also $S_{k-1}$ is isotropic. Every vector in $R_k$ is contained in a subspace $S'_k$ containing $S_{k-1}$. Because $S'_k$ is isotropic, $S'_k$ is contained in $S_{k-1}^\perp$. Therefore $R_k$ is contained in $S_{k-1}^\perp$. By Lemma 3.11, $[S_{k-1} \subset S_k \subset R_k \subset W]$ is contained in the image of $\iota$. □

**Corollary 3.14.** *Assume $n$ is at least $2k$. The pullback under $\iota$ of $T_{ev}$ admits a short exact sequence,*

$$0 \to [(R_{k+1}/S_k)^\vee \otimes (S_{k-1}^\perp/R_{k+1})] \to \iota^* T_{ev} \to [(S_k/S_{k-1}) \otimes S_{k-1}^\vee] \to 0.$$

*And the pullback of $\psi^\vee$ under $\iota$ admits an isomorphism,*

$$\iota^*\psi^\vee \cong (R_{k+1}/S_k) \otimes (S_k/S_{k-1})^\vee.$$

*Proof.* This is very similar to the proof of Corollary 3.4. □

## 4. Maps of vector bundles on the projective line

Using the propositions of Subsections 2 and 3, a very twisting family of pointed lines on a classical or isotropic Grassmannian is equivalent to a flag of locally direct summands of $W \otimes_\kappa \mathcal{O}_{\mathbb{P}^2}$ such that the associated locally free sheaf $\zeta^* T_{\text{ev}}$ is ample and the associated invertible sheaf $\zeta^*\psi^\vee$ has nonnegative degree. However, in the isotropic case, the flags are difficult to construct directly. This subsection contains the proof of a fact about maps of vector bundles on the projective line which will be useful for constructing flags.

Let $a \le b$ be nonnegative integers. Define $H$ to be the rank $ab(b+1-a)$ vector space,

$$H = \text{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{O}_{\mathbb{P}^1}^{\oplus a}, \mathcal{O}_{\mathbb{P}^1}(b-a)^{\oplus b}).$$

There is an open subset of $H$ parametrizing maps whose cokernel is locally free of rank $b-a$. There is an open subset of this subset parametrizing maps whose cokernel is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(b)^{\oplus(b-a)}$. Denote by $H^o$ this open subset of $H$.

**Proposition 4.1.** *The open subset $H^o$ is not empty.*

*Proof.* If $a$ is zero, this is vacuous. Thus assume $a \ge 1$. Let $U$ be a rank 2 vector space and identify $\mathbb{P}^1$ with $\mathbb{P}(U)$. The homogeneous coordinate ring of $\mathbb{P}^1$ is $S := \text{Sym}^\bullet(U^\vee)$. Denote the graded pieces by $S_k$, i.e., $S_k = \text{Sym}^k(U^\vee)$. By convention, define $S_{-1}$ to be the zero vector space. Denote by $A$ the associative, unital $\kappa$-algebra of linear maps from $S$ to $S$, $A = \text{Hom}(S, S)$. This has a natural structure of left $S$-module by $(p \cdot L)(q) = pL(q)$ for every $p, q$ in $S$ and every $L$ in $A$. Denote by Diff the subalgebra of $\text{Hom}(S, S)$ of differential operators on $S$. This is a left $S$-submodule of $A$.

Denote by $d$ the unique $\kappa$-derivation,

$$d : S \to S \otimes_\kappa U^\vee,$$

such that $d|_{S_1} : U^\vee \to S \otimes_\kappa U^\vee$ factors as

$$U^\vee \xrightarrow{\text{Id}} \kappa \otimes_\kappa U^\vee = S_0 \otimes_\kappa U^\vee \subset S \otimes_\kappa U^\vee.$$

The derivation $d$ identifies $U$ as a linear subspace of Diff. Define $\text{Diff}_{a-1}$ to be the linear subspace of Diff generated by $U^{\otimes(a-1)}$. This, of course, is just an isomorphic copy of $\text{Sym}^{a-1}(U)$. In particular it has rank $a$. Also every element in $\text{Diff}_{a-1}$ has order $a-1$.

There is a canonical map $c : A \to (\bigwedge^2 U) \otimes_\kappa A$ defined as follows. Choose an ordered basis $\mathbf{e}_0, \mathbf{e}_1$ for $U$, and denote by $T_0, T_1$ the dual ordered basis for $U^\vee$. For every linear map $L : S \to S$, define $c(L) : S \to (\bigwedge^2 U) \otimes S$ to be,

$$c(L)(p) = (\mathbf{e}_0 \wedge \mathbf{e}_1) \otimes (T_1 L(T_0 \cdot p) - T_0 L(T_1 \cdot p)).$$

It is straightforward to check this is independent of the choice of basis. The map $c$ is a morphism of left $S$-modules (where $\bigwedge^2 U$ is given the trivial $S$-module structure). More importantly, if $L$ is a differential operator of order $k+1$, then $c(L)$ is a differential operator of order $k$. Define $c^l : A \to (\bigwedge^2 U)^{\otimes l} \otimes_\kappa A$ to be the $l$-fold composition of $c$ in the obvious way. Then $\mathrm{Ker}(c^l)$ contains the subspace of differential operators of order $\leq l-1$.

For every integer $k$, define $Q_k$ to be the quotient $\mathrm{Hom}(S_k, S)$ of $A$, and denote by $\pi_k : A \to Q_k$ the quotient map. By convention, define $Q_{-1}$ to be the zero vector space. The space $Q_k$ has a natural left $S$-module structure by $(q \cdot L)(p) = qL(p)$ and $\pi_k$ is a morphism of left $S$-modules. Make $Q_k$ a graded $S$-module by defining $(Q_k)_l = \mathrm{Hom}(S_k, S_{k+l})$ for every integer $l$. The associated sheaf on $\mathbb{P}(U)$ is $S_k^\vee \otimes_\kappa \mathcal{O}_{\mathbb{P}(U)}(k)$. For every integer $0 \leq l \leq k+1$, there is a unique degree $2l$ map of graded $S$-modules, $C_l : Q_k \to (\bigwedge^2 U)^{\otimes l} \otimes_\kappa Q_{k-l}$, such that the following diagram commutes,

$$\begin{array}{ccc} A & \xrightarrow{c^l} & (\bigwedge^2 U)^{\otimes l} \otimes_\kappa A \\ {\scriptstyle \pi_k}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}\otimes\pi_{k-l}} \\ Q_k & \xrightarrow{C_l} & (\bigwedge^2 U)^{\otimes l} \otimes_\kappa Q_{k-l}. \end{array}$$

This induces a map of associated sheaves,

$$\widetilde{C}_l : S_k^\vee \otimes_\kappa \mathcal{O}_{\mathbb{P}(U)}(k) \to [(\bigwedge^2 U)^{\otimes l} \otimes_\kappa S_{k-l}^\vee] \otimes_\kappa \mathcal{O}_{\mathbb{P}(U)}(k+l).$$

The composite map $\mathrm{Diff}_{a-1} \hookrightarrow A \xrightarrow{\pi_{b-1}} Q_{b-1}$ has image in the subspace $(Q_{b-1})_{-(a-1)}$, and so induces a map of associated sheaves,

$$\phi_{a,b} : \mathrm{Diff}_{a-1} \otimes_\kappa \mathcal{O}_{\mathbb{P}(U)} \to S_{b-1}^\vee \otimes_\kappa \mathcal{O}_{\mathbb{P}(U)}(b-a).$$

Twisting the map $\widetilde{C}_a$ appropriately gives a map of associated sheaves,

$$\psi_{a,b} : S_{b-1}^\vee \otimes_\kappa [(\bigwedge^2 U)^{\otimes l} \otimes S_{b-a-1}^\vee] \otimes_\kappa \mathcal{O}_{\mathbb{P}(U)}(b).$$

Since $\dim(\mathrm{Diff}_{a-1})$ equals $a$, $\dim(S_{b-1}^\vee)$ equals $b$, and $\dim((\bigwedge^2 U)^{\otimes l} \otimes S_{b-a-1}^\vee)$ equals $b-a$, the proposition is implied by the following.

**Claim 4.2.** *The following sequence of sheaves on $\mathcal{O}_{\mathbb{P}(U)}$ is exact,*

$$0 \to \mathit{Diff}_{a-1} \otimes_\kappa \mathcal{O}_{\mathbb{P}(U)} \xrightarrow{\phi_{a,b}} S_{b-1}^\vee \otimes_\kappa \mathcal{O}_{\mathbb{P}(U)}(b-a) \xrightarrow{\psi_{a,b}} [(\bigwedge^2 U)^\vee \otimes S_{a+b-1}^\vee] \otimes_\kappa \mathcal{O}_{\mathbb{P}(U)}(b) \to 0.$$

First of all, because $\mathrm{Diff}_{a-1}$ is contained in the kernel of $c^a$, $\psi_{a,b} \circ \phi_{a,b}$ is the zero map. There is a natural action of $\mathbf{GL}(U)$ on $\mathbb{P}(U)$. Each vector bundle in the claim has a natural $\mathbf{GL}(U)$-linearization and each of $\phi_{a,b}$ and $\psi_{a,b}$ is $\mathbf{GL}(U)$-equivariant. Thus to prove the sequence is exact, it suffices to prove it is exact at one point of $\mathbb{P}(U)$. With respect to the bases $\mathbf{e}_0, \mathbf{e}_1$ of $U$ and the dual basis $T_0, T_1$, denote by $\partial_0$, $\partial_1$ the elements of $\mathrm{Diff}_1$ corresponding to $\mathbf{e}_0, \mathbf{e}_1$. Then an ordered basis for $\mathrm{Diff}_{a-1}$ consists of,

$$\frac{1}{(a-1)! \cdot 0!}\partial_0^{a-1}, \ldots, \frac{1}{(a-k)! \cdot (k-1)!}\partial_0^{a-k}\partial_1^{k-1}, \ldots, \frac{1}{0! \cdot (a-1)!}\partial_1^{a-1}.$$

An ordered basis for $S_{b-1}$ consists of,

$$T_0^{b-1}, \dots, T_0^{b-j}T_1^{j-1}, \cdots, T_1^{b-1},$$

and this gives a dual ordered basis for $S_{b-1}^\vee$. Similarly, an ordered basis for $S_{b-a-1}$ consists of,

$$T_0^{a+b-1}, \dots, T_0^{a+b-1-i}T_1^{i-1}, \dots, T_1^{a+b-1},$$

and, tensored with $(\mathbf{e}_0 \wedge \mathbf{e}_1)^{\otimes l}$, this gives a dual ordered basis for $(\bigwedge^2 U)^{\otimes l} \otimes S_{a+b-1}^\vee$. With respect to these ordered bases, the entries of the matrix of $\phi_{a,b}$ equal,

$$(\phi_{a,b})_{j,k} = \binom{b-j}{a-k}\binom{j-1}{k-1} T_0^{b-1+k-j} T_1^{j-k},$$

and the entries of the matrix of $\psi_{a,b}$ equal,

$$(\psi_{a,b})_{i,j} = (-1)^{j-i}\binom{a}{j-i} T_0^{j-i} T_1^{a+i-j}.$$

Plugging in $T_0 = 1, T_1 = 0$ gives the matrices,

$$\phi_{a,b}|_{[1,0]} = \left( \frac{D_{a,a}}{0_{b-a,a}} \right),$$

and

$$\psi_{a,b}|_{[1,0]} = (-1)^a \cdot \left( \begin{array}{c|c} 0_{a,b-a} & I_{b-a,b-a} \end{array} \right),$$

where $0_{k,l}$ is the zero $k \times l$ matrix, $I_{k,k}$ is the $k \times k$ identity matrix, and $D_{a,a}$ is the invertible $a \times a$ diagonal matrix with entries $(D_{a,a})_{j,j} = \binom{b-j}{a-j}$. These matrices visibly give a short exact sequence of $\kappa$-vector spaces. $\square$

Because the entries of $\phi_{a,b}$ and $\psi_{a,b}$ are monomials, the very twisting families constructed later are more likely to be orbit curves.

## 5. Application to isotropic subspaces

This subsection applies Proposition 4.1 to construct some useful isotropic subspaces of $W \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}$.

**Hypothesis 5.1.** Let $a$ and $b$ be positive integers. Assume that $b$ is at least $2a$.

Let $W_{b,+}$ be an $b$-dimensional $\kappa$-vector space. Denote the dual vector space by $W_{b,-}$. By Proposition 4.1, there exists a map,

$$\phi_{a,b,+} : \mathcal{O}_{\mathbb{P}^1}(-(b-a))^{\oplus a} \to W_{b,+} \otimes_\kappa \mathcal{O}_{\mathbb{P}^1},$$

whose cokernel is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(a)^{\oplus(b-a)}$. Thus the annihilator in $W_{b,-} \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}$ of the image of $\phi_{a,b,+}$ is the image of a map,

$$\psi_{a,b,+}^\dagger : \mathcal{O}_{\mathbb{P}^1}(-a)^{\oplus(b-a)} \to W_{b,-} \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}.$$

By hypothesis, $b-a$ is at least $a$. Thus, by Proposition 4.1, there exists a map,

$$\phi_{a,b-a} : \mathcal{O}_{\mathbb{P}^1}(-(b-a))^{\oplus a} \to \mathcal{O}_{\mathbb{P}^1}(-a)^{\oplus(b-a)},$$

whose cokernel is isomorphic to $\mathcal{O}_{\mathbb{P}^1}^{\oplus(b-2a)}$. Define $\phi_{a,b,-}$ to be the composite map,

$$\psi_{a,b,+}^\dagger \circ \phi_{a,b-a} : \mathcal{O}_{\mathbb{P}^1}(-(b-a))^{\oplus a} \to W_{b,-} \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}.$$

Define $W_{2b}$ to be $W_{b,+} \oplus W_{b,-}$. Thus $(W_{2b})^\vee$ is canonically isomorphic to $W_{b,-} \oplus W_{b,+}$. In the symmetric case, define $\beta_{2b} : W_{2b} \to W_{2b}^\vee$ to be the linear map $(W_{b,+} \oplus W_{b,-}) \to (W_{b,-} \oplus W_{b,+})$ with matrix,

$$\beta_{2b} = \left( \begin{array}{c|c} 0 & \mathrm{Id}_{W_{b,-}} \\ \hline \mathrm{Id}_{W_{b,+}} & 0 \end{array} \right).$$

In the skew-symmetric case, define $\beta_{2b}$ to be the linear map with matrix,

$$\beta_{2b} = \left( \begin{array}{c|c} 0 & -\mathrm{Id}_{W_{b,-}} \\ \hline \mathrm{Id}_{W_{b,+}} & 0 \end{array} \right).$$

Define $E_{2a,2b}$ to be the image in $W_{2b} \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}$ of the sheaf map,

$\phi_{a,b,+} \oplus \phi_{a,b,-} : \mathcal{O}_{\mathbb{P}^1}(-(b-a))^{\oplus a} \oplus \mathcal{O}_{\mathbb{P}^1}(-(b-a))^{\oplus a} \to (W_{b,+} \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}) \oplus (W_{b,-} \otimes_\kappa \mathcal{O}_{\mathbb{P}^1})$.

Denote by $E_{2a,2b}^\perp$ the annihilator of $E_{2a,2b}$ for $\beta_{2b}$.

**Lemma 5.2.** *The subsheaf $E_{2a,2b}$ is a rank $2a$ locally direct summand of $W_{2b} \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}$ isotropic for $\beta_{2b}$. The quotient $(E_{2a,2b})^\perp / E_{2a,2b}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}^{\oplus(2b-4a)}$. And, $E_{2a,2b}^\vee$ is an ample vector bundle on $\mathbb{P}^1$.*

*Proof.* Denote by $F$ the rank $2b-4a)$ locally free sheaf $E_{2a,2b}^\perp / E_{2a,2b}$. Associated to $\beta$, there is a symmetric, resp. skew-symmetric, pairing $\beta_F$ for $F$. By construction, $G := \mathrm{Image}(\psi_{a,b,+}^\dagger)/\mathrm{Image}(\phi_{a,b,-})$ is an rank $b-2a$ locally direct summand of $F$, isotropic for $\beta_F$. Therefore $F$ is isomorphic to $G \oplus G^\vee$. By construction, $G$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}^{\oplus(b-2a)}$. Therefore $F$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}^{\oplus(2b-4a)}$.

By construction, $E_{2a,2b}^\vee$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(b-a)^{\oplus 2a}$. By hypothesis, $a$ is positive, and $b-2a$ is nonnegative, so also $b-a = (b-2a)+a \geq a > 0$. Thus $\mathcal{O}_{\mathbb{P}^1}(b-a)$ is ample. Since $2a > 0$, $\mathcal{O}_{\mathbb{P}^1}(b-a)^{\oplus 2a}$ is ample. □

## 6. The classical Grassmannian

Let $n > 2$ be an integer and let $k$ be an integer $0 < k < n$. Let $V$ be an $n$-dimensional $\kappa$-vector space, and denote by $(X, \mathcal{O}_X(1))$ the Grassmannian $\mathrm{Flag}(k,V)$ and the Plücker invertible sheaf. Replacing $\mathrm{Flag}(k,V)$ by the isomorphic scheme $\mathrm{Flag}(n-k,V^\vee)$ if necessary, assume $k \leq n/2$. Of course $\mathrm{Flag}(k,V)$ equals $\mathbf{SL}_n/P_k$, where $P_k$ is the maximal parabolic group corresponding to the $k^{\text{th}}$ node of the Dynkin diagram $A_{n-1}$.

By Proposition 2.2, a morphism $\zeta : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,1}(\mathrm{Flag}(k,V),1)$ is equivalent to a $(k-1,k,k+1)$-flag of locally direct summands,

$$E_{k-1} \subset E_k \subset E_{k+1} \subset V \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}.$$

By Corollary 2.3, the morphism $\zeta$ is very twisting iff

(i) the bundle,
$$[(E_{k+1}/E_k)^\vee \otimes ((V \otimes \mathcal{O}_{\mathbb{P}^1})/E_{k+1})] \oplus [(E_k/E_{k-1}) \otimes E_{k-1}^\vee],$$
is ample, and

(ii) the bundle,
$$(E_{k+1}/E_k) \otimes (E_k/E_{k-1})^\vee,$$
has nonnegative degree.

**Proposition 6.1.** *There exists a very twisting morphism $\zeta : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,1}(X,1)$.*

*Proof.* It is equivalent to prove there exists a flag $E_{k-1} \subset E_k \subset E_{k+1} \subset V \otimes \mathcal{O}_{\mathbb{P}^1}$ satisfying Conditions (i)–(ii). Let $T_0, T_1$ denote homogeneous coordinates on $\mathbb{P}^1$.

Let $V'_{2k-2}$ be a rank $2k-2$ vector space, let $E'_{k-1}$ be $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus(k-1)}$, and let $\phi'_{k-1,2k-2} : E'_{k-1} \to V'_{2k-2} \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}$ be a morphism whose cokernel is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(k-1)}$, as in Proposition 4.1. Let $V''_{n+1-2k}$ be a rank $n+1-2k$ vector space, let $E''_1$ be $\mathcal{O}_{\mathbb{P}^1}(-(n-2k))$, and let $\phi''_{1,n+1-2k} : E''_1 \to V''_{n+1-2k} \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}$ be a morphism whose cokernel is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(1)^{n-2k}$, as in Proposition 4.1. Finally, let $V'''_1$ be a rank 1 vector space, let $E'''_1$ be $V'''_1 \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}$, and let $\phi'''_{1,1}$ be the identity map.

Define $V$ to be the direct sum of $V'_{2k-2}$, $V''_{n+1-2k}$ and $V'''_1$. Define $E_{k+1}$ to be the image of $\phi'_{k-1,2k-1} \oplus \phi''_{1,n+1-2k} \oplus \phi'''_{1,1}$ in $V \otimes \mathcal{O}_{\mathbb{P}^1}$. The cokernel is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(k-1)} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(n-2k)}$, i.e., $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(n-k-1)}$.

Define $E_k$ to be the image of $\phi'_{k-1,2k-2} \oplus \phi''_{1,n+1-2k}$. The quotient $E_{k+1}/E_k$ equals $E'''_1 \cong \mathcal{O}_{\mathbb{P}^1}$. In particular, $(E_{k+1}/E_k)^\vee \otimes ((V \otimes \mathcal{O}_{\mathbb{P}^1})/E_{k+1})$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(1)^{n-k-1}$, which is ample. This is half of Condition (i). For $E_{k-1}$, there are two cases.

**Case I:** $k = 1$**.** In this case, define $E_{k-1}$ to be $(0)$. The quotient $E_k/E_{k-1}$ equals $E''_1 \cong \mathcal{O}_{\mathbb{P}^1}(-(n-2k))$. So $\deg(E_{k+1}/E_k)$ equals 0 and $\deg(E_k/E_{k-1})$ equals $-(n-2k)$. Since $n \geq 2k$, $-(n-2k) \leq 0$, i.e., Condition (ii) holds. Since $k = 1$, Condition (i) holds.

**Case II:** $k > 1$**.** Decompose $E'_{k-1}$ as $E'_{k-2,a} \oplus E'_{1,b}$, where $E'_{k-2,a}$ is the first $k-2$ summands and $E'_{1,b}$ is the last summand. Define $E''''_1$ to be $\mathcal{O}_{\mathbb{P}^1}(-(n+1-2k))$. Define $\phi'_{k-2,a} : E'_{k-2,a} \to E'_{k-2,a}$ to be the identity map. And define $\phi''''_{1,2} : E''''_1 \to E'_{1,b} \oplus E''_1$, i.e., $\mathcal{O}_{\mathbb{P}^1}(-(n+1-2k)) \to \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-(n-2k))$, to be the map with matrix,

$$\phi''''_{1,2} = \begin{pmatrix} T_0^{n-2k} \\ T_1 \end{pmatrix}$$

Define $E_{k-1}$ to be $E'_{k-2,a} \oplus E''''_1$, and define $\phi_{k-1} : E_{k-1} \to E_k$ to be $\phi'_{k-2,a} \oplus \phi''''_{1,2}$. The map $\phi''''_{1,2}$ is injective with cokernel $\mathcal{O}_{\mathbb{P}^1}$. Thus $\phi_{k-1}$ is injective and $E_k/E_{k-1}$ equals $\mathcal{O}_{\mathbb{P}^1}$. Since both $E_{k+1}/E_k$ and $E_k/E_{k-1}$ have degree 0, Condition (ii) holds. Also, $E_{k-1}^\vee \otimes (E_k/E_{k-1})$ equals $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(k-2)} \oplus \mathcal{O}_{\mathbb{P}^1}(n+1-2k)$. Since $n \geq 2k$, this is an ample bundle. Therefore Condition (i) holds. □

**Claim 6.2.** *The very twisting family in the proof of Proposition 6.1 can be chosen to be an orbit curve.*

*Proof.* For simplicity, assume $k > 1$; the case $k = 1$ is similar and easier. Choose $\phi'_{k-1,2k-2}$ to be the map with matrix,

$$\phi'_{k-1,2k-2} = \begin{pmatrix} T_0 & 0 & \dots & 0 \\ T_1 & 0 & \dots & 0 \\ 0 & T_0 & \dots & 0 \\ 0 & T_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & T_0 \\ 0 & 0 & \dots & T_1 \end{pmatrix}$$

Choose $\phi''_{1,n+1-2k}$ to be the map with matrix,

$$\phi''_{1,n+1-2k} = \begin{pmatrix} T_0^{n-2k} \\ \vdots \\ T_0^{n-2k-j}T_1^j \\ \vdots \\ T_1^{n-2k} \end{pmatrix}$$

Let $\lambda' : \mathbf{G}_m \times V \to V$ be the linear action compatible with the direct sum decomposition $V'_{2k-2} \oplus V''_{n+1-2k} \oplus V'''_1$ given by the diagonal matrices $D_1$, $D_2$,

$$D_1 = \left(\begin{array}{cc|cc|c|cc} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & t & 0 & 0 & \dots & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & t & \dots & 0 & 0 \\ \hline \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & t \end{array}\right)$$

$$D_2 = \begin{pmatrix} t & 0 & \dots & 0 \\ 0 & t^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t^{n+1-2k} \end{pmatrix}$$

and $D_3$ is the $1 \times 1$ matrix $t^c$ for $c = -[(k-1) + (n+2-2k)(n+1-2k)/2]$. Let $E'_{k-1}$ be the subspace of $V'_{2k-2}$ which is the image of the matrix,

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

Let $E''_1$ be the subspace of $V''_{n+1-2k}$ spanned by the vector,

$$\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

And let $E'''_1$ equals $V'''_1$. Define $E_{k+1} = E'_{k-1} \oplus E''_1 \oplus E'''_1$. Define $E_k$ to be $E'_{k-1} \oplus E''_1$.

Decompose $V'_{2k-2}$ as $V'_{2k-4,a} \oplus V'_{2,b}$ where $V'_{2k-4,a}$ is the first $2k-4$ summands, and $V'_{2,b}$ is the last 2 summands. Define $E'_{k-2,a}$ to be the subspace of $V'_{2k-4,a}$ which is the image of the matrix,

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

Define $E_1''''$ to be the subspace of $V'_{2,b} \oplus V''_{n+1-2k}$ spanned by the vector,

$$\begin{pmatrix} 1 \\ \vdots \\ 1 \\ \hline 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Define $E_{k-1}$ to be $E'_{k-2,a} \oplus E_1''''$. This gives a flag of subbundle of $V$, $E_{k-1} \subset E_k \subset E_{k+1} \subset V$. Define $P$ to be the maximal parabolic subgroup of $\mathbf{SL}(V)$ that is the stabilizer of the flag $E_k \subset V$ and define $P'$ to be the stabilizer of the flag $E_{k-1} \subset E_k \subset E_{k+1} \subset V$. The 1-parameter subgroup $\lambda : \mathbf{G}_m \to \mathbf{SL}(V)$ is defined above. The rational curve $\zeta : \mathbb{P}^1 \to \text{Flag}(k-1, k, k+1, V)$ equals the orbit curve associated to $\lambda$ and the flag $[E_{k-1} \subset E_k \subset E_{k+1} \subset V]$. □

## 7. Isotropic Grassmannians, Case I

Let $(W, \beta)$ be a symmetric or skew-symmetric pairing of dimension $n$. If $n = 2$ or 3 there is no very twisting family of pointed lines to an isotropic Grassmannian of $(W, \beta)$. Thus assume $n \geq 4$. This subsection proves existence of a very twisting family of pointed lines on the Grassmannian of isotropic $k$-planes when $k$ is odd and $n \geq 2k + 2$.

**Hypothesis 7.1.** The pairing $(W, \beta)$ is symmetric of dimension $2m$ or $2m + 1$ or the pairing $(W, \beta)$ is skew-symmetric of dimension $2m$, and $k = 2l + 1$. Assume that $l$ is nonnegative and $m$ is at least $\max(2, 2l + 2)$.

The last inequality is equivalent to $k$ is positive, $n \geq 4$, and $n \geq 2k + 2$.

Let $(W'_4, \beta_4)$ and $E'_{2,4}$ be as in Subsection 5 for $a = 1$ and $b = 2$. In particular, $E'_{2,4}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$.

**Case Ia:** $k = 1$**.** Assume that $k = 1$. Let $(W''_{n-4}, \beta'')$ be a symmetric pairing, resp. skew-symmetric pairing, of dimension $n - 4$. Define $(W, \beta)$ to be the orthogonal direct sum of $(W'_4, \beta'_4)$ and $(W''_{n-4}, \beta'')$. Define $E_2$ to be $E'_{2,4}$ considered as a locally direct summand of $W \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}$ via the embedding $W'_4 \otimes_\kappa \mathcal{O}_{\mathbb{P}^1} \hookrightarrow W \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}$. Define $E_1$ to be a direct summand $\mathcal{O}_{\mathbb{P}^1}(-1)$ in $E_2$, and define $E_0$ to be the zero sheaf.

**Lemma 7.2.** *Assume $k = 1$ and $n \geq 4$. The flag $E_0 \subset E_1 \subset E_2 \subset W \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}$ is a $(0, 1, 2)$-flag of isotropic locally direct summands for $\beta$. The cokernel $E_2^\perp / E_2$ is isomorphic to $W''_{n-4} \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}$. The cokernels $E_2/E_1$ and $E_1/E_0$ are isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1)$. And $E_0^\vee$ is the zero sheaf.*

*Proof.* By construction, $E_2$ is isotropic of rank 2. Since $E_1$ is contained in $E_2$, it is isotropic. Of course (0) is isotropic. By construction, the annihilator $E_2^{\perp}$ of $E_2$ with respect to $\beta$ equals the direct sum of the annihilator $(E'_{2,4})^{\perp}$ of $E'_{2,4}$ with respect to $\beta'_{2,4}$ and $W''_{n-4} \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}$. By Lemma 5.2, $(E'_{2,4})^{\perp}$ equals $E'_{2,4}$. Therefore $E_2^{\perp}/E_2$ equals $W''_{n-4} \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}$. By definition of $E_1$ and $E_0$, $E_2/E_1 \cong E_1/E_0 \cong \mathcal{O}_{\mathbb{P}^1}(-1)$. The dual of the zero sheaf is the zero sheaf. □

**Proposition 7.3.** *Assume $k = 1$. In the skew-symmetric case, assume $n \geq 4$. In the symmetric case, assume $n \geq 5$. The morphism $\zeta : \mathbb{P}^1 \to M$ associated to the flag in Lemma 7.2 is a very twisting family of pointed lines on $Flag_1(W, \beta)$.*

*Proof.* By Corollary 3.4 and Corollary 3.14, in both the symmetric and skew-symmetric case $\iota^*\psi^\vee$ equals $(E_{k+1}/E_k) \otimes (E_k/E_{k-1})^\vee$. By Lemma 7.2, this equals $\mathcal{O}_{\mathbb{P}^1}$, and so has nonnegative degree. This is (iii) of Definition 1.1. In the skew-symmetric case, by Corollary 3.14 and since $E_0^\vee$ is the zero sheaf, $\iota^* T_{\text{ev}}$ is isomorphic to $(E_2/E_1)^\vee \otimes (E_0^{\perp}/E_2)$. Also since $E_0$ is the zero sheaf, $E_0^{\perp}$ equals $W \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}$. In particular, $E_0^{\perp}/E_2$ has rank $n - 2$, which is positive by the hypothesis that $n \geq 4$. Since $W \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}$ is globally generated, the quotient $E_0^{\perp}/E_2$ is globally generated of positive rank. The tensor product of an ample bundle and a globally generated, positive rank bundle is an ample bundle. Since $(E_2/E_1)^\vee$ is ample, the tensor product $(E_2/E_1)^\vee \otimes (E_0^{\perp}/E_2)$ is ample.

The argument in the symmetric case is the same, except $E_0^{\perp}/E_2$ is replaced by $E_2^{\perp}/E_2$, which is globally generated of rank $n - 4$ by Lemma 7.2. The rank $n - 4$ is positive by the hypothesis that $n \geq 5$. □

An argument similar to the proof of Claim 6.2 proves the very twisting family can be chosen to be an orbit curve.

**Case Ib:** $k > 1$**.** Assume now that $k > 1$, i.e., $l \geq 1$. Let $(W''_{2m-4}, \beta''_{2m-4})$ and $E''_{2l,2m-4,\text{pre}}$ be as in Subsection 5 for $a = l$ and $b = m - 2$. Hypothesis 7.1 implies $b \geq 2a$, i.e., Hypothesis 5.1 holds. By Lemma 5.2, $(E''_{2l,2m-4,\text{pre}})^\vee$ is ample. Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be any finite morphism such that $f^*[(E''_{2l,2m-4,\text{pre}})^\vee] \otimes \mathcal{O}_{\mathbb{P}^1}(-1)$ is ample. In every case except $(l, m) = (1, 4)$, it suffices to take $f$ to be the identity map. If $(l, m) = (1, 4)$, it suffices to take $f$ to be any finite morphism of degree $\geq 2$. At any rate, define $E''_{2l,2m-4}$ to be $f^*(E''_{2l,2m-4,\text{pre}})$ considered as a subsheaf of $f^*(W''_{2m-4} \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}) = W''_{2m-4} \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}$.

If $n = 2m$, define $(W, \beta)$ to be the orthogonal direct sum of $(W'_4, \beta'_4)$ and $(W''_{2m-4}, \beta''_{2m-4})$. If $n = 2m + 1$, which can only occur in the symmetric case, let $(\mathbf{1}, \beta_{\mathbf{1}})$ be a symmetric pairing of dimension 1, and define $(W, \beta)$ to be the orthogonal direct sum of $(W'_4, \beta'_4)$, $(W''_{2m-4}, \beta''_{2m-4})$ and $(\mathbf{1}, \beta_{\mathbf{1}})$. Define $E_{2l+2}$ to be the direct sum $E'_{2,4}$ and $E''_{2l,2m-4}$. Define $E_{2l+1}$ to be the direct sum of one direct summand $\mathcal{O}_{\mathbb{P}^1}(-1)$ of $E'_{2,4}$ and $E''_{2l,2m-4}$. Finally, define $E_{2l}$ to be $E''_{2l,2m-4}$.

**Lemma 7.4.** *Assume $l \geq 1$ and $m \geq 2l+2$. The flag $E_{2l} \subset E_{2l+1} \subset E_{2l+2} \subset W \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}$ is a $(k-1, k, k+1)$-flag of isotropic locally direct summands for $\beta$. The cokernel $E_{2l+2}^{\perp}/E_{2l+2}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}^{\oplus(2m-4l-4)}$ if $n = 2m$, respectively $\mathcal{O}_{\mathbb{P}^1}^{\oplus(2m-4l-3)}$ if $n = 2m + 1$. The cokernel $E_{2l}^{\perp}/E_{2l+2}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus(2m-4l-4)}$ if $n = 2m$, respectively $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus(2m-4l-3)}$ if $n = 2m + 1$. The cokernels*

*$E_{2l+2}/E_{2l+1}$ and $E_{2l+1}/E_{2l}$ are each isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1)$. And $E_{2l}^\vee \otimes (E_{2l+1}/E_{2l})$ is ample.*

*Proof.* Since $E'_{2,4}$ is isotropic for $\beta'_{2,4}$ and $E''_{2l,2m-4}$ is isotropic for $\beta''_{2l,2m-4}$, $E_{2l+2}$ is isotropic for $\beta$. Since $E_{2l+1}$ and $E_{2l}$ are contained in $E_{2l+2}$, they are also isotropic for $\beta$. The annihilator $E_{2l+2}^\perp$ of $E_{2l+2}$ with respect to $\beta$ is the direct sum of the annihilator $(E'_{2,4})^\perp$ of $E'_{2,4}$ with respect to $\beta'_{2,4}$, the annihilator $(E''_{2l,2m-4})^\perp$ of $E''_{2l,2m-4}$ with respect to $\beta''_{2l,2m-4}$, and also $\mathcal{O}_{\mathbb{P}^1}$ if $n = 2m+1$. Therefore $E_{2l+2}^\perp/E_{2l+2}$ equals the direct sum of $(E'_{2,4})^\perp/E'_{2,4}$, $(E''_{2l,2m-4})^\perp/E''_{2l,2m-4}$, and also $\mathcal{O}_{\mathbb{P}^1}$ if $n = 2m+1$. By Lemma 5.2, $E'_{2,4}$ is its own annihilator, and $(E''_{2l,2m-4})^\perp/E''_{2l,2m-4}$ equals $\mathcal{O}_{\mathbb{P}^1}^{\oplus(2m-4l-4)}$. Therefore $E_{2l+2}^\perp/E_{2l+2}$ equals $\mathcal{O}_{\mathbb{P}^1}^{\oplus(2m-4l-4)}$ if $n = 2m$, and equals $\mathcal{O}_{\mathbb{P}^1}^{\oplus(2m-4l-3)}$ if $n = 2m+1$.

The computation of $E_{2l}^\perp/E_{2l+2}$ is similar, except the summand $(E'_{2,4})^\perp/E'_{2,4}$ is replaced by $W'_4 \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}/E'_{2,4}$. Since $E'_{2,4}$ equals its own annihilator, $W'_4 \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}/E'_{2,4}$ equals the dual of $E'_{2,4}$. Thus $W'_4 \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}/E'_{2,4}$ equals $(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2})^\vee$, i.e., $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$. Therefore $E_{2l}^\perp/E_{2l+2}$ equals $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus(2m-4l-4)}$ if $n = 2m$, and $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus(2m-4l-3)}$ if $n = 2m+1$.

By definition, $E_{2l+2}/E_{2l+1}$ and $E_{2l+1}/E_{2l}$ are isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1)$ and $E_{2l}^\vee \otimes (E_{2l+1}/E_{2l})$, i.e., $E_{2l}^\vee \otimes \mathcal{O}_{\mathbb{P}^1}(-1)$, is ample. □

**Proposition 7.5.** *Assume $l \geq 1$ and $m \geq 2l+1$. The morphism $\zeta : \mathbb{P}^1 \to M$ associated to the flag in Lemma 7.4 is a very twisting family of pointed lines on Flag$_k(W, \beta)$.*

*Proof.* This is very similar to the proof of Proposition 7.3. □

It seems likely $\zeta$ can be chosen to be an orbit curve. However, since the entries of the matrix for $\phi''_{2l,2m-4}$ are typically not monomials, it is not certain.

## 8. Isotropic Grassmannians, Case II

Let $(W, \beta)$ be a symmetric or skew-symmetric pairing of dimension $n$. This subsection proves existence of a very twisting family of pointed lines on the Grassmannian of isotropic $k$-planes when $k$ is even and $n \geq 2k+2$.

**Hypothesis 8.1.** The pairing $(W, \beta)$ is symmetric of dimension $2m$ or $2m+1$ or the pairing $(W, \beta)$ is skew-symmetric of dimension $2m$, and $k = 2l$. Assume that $l$ is positive and $m$ is at least $2l+1$.

The last inequality is equivalent to $k$ is positive and $n \geq 2k+2$. In particular, observe that $m$ is at least 3.

Define $(W'_6, \beta'_6)$ as in Subsection 5 for $b = 3$. Choose an ordered basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ of $W'_{3,+}$. Denote the dual ordered basis of $W'_{3,-}$ by $x_1, x_2, x_3$. Define $E'_{3,6}$ to be the rank 3 locally free $\mathcal{O}_{\mathbb{P}^1}$-module $\mathcal{O}_{\mathbb{P}^1}(-2)\mathbf{f} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)\mathbf{g}_1 \oplus \mathcal{O}_{\mathbb{P}^1}(-1)\mathbf{g}_2$, where the symbols $\mathbf{f}, \mathbf{g}_1, \mathbf{g}_2$ are simply place-holders. There is an isomorphism $\phi'_{3,6}$ of $E'_{3,6}$ to an isotropic locally direct summand of $W'_6 \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}$. The definition of $\phi'_{3,6}$ is different in the symmetric and skew-symmetric case.

**Symmetric case.** Define $\phi'_{3,6} : E'_{3,6} \to W'_6 \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}$ to be the unique $\mathcal{O}_{\mathbb{P}^1}$-module homomorphism satisfying,

$$\begin{array}{rcl} \phi'_{3,6}(\mathbf{g}) & = & T_0^2\mathbf{e}_1 + T_0T_1\mathbf{e}_2 + T_1^2 x_3 \\ \phi'_{3,6}(\mathbf{f}_1) & = & T_0\mathbf{e}_3 - T_1 x_2 \\ \phi'_{3,6}(\mathbf{f}_2) & = & T_1 x_1 - T_0 x_2 \end{array}$$

The proof of the following lemma is a straightforward computation.

**Lemma 8.2.** *Let $(W'_6, \beta'_6)$ be the symmetric pairing from above. The image of $\phi'_{3,6}$ is a rank 3 locally direct summand of $W'_6 \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}$ isotropic for $\beta'_6$. It equals its own annihilator with respect to $\beta'_6$.*

**Skew-symmetric case.** Define $\phi'_{3,5} : E'_{3,6} \to W'_6 \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}$ to be the unique $\mathcal{O}_{\mathbb{P}^1}$-module homomorphism satisfying,

$$\begin{array}{rcl} \phi'_{3,6}(\mathbf{g}) & = & -T_0^2\mathbf{e}_1 + T_0T_1(\mathbf{e}_2 - x_2) + T_1^2 x_3 \\ \phi'_{3,6}(\mathbf{f}_1) & = & T_0(\mathbf{e}_2 + x_2) + 2T_1 x_1 \\ \phi'_{3,6}(\mathbf{f}_2) & = & T_1(\mathbf{e}_2 + x_2) + 2T_0\mathbf{e}_3 \end{array}$$

The proof of the following lemma is a straightforward computation.

**Lemma 8.3.** *Let $(W'_6, \beta'_6)$ be the skew-symmetric pairing from above. The image of $\phi'_{3,6}$ is a rank 3 locally direct summand of $W'_6 \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}$ isotropic for $\beta'_6$. It equals its own annihilator with respect to $\beta'_6$.*

**Case IIa:** $k = 2$**.** Assume that $k = 2$, i.e., $l = 1$. Then Hypothesis 8.1 is equivalent to $n \geq 6$. Let $(W''_{n-6}, \beta'')$ be a symmetric pairing, resp. skew-symmetric pairing, of dimension $n - 6$. Define $(W, \beta)$ to be the orthogonal direct sum of $(W'_6, \beta'_6)$ and $(W''_{n-6}, \beta'')$. Define $E_3$ to be $E'_{3,6}$. Define $E_2$ to be direct summand $\mathcal{O}_{\mathbb{P}^1}(-2)\mathbf{g} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)\mathbf{f}_1$, and define $E_1$ to be $\mathcal{O}_{\mathbb{P}^1}(-2)\mathbf{g}$.

**Lemma 8.4.** *Assume $k = 2$ and $n \geq 6$, either the symmetric or the skew-symmetric case. The flag $E_1 \subset E_2 \subset E_3 \subset W \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}$ is a $(1, 2, 3)$-flag of isotropic locally direct summands for $\beta$. The cokernel $E_3^\perp/E_3$ is isomorphic to $W'_{n-6} \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}$. The cokernel $E_1^\perp/E_3$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \oplus W'_{n-4} \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}$. The cokernels $E_3/E_2$ and $E_2/E_1$ are isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1)$. And $E_1^\vee \otimes (E_2/E_1)$ is isomorphic to the ample invertible sheaf $\mathcal{O}_{\mathbb{P}^1}(1)$.*

*Proof.* By Lemma 8.2, resp. Lemma 8.3, $E_3$ is isotropic of rank 3. Since $E_2$ and $E_1$ are contained in $E_3$, they are isotropic. By construction, the annihilator $E_3^\perp$ of $E_3$ with respect to $\beta$ is the direct sum of the annihilator $(E'_{3,6})^\perp$ of $E'_{3,6}$ with respect to $\beta'_{3,6}$ and $W''_{n-6} \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}$. By Lemma 8.2, resp. Lemma 8.3, $(E'_{3,6})^\perp$ equals $E'_{3,6}$. Therefore $E_3^\perp/E_3$ equals $W''_{n-6} \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}$. Similarly, $E_1^\perp$ is the direct sum of the annihilator of $\mathcal{O}_{\mathbb{P}^1}(-2)\mathbf{g}$ with respect to $\beta'_{3,6}$ and $W''_{n-6} \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}$. Since $E'_{3,6}$ equals its own annihilator, $(\mathcal{O}_{\mathbb{P}^1}(-2)\mathbf{g})^\perp/E'_{3,6}$ equals the dual of $E'_{3,6}/\mathcal{O}_{\mathbb{P}^1}(-2)\mathbf{g}$. Thus $(\mathcal{O}_{\mathbb{P}^1}(-2)\mathbf{g})^\perp/E'_{3,6}$ equals $(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2})^\vee$, i.e., $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$. Therefore $E_1^\perp/E_3$ equals $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \oplus W''_{n-6} \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}$.

By the definition of $E_2$ and $E_1$, $E_3/E_2 \cong E_2/E_1 \cong \mathcal{O}_{\mathbb{P}^1}(-1)$. Since $E_1 \cong \mathcal{O}_{\mathbb{P}^1}(-2)$, $E_1^\vee \otimes (E_2/E_1)$ equals $\mathcal{O}_{\mathbb{P}^1}(2) \otimes \mathcal{O}_{\mathbb{P}^1}(-1)$, i.e., $\mathcal{O}_{\mathbb{P}^1}(1)$. ☐

**Proposition 8.5.** *Assume $k = 2$ and $n \geq 6$, either the symmetric or the skew-symmetric case. The morphism $\zeta : \mathbb{P}^1 \to M$ associated to the flag in Lemma 8.4 is a very twisting family of pointed lines on $Flag_2(W, \beta)$.*

*Proof.* This is very similar to the proof of Proposition 7.3. Of course now the term $E_1^\vee \otimes (E_2/E_2)$ in $\zeta^* T_{\text{ev}}$ is nonzero. But by the last part of Lemma 8.4, this is ample. □

An argument similar to the proof of Claim 6.2 proves the very twisting family can be chosen to be an orbit curve.

**Case IIb: $k > 2$.** Assume now that $k > 2$, i.e., $l \geq 2$. In both the symmetric case and the skew-symmetric case, let $(W''_{2m-6}, \beta''_{2m-6})$ and $E''_{2l-2,2m-6,\text{pre}}$ be as in Subsection 5 for $a = l-1$ and $b = m-3$. Because $l \geq 2$, $a$ is positive. By Hypothesis 8.1, $b \geq 2a$. In particular, $b$ is positive and Hypothesis 5.1 holds. By Lemma 5.2, $(E''_{2l-2,2m-6}, \text{pre})^\vee$ is ample. Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be any finite morphism such that $f^*[(E''_{2l-2,2m-6,\text{pre}})^\vee] \otimes \mathcal{O}_{\mathbb{P}^1}(-1)$ is ample. In every case except $(l,m) = (2,5)$, it suffices to take $f$ to be the identity map. If $(l,m) = (2,5)$, it suffices to take $f$ to be any finite morphism of degree $\geq 2$. At any rate, define $E''_{2l-2,2m-6}$ to be $f^*(E''_{2l-2,2m-6,\text{pre}})$ considered as a subsheaf of $f^*(W''_{2m-6} \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}) = W''_{2m-6} \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}$.

If $n = 2m$, define $(W, \beta)$ to be the orthogonal direct sum of $(W'_6, \beta'_6)$ and $(W''_{2m-6}, \beta''_{2m-6})$. If $n = 2m+1$, which can only occur in the symmetric case, let $(\mathbf{1}, \beta_{\mathbf{1}})$ be a symmetric pairing of dimension 1, and define $(W, \beta)$ to be the orthogonal direct sum of $(W'_6, \beta'_6)$, $(W''_{2m-6}, \beta''_{2m-6})$ and $(\mathbf{1}, \beta_{\mathbf{1}})$. Define $E_{2l+1}$ to be the direct sum $E'_{3,6}$ and $E''_{2l-2,2m-6}$. Define $E_{2l}$ to be the direct sum of $\mathcal{O}_{\mathbb{P}^1}(-2)\mathbf{g} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)\mathbf{f}_1$ and $E''_{2l-2,2m-6}$. Finally, define $E_{2l-1}$ to be the direct sum of $\mathcal{O}_{\mathbb{P}^1}(-2)\mathbf{g}$ and $E''_{2l-2,2m-6}$.

**Lemma 8.6.** *Assume $l \geq 2$ and $m \geq 2l+1$. The flag $E_{2l-1} \subset E_{2l} \subset E_{2l+1} \subset W \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}$ is a $(k-1,k,k+1)$-flag of isotropic locally direct summands for $\beta$. The cokernel $E_{2l+1}^\perp / E_{2l+1}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}^{\oplus(2m-4l-2)}$ if $n = 2m$, respectively $\mathcal{O}_{\mathbb{P}^1}^{\oplus(2m-4l-1)}$ if $n = 2m+1$. The cokernel $E_{2l-1}^\perp / E_{2l+1}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus(2m-4l-2)}$ if $n = 2m$, respectively $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus(2m-4l-1)}$ if $n = 2m+1$. The cokernels $E_{2l+1}/E_{2l}$ and $E_{2l}/E_{2l-1}$ are each isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1)$. And $E_{2l-1}^\vee \otimes (E_{2l}/E_{2l-1})$ is ample.*

*Proof.* Since $E'_{3,6}$ is isotropic for $\beta'_{3,6}$ and $E''_{2l-2,2m-6}$ is isotropic for $\beta''_{2l-2,2m-6}$, $E_{2l+1}$ is isotropic for $\beta$. Since $E_{2l}$ and $E2l-1$ are contained in $E_{2l+1}$, they are also isotropic for $\beta$. The annihilator $E_{2l+1}^\perp$ of $E_{2l+1}$ with respect to $\beta$ is the direct sum of the annihilator $(E'_{3,6})^\perp$ of $E'_{3,6}$ with respect to $\beta'_{3,6}$, the annihilator $(E''_{2l-2,2m-6})^\perp$ of $E''_{2l-2,2m-6}$ with respect to $\beta''_{2l-2,2m-6}$, and also $\mathcal{O}_{\mathbb{P}^1}$ if $n = 2m+1$. Therefore $E_{2l+1}^\perp / E_{2l+2}$ equals the direct sum of $(E'_{3,6})^\perp / E'_{3,6}$, $(E''_{2l-2,2m-6})^\perp / E''_{2l-2,2m-6}$, and also $\mathcal{O}_{\mathbb{P}^1}$ if $n = 2m+1$. By Lemma 8.2, resp. Lemma 8.3, $E'_{3,6}$ is its own annihilator. By Lemma 5.2, $(E''_{2l-2,2m-6})^\perp / E''_{2l-2,2m-6}$ equals $\mathcal{O}_{\mathbb{P}^1}^{\oplus(2m-4l-2)}$. Therefore $E_{2l+1}^\perp / E_{2l+1}$ equals $\mathcal{O}_{\mathbb{P}^1}^{\oplus(2m-4l-2)}$ if $n = 2m$, and equals $\mathcal{O}_{\mathbb{P}^1}^{\oplus(2m-4l-1)}$ if $n = 2m+1$.

The computation of $E_{2l-1}^\perp / E_{2l+1}$ is similar, except the summand $(E'_{3,6})^\perp / E'_{3,6}$ is replaced by $(\mathcal{O}_{\mathbb{P}^1}(-2)\mathbf{g})^\perp / E'_{3,6}$. Since $E'_{3,6}$ equals its own annihilator, $(\mathcal{O}_{\mathbb{P}^1}(-2)\mathbf{g})^\perp / E'_{3,6}$ equals the dual of $E'_{3,6} / \mathcal{O}_{\mathbb{P}^1}(-2)\mathbf{g}$. Thus $(\mathcal{O}_{\mathbb{P}^1}(-2)\mathbf{g})^\perp / E'_{3,6}$ equals $(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2})^\vee$, i.e., $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$. Therefore $E_{2l-1}^\perp / E_{2l+1}$ equals $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus(2m-4l-2)}$ if $n = 2m$, and $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus(2m-4l-1)}$ if $n = 2m+1$.

By definition, $E_{2l+1}/E_{2l}$ and $E_{2l}/E_{2l-1}$ are isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1)$. Since $(E''_{2l-2,2m-6})^\vee \otimes \mathcal{O}_{\mathbb{P}^1}(-1)$ is ample, also $E_{2l-1}^\vee \otimes (E_{2l}/E_{2l-1})$, which equals the direct sum of $\mathcal{O}_{\mathbb{P}^1}(1)$ and $(E''_{2l-2,2m-6})^\vee \otimes \mathcal{O}_{\mathbb{P}^1}(-1)$, is ample. □

**Proposition 8.7.** *Assume $l \geq 1$ and $m \geq 2l+1$. The morphism $\zeta : \mathbb{P}^1 \to M$ associated to the flag in Lemma 8.6 is a very twisting family of pointed lines on $Flag_k(W, \beta)$.*

*Proof.* The proof is very similar to the proof of Proposition 8.5. □

It seems likely $\zeta$ can be chosen to be an orbit curve. However, since the entries of the matrix for $\phi''_{2l,2m-4}$ are typically not monomials, it is not certain.

## 9. Isotropic Grassmannians, Case III

Let $(W, \beta)$ be a symmetric pairing of dimension $n = 2k$. This subsection proves existence of a very twisting family of pointed lines on the Grassmannian of isotropic $k$-planes when $k \geq 3$. There is no very twisting family if $k = 1$ or $k = 2$.

**Case IIIa, $k$ even.** Let $l \geq 2$ be an integer, and let $k$ equal $2l$. Let $n$ equal $2k$, i.e., $4l$. Let $(W'_4, \beta'_4)$ and $E'_{2,4}$ be as in Subsection 5 for $a = 1$ and $b = 2$. In particular, $E'_{2,4}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$.

Let $(W''_{4l-4}, \beta''_{4l-4})$ and $E''_{2l-2,4l-4,\text{pre}}$ be as in Subsection 5 for $a = l-1$ and $b = 2l-2$. Since $l \geq 2$, $a$ is positive. And, of course, $b$ equals $2a$. Thus Hypothesis 5.1 holds. By Lemma 5.2, $(E''_{2l-2,4l-4,\text{pre}})^\vee$ is ample. Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be any finite morphism such that $f^*[(E''_{2l-2,4l-4,\text{pre}})^\vee] \otimes \mathcal{O}_{\mathbb{P}^1}(-1)$ is ample. In every case except $l = 2$, it suffices to take $f$ to be the identity map. If $l = 2$, it suffices to take $f$ to be any finite morphism of degree $\geq 2$. At any rate, define $E''_{2l-2,4l-4}$ to be $f^*(E''_{2l-2,4l-4,\text{pre}})$ considered as a subsheaf of $f^*(W''_{4l-4} \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}) = W''_{4l-4} \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}$.

Define $(W, \beta)$ to be the orthogonal direct sum of $(W'_4, \beta'_4)$ and $(W''_{4l-4}, \beta''_{4l-4})$. Define $E_{2l}$ to be the direct sum of $E'_{2,4}$ and $E''_{2l-2,4l-4}$. And define $E_{2l-2}$ to be $E''_{2l-2,4l-4}$.

**Lemma 9.1.** *Assume $l \geq 2$. Let $k$ equal $2l$ and let $n$ equal $4l$. The flag $E_{2l-2} \subset E_{2l} \subset W \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}$ is a $(k-2,k)$-flag of isotropic locally direct summands for $\beta$. The cokernel $E_{2l}/E_{2l-2}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$. The determinant $\bigwedge^2 (E_{2l}/E_{2l-2})^\vee$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(2)$. And $(E_{2l}/E_{2l-2}) \otimes E_{2l-2}^\vee$ is ample.*

*Proof.* The proof is very similar to the proof of Lemma 7.4. The novel feature is the computation of $\bigwedge^2 (E_{2l}/E_{2l-2})^\vee$, which is obviously $\mathcal{O}_{\mathbb{P}^1}(2)$ since $E_{2l}/E_{2l-2}$ equals $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$. □

**Proposition 9.2.** *Assume $l \geq 2$. Let $k$ equal $2l$ and let $n$ equal $4l$. The morphism $\zeta : \mathbb{P}^1 \to M$ associated to the flag in Lemma 9.1 is a very twisting family of pointed lines on $Flag_k(W, \beta)$.*

*Proof.* The proof is very similar to the proof of Proposition 7.3. Instead of Corollary 3.4, use Corollary 3.10. □

An argument similar to the proof of Claim 6.2 proves the very twisting family can be chosen to be an orbit curve.

**Case IIIb, $k$ odd.** Let $l \geq 1$ be an integer, let $k$ equal $2l+1$, and let $n$ equal $2k = 4l+2$. For $(n,k) = (2,1)$, there is no very twisting family of lines. Let

$(W'_6, \beta'_6)$ and $E'_{3,6}$ be as in Subsection 8 for the symmetric pairing. In particular, $E'_{3,6}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-2)\mathbf{g} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)\mathbf{f}_1 \oplus \mathcal{O}_{\mathbb{P}^1}(-1)\mathbf{f}_2$. If $l = 1$, i.e., $k = 3$ and $n = 6$, define $(W, \beta)$ to be $(W'_6, \beta'_6)$, define $E'_k$ to be $E'_{3,6}$ and define $E'_{k-2}$ to be $\mathcal{O}_{\mathbb{P}^1}(-2)\mathbf{g}$.

Next assume $l \geq 2$. Let $(W''_{4l-4}, \beta''_{4l-4})$ and $E''_{2l-2,4l-4,\text{pre}}$ be as in Subsection 5 for $a = l - 1$ and $b = 2l - 2$. Since $l \geq 2$, $a$ is positive. And, of course, $b = 2a$. Thus Hypothesis 5.1 holds. By Lemma 5.2, $(E''_{2l-2,4l-4,\text{pre}})^\vee$ is ample. Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be any finite morphism such that $f^*[(E''_{2l-2,4l-4,\text{pre}})^\vee] \otimes \mathcal{O}_{\mathbb{P}^1}(-1)$ is ample. In every case except $l = 2$, it suffices to take $f$ to be the identity map. If $l = 2$, it suffices to take $f$ to be any finite morphism of degree $\geq 2$. At any rate, define $E''_{2l-2,4l-4}$ to be $f^*(E''_{2l-2,4l-4,\text{pre}})$ considered as a subsheaf of $f^*(W''_{4l-4} \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}) = W''_{4l-4} \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}$.

Define $(W, \beta)$ to be the orthogonal direct sum of $(W'_6, \beta'_6)$ and $(W''_{4l-4}, \beta''_{4l-4})$. Define $E_{2l+1}$ to be the direct sum of $E'_{3,6}$ and $E''_{2l-2,4l-4}$. And define $E_{2l-1}$ to be the direct sum of $\mathcal{O}_{\mathbb{P}^1}(-2)\mathbf{g}$ and $E''_{2l-2,4l-4}$.

**Lemma 9.3.** *Assume $l \geq 1$. Let $k$ equal $2l + 1$ and let $n$ equal $4l + 2$. The flag $E_{2l-1} \subset E_{2l+1} \subset W \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}$ is a $(k-2, k)$-flag of isotropic locally direct summands for $\beta$. The cokernel $E_{2l+1}/E_{2l-1}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$. The determinant $\bigwedge^2(E_{2l+1}/E_{2l-1})^\vee$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(2)$. And $(E_{2l+1}/E_{2l-1}) \otimes E^\vee_{2l-1}$ is ample.*

*Proof.* The proof is very similar to the proof of Lemma 9.1. ☐

**Proposition 9.4.** *Assume $l \geq 1$. Let $k$ equal $2l + 1$ and let $n$ equal $4l + 2$. The morphism $\zeta : \mathbb{P}^1 \to M$ associated to the flag in Lemma 9.1 is a very twisting family of pointed lines on $\mathit{Flag}_k(W, \beta)$.*

*Proof.* The proof is very similar to the proof of Proposition 9.2. ☐

An argument similar to the proof of Claim 6.2 proves the very twisting family can be chosen to be an orbit curve.

## 10. Isotropic Grassmannians, Case IV

Let $(W, \beta)$ be a skew-symmetric pairing of dimension $n = 2k$. This subsection proves existence of a very twisting family of pointed lines on the Grassmannian of isotropic $k$-planes when $k \geq 2$. For $k = 1$ there is no very twisting family of pointed lines.

**Case IVa, $k$ even.** Let $l \geq 1$ be an integer, and let $k$ equal $2l$. Let $n$ equal $2k$, i.e., $4l$. Let $(W'_4, \beta'_2)$ be $(W'_{2,+} \oplus W'_{2,-}, \beta'_4)$ as in Subsection 5 for $b = 2$. Choose an ordered basis $\mathbf{e}_1, \mathbf{e}_2$ for $W'_{2,+}$, and let $x_1, x_2$ be the dual ordered basis for $W'_{2,-}$. Define $R'_3$ to be the image of the sheaf,

$$\phi'_3 : \mathcal{O}_{\mathbb{P}^1}\{\mathbf{a}\} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)\{\mathbf{b}_+, \mathbf{b}_-\} \to W'_4 \otimes \mathcal{O}_{\mathbb{P}^1},$$

$$\phi'_3(\mathbf{a}) = \mathbf{e}_2 - x_2,\ \phi'_3(\mathbf{b}_+) = T_0\mathbf{e}_1 + T_1\mathbf{e}_2,\ \phi'_3(\mathbf{b}_-) = -T_1x_1 + T_0x_2.$$

Define $E'_2$ and $E'_1$ to be the subsheaves of $R'_3$ given by,

$$E'_2 = \phi'_3(\mathcal{O}_{\mathbb{P}^1}(-1)\{\mathbf{b}_+, \mathbf{b}_-\}),$$
$$E'_1 = \phi'_3(\mathcal{O}_{\mathbb{P}^1}(-2)\{T_0\mathbf{b}_+ - T_1\mathbf{b}_-\}) = \mathcal{O}_{\mathbb{P}^1}(-2)\{T_0^2\mathbf{e}_1 + T_0T_1(\mathbf{e}_2 - x_2) + T_1^2x_2\}.$$

**Lemma 10.1.** *Let $(W'_4, \beta'_4)$ be the skew-symmetric pairing from above. The sheaf $E'_2$ is a rank 2 isotropic locally direct summand of $W'_4 \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}$. The subsheaf $E'_1$ is a rank 1 isotropic locally direct summand. The annihilator of $E'_1$ with respect to $\beta'_4$ equals $R'_3$. The cokernels $R'_3/E'_2$ and $E'_2/E'_1$ are both isomorphic to $\mathcal{O}_{\mathbb{P}^1}$. And $(E'_1)^\vee$ is an ample invertible sheaf.*

*Proof.* Since $\mathbf{b}_+$ maps into $W'_{2,+} \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}$ and $\mathbf{b}_-$ maps into $W'_{2,-} \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}$, the images are mutually orthogonal. Therefore $E'_2$ is isotropic. It is straightforward to compute that $E'_2$ and $E'_1$ are locally direct summands of rank 2, respectively rank 1. It is also straightforward to compute that $R'_3$ is a rank 3 locally direct summand that annihilates $E'_1$. Thus it is all of the annihilator of $E'_1$. The cokernels $R'_3/E'_2$ and $E'_2/E'_1$ are invertible sheaves. Comparing the degrees of $E'_1$, $E'_2$ and $R'_3$, the cokernels have degree 0, thus are isomorphic to $\mathcal{O}_{\mathbb{P}^1}$. Of course $(E'_1)^\vee$ is isomorphic to the ample invertible sheaf $\mathcal{O}_{\mathbb{P}^1}(2)$. □

If $l = 1$, define $W''_0$ to be the zero vector space, and define $E''_0$ to be the zero sheaf on $\mathbb{P}^1$. If $l > 1$, let $(W''_{4l-4}, \beta''_{4l-4})$ and $E''_{2l-2,4l-4}$ be as in Subsection 5 for $a = l-1$ and $b = 2l-2$. Since $l > 1$, $a$ is positive. And, of course, $b$ equals $2a$. Thus Hypothesis 5.1 holds. By Lemma 5.2, $(E''_{2l-2,4l-4})^\vee$ is ample.

Define $(W, \beta)$ to be the orthogonal direct sum of $(W'_2, \beta'_2)$ and $(W''_{4l-4}, \beta''_{4l-4})$, which is just $(W'_2, \beta'_2)$ if $l$ equals 1. Define $R_{2l+1}$ to be the direct sum of $R'_3$ and $E''_{2l-2,4l-4}$. Define $E_{2l}$ to be the direct sum of $E'_2$ and $E''_{2l-2,4l-4}$. And define $E_{2l-1}$ to be the direct sum of $E'_1$ and $E''_{2l-2,4l-4}$.

**Lemma 10.2.** *Assume $l \geq 2$. Let $k$ equal $2l$ and let $n$ equal $4l$. The flag $E_{2l-1} \subset E_{2l} \subset R_{2l+1} \subset W \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}$ is a $(k-1, k, k+1)$-flag parametrized by a morphism $\zeta : \mathbb{P}^1 \to M$. The annihilator of $E_{2l-1}$ equals $R_{2l+1}$. The cokernels $R_{k+1}/E_k$ and $E_k/E_{k-1}$ are each isomorphic to $\mathcal{O}_{\mathbb{P}^1}$. And $(E_k/E_{k-1}) \otimes E^\vee_{k-1}$ is ample.*

*Proof.* This follows by combining Lemma 10.1 with the method of proof of Lemma 7.2. The novelty is that $R_{2l+1}$ is not isotropic. However, it does equal the annihilator of $E_{2l-1}$ with respect to $\beta_{2l}$. □

**Proposition 10.3.** *Assume $l \geq 2$. Let $k$ equal $2l$ and let $n$ equal $4l$. The morphism $\zeta : \mathbb{P}^1 \to M$ associated to the flag in Lemma 10.2 is a very twisting family of pointed lines on $\mathrm{Flag}_k(W, \beta)$.*

*Proof.* This is very similar to the proof of Proposition 7.3. □

**Case IVb, $k$ odd.** Let $l \geq 1$ be an integer, let $k$ equal $2l+1$, and let $n$ equal $2k = 4l+2$. For $(n, k) = (2, 1)$, there is no very twisting family of lines. Let $(W'_2, \beta'_2)$ be as in 5 for $b = 1$. In particular, $W'_2$ is the direct sum of $W'_{1,+}$ and $W'_{1,-}$.

Let $(W''_{4l}, \beta''_{4l})$ and $E''_{2l,4l}$ be as in Subsection 5 for $a = l$ and $b = 2l$. Since $l \geq 1$, $a$ is positive. And, of course, $b$ equals $2a$. Thus Hypothesis 5.1 holds.

Define $(W, \beta)$ to be the orthogonal direct sum of $(W'_2, \beta'_2)$ and $(W''_{4l}, \beta''_{4l})$. Define $R_{2l+2}$ to be the direct sum of $W'_2$ and $E''_{2l,4l}$. Define $E_{2l+1}$ to be the direct sum of $W'_{1,+}$ and $E''_{2l,4l}$. And define $E_{2l}$ to be $E''_{2l,4l}$.

**Lemma 10.4.** *Assume $l \geq 1$. Let $k$ equal $2l+1$ and let $n$ equal $4l+2$. The flag $E_{2l} \subset E_{2l+1} \subset R_{2l+2} \subset W \otimes_\kappa \mathcal{O}_{\mathbb{P}^1}$ is a $(k-1,k,k+1)$-flag parametrized by a morphism $\zeta : \mathbb{P}^1 \to M$. The annihilator of $E_{2l}$ equals $R_{2l+2}$. The cokernels $R_{k+1}/E_k$ and $E_k/E_{k-1}$ are each isomorphic to $\mathcal{O}_{\mathbb{P}^1}$. And $(E_k/E_{k-1}) \otimes E_{k-1}^\vee$ is ample.*

*Proof.* This is similar to the proof of Lemma 10.2. □

**Proposition 10.5.** *Assume $l \geq 1$. Let $k$ equal $2l+1$ and let $n$ equal $4l+2$. The morphism $\zeta : \mathbb{P}^1 \to M$ associated to the flag in Lemma 10.4 is a very twisting family of pointed lines on $\mathit{Flag}_k(W,\beta)$.*

*Proof.* This is similar to the proof of Proposition 10.5. □

An argument similar to the proof of Claim 6.2 proves the very twisting family can be chosen to be an orbit curve.

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