# DEGENERATIONS OF RATIONALLY CONNECTED VARIETIES AND PAC FIELDS

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Abstract. A degeneration of a separably rationally connected variety over a field $k$ contains a geometrically irreducible subscheme if $k$ contains the algebraic closure of its subfield. If $k$ is a perfect PAC field, the degeneration has a $k$-point. This generalizes [FJ05, Theorem 21.3.6(a)]: a degeneration of a Fano complete intersection over $k$ has a $k$-point if $k$ is a perfect PAC field containing the algebraic closure of its prime subfield.

## 1. Statement of results

Recently, a number of fields long known to be $C_1$ were proved to satisfy an *a priori* stronger property related to rationally connected varieties.

(i) Every rationally connected variety defined over the function field of a curve over a characteristic 0 algebraically closed field has a rational point, [GHS03].
(ii) Every separably rationally connected variety defined over the function field of a curve over an algebraically closed field of arbitrary characteristic has a closed point, [dJS03].
(iii) Every smooth, rationally chain connected variety over a finite field has a rational point, [Esn03].

Moreover, in each of these cases, degenerations of these varieties also have rational points, at least under some mild hypotheses on the degeneration.

This article considers the same problem for perfect PAC fields containing an algebraically closed field. Such fields are known to be $C_1$, [FJ05, Theorem 21.3.6(a)]. The main theorem is the following.

**Theorem 1.1.** *Let $k$ be a perfect PAC field containing the algebraic closure of its prime subfield. Let $X_k$ be a $k$-scheme which is the closed fiber of a proper, flat scheme over a DVR whose geometric generic fiber is separably rationally connected (in the sense of* [dJS03]*). Then $X_k$ has a $k$-point.*

**Remark 1.2.** This gives a new proof of [FJ05, Theorem 21.3.6(a)], i.e., every perfect PAC field containing an algebraically closed field is $C_1$.

This should be compared to the following theorems of Kollár and de Jong respectively.

**Theorem 1.3.** [Kol05] *Let $k$ be a characteristic 0 PAC field. Let $X_k$ be a $k$-scheme whose base-change $X \otimes_k \overline{k}$ is the closed fiber of a projective, flat scheme over a DVR whose geometric generic fiber is a Fano manifold. Then $X_k$ has a $k$-point. In particular, $k$ is $C_1$.*

**Theorem 1.4** (de Jong)**.** *Let $k$ be a characteristic 0 field having a point in every rationally connected $k$-scheme and containing $\overline{\mathbb{Q}}$. Let $X_k$ be a $k$-scheme which is* *the closed fiber of a proper, flat scheme over a DVR whose geometric generic fiber is rationally connected. Then $X_k$ has a $k$-point. In particular, $k$ is $C_1$.*

Theorem 1.1 is a consequence of the following more precise result.

**Theorem 1.5.** *Let $k$ be a field containing the algebraic closure of its prime subfield. Let $X$ be a proper $k$-scheme which is the closed fiber of a proper, flat scheme over a DVR whose geometric generic fiber is separably rationally connected (in the sense of* [dJS03]*). There exists a closed subscheme $Y$ of $X$ such that $Y \otimes_k \overline{k}$ is irreducible.*

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## 2. Proofs

**Definition 2.1.** Let $R$ be a DVR and let $X_R$ be an $R$-scheme. A *finite type model* of $(R, X_R)$ consists of a datum

$$((P, D) \to (S, \mathfrak{s}), \mathrm{Spec}\ R \to P, X_P)$$

of

(i) a Dedekind domain $S$ of finite type over $\mathbb{Z}$ or over $\mathbb{Q}$ and a maximal ideal $\mathfrak{s}$ of $S$,
(ii) a normal, flat, projective $S$-scheme $P$ with geometrically integral generic fiber and an integral Weil divisor $D$ of $P$ contained in $P_{\mathfrak{s}}$,
(iii) a dominant morphism $\mathrm{Spec}\ R \to P$,
(iv) and a $P$-scheme $X_P$

such that the inverse image of $D$ equals $\mathfrak{m}_R$ and the base change of $X_P$ equals $X_R$.

**Lemma 2.2.** *If $X_R$ is a finite type $R$-scheme, there exists a finite type model such that $X_P$ is of finite type over $P$.*

*If $X_R$ is proper, resp. projective, there exists a finite type model such that $X_P$ is proper, resp. projective.*

*Proof.* This follows easily from [Gro63, §8], results about normalization, e.g., [Gro67, Scholie 7.8.3], and Nagata compactification, [Lüt93]. □

Let $B$ be a scheme. As in [Jou83], for each integer $e$ denote by $\mathrm{Gr}(e, N)$ the Grassmannian over $B$ parametrizing codimension-$e$ linear subspaces of fibers of $\mathbb{P}^N_B$. Denote by $\Lambda_e$ the universal codimension-$e$ linear subscheme of $\mathrm{Gr}(e, N) \times_B \mathbb{P}^N_B$.

**Notation 2.3.** For every $B$-scheme $T$ and $B$-morphism $i : T \to \mathbb{P}^N_B$, denote by $\mathcal{Z}_{i,e}$ the fiber product $T \times_{\mathbb{P}^N_B} \Lambda_e$. If $i$ is inclusion of a subscheme, denote this by $\mathcal{Z}_{T,e}$. Observe that for a $k$-morphism $j : U \to T$, $\mathcal{Z}_{i \circ j, e}$ equals $U \times_T \mathcal{Z}_{i,e}$. Denote by $\mathrm{pr}_G : \mathcal{Z}_{i,e} \to \mathrm{Gr}(e, N)$ and $\mathrm{pr}_T : \mathcal{Z}_{i,e} \to T$ the 2 projections.

Denote by $G_{i,e}$ the maximal open subscheme of $\mathrm{Gr}(e, N)$ such that $G_{i,e} \times_{\mathrm{Gr}(e,N)} \mathcal{Z}_{i,e} \to G_{i,e}$ is flat. Denote by $\mathcal{Z}_{i,e,G} \to G_{i,e}$ the base change of $\mathrm{pr}_G$ to $G_{i,e}$.

Let $k$ be a field and let $I$ be a geometrically irreducible, locally closed subscheme of $\mathbb{P}^N_k$. Denote by $d + 1$ the dimension of $I$. Denote $G_{I,d}$ by $G$ and denote $\mathcal{Z}_{I,d,G}$ by $\mathcal{Z}$. Denote by $\mathrm{pr}_G : \mathcal{Z} \to G$ and $\mathrm{pr}_I : \mathcal{Z} \to I$ the 2 projections.

Let $H$ be a $k$-scheme and let $f : H \to G$ be a morphism. Denote by $\mathcal{Z}_H$ the fiber product $H \times_G \mathcal{Z}$. Denote by $\mathrm{pr}_H : \mathcal{Z}_H \to H$ and $\mathrm{pr}_{\mathcal{Z}} : \mathcal{Z}_H \to \mathcal{Z}$ the 2 projections. Denote by $p : \mathcal{Z}_H \to I$ the composition $\mathrm{pr}_I \circ \mathrm{pr}_{\mathcal{Z}}$.

**Lemma 2.4.** *If the dimension of $I$ is positive, if $H$ is geometrically irreducible and if $f$ is dominant, then $\mathcal{Z}_H$ is geometrically irreducible, and the geometric generic fiber of $p$ is irreducible.*

*Proof.* By definition of $G$, $\mathcal{Z}$ is flat over $G$ of relative dimension 1. By [Jou83, Théorème I.6.10.2], the geometric generic fiber of $\mathcal{Z} \to G$ is irreducible. Since $f$ is dominant, $\mathcal{Z}_H \to H$ is flat of relative dimension 1 with irreducible geometric generic fiber. Since $H$ is also geometrically irreducible, $\mathcal{Z}_H$ is geometrically irreducible.

Because $f$ is dominant and $\mathrm{pr}_I$ are dominant, also $\mathrm{pr}_{\mathcal{Z}}$, and thus the composition $p$, is dominant. Thus, to prove the geometric generic fiber of $p$ is irreducible, it suffices to prove the geometric generic fiber of

$$\mathrm{pr}_1 : \mathcal{Z}_H \times_I \mathcal{Z}_H \to \mathcal{Z}_H$$

is irreducible.

The morphism $\mathrm{pr}_H \circ \mathrm{pr}_1 : \mathcal{Z}_H \times_I \mathcal{Z}_H \to H$ is the base-change of $\mathcal{Z}_{p,d} \to \mathrm{Gr}(d,N)$ by $H \to \mathrm{Gr}(d,N)$. The image of $p$ is dense in $I$, thus has dimension $d+1$. Thus, by [Jou83, Théorème I.6.10.2], the geometric generic fiber of $\mathrm{pr}_G : \mathcal{Z}_{p,d} \to \mathrm{Gr}(d,n)$ is irreducible. Since $H \to \mathrm{Gr}(d,N)$ is dominant, also the geometric generic fiber of $\mathrm{pr}_H \circ \mathrm{pr}_1$ is irreducible. Denote the geometric generic fiber by $X$.

Denote by $Y$ the geometric generic fiber of $\mathrm{pr}_H : \mathcal{Z}_H \to H$. By the argument above, $Y$ is also irreducible. The morphism $\mathrm{pr}_1$ induces a dominant morphism $q : X \to Y$. The geometric generic fiber of $p$ is irreducible if and only if the geometric generic fiber of $q$ is irreducible.

Denote by $\widetilde{q} : \widetilde{Y} \to Y_{\mathrm{red}}$ the integral closure of $Y_{\mathrm{red}}$ in the fraction field of $X_{\mathrm{red}}$. By the Noether normalization theorem, $\widetilde{q}$ is finite. And the geometric generic fiber of $X_{\mathrm{red}} \to \widetilde{Y}$ is integral. Thus the geometric generic fiber of $p$ is irreducible if and only if the geometric generic fiber of $\widetilde{q}$ is irreducible.

Because $X$ is irreducible, also $\widetilde{Y}$ is irreducible. The diagonal morphism $\Delta : \mathcal{Z}_H \to \mathcal{Z}_H \times_I \mathcal{Z}_H$ gives a section of $\mathrm{pr}_1$. This induces a section $s : Y \to X$ of $q$. This induces a section $\widetilde{s} : Y_{\mathrm{red}} \to \widetilde{Y}$ of $\widetilde{q}$.

Since $\widetilde{q}$ is finite, $\dim(\widetilde{Y})$ equals $\dim(Y_{\mathrm{red}})$. The image of $\widetilde{s}$ is an irreducible closed subscheme of $\widetilde{Y}$ whose dimension equals $\dim(Y_{\mathrm{red}})$, i.e., $\dim(\widetilde{Y})$. Therefore, the image is an irreducible component of $\widetilde{Y}$. Since $\widetilde{Y}$ is irreducible, the image of $\widetilde{s}$ is all of $\widetilde{Y}$. Thus $\widetilde{s}$ is an inverse of $\widetilde{q}$. Therefore the geometric generic fiber of $\widetilde{q}$ is a point, which is irreducible. □

*Proof of Theorem 1.5.* Let $R$ be a DVR whose residue field $k$ contains the algebraic closure of its prime subfield. Let $X_R$ be a proper $R$-scheme whose geometric generic fiber is separably rationally connected. By Lemma 2.2, there exists a finite type model with $X_P$ proper. By hypothesis, $k$ contains the algebraic closure of the residue field $\overline{\kappa(\mathfrak{s})}$.

If $P$ has relative dimension 0 over $S$, then $P$ equals $S$. Thus $X_k$ is the base change of the proper $\kappa(\mathfrak{s})$-scheme $X_{\mathfrak{s}}$. Since $\kappa(\mathfrak{s})$ is a finite extension of the prime subfield of $k$, the algebraic closure $\overline{\kappa(\mathfrak{s})}$ is contained in $k$. The base change $Y$ of any $\overline{\kappa(\mathfrak{s})}$-point of $X_{\mathfrak{s}}$ is a geometrically irreducible subvariety of $X_k$.

Thus, assume the relative dimension of $P$ over $S$ is positive, $d+1$. Let $P \hookrightarrow \mathbb{P}^N_S$ be a closed immersion. Denote by $G_{P,d}$ and $\mathcal{Z}_{P,d,G}$ the schemes over $S$ from Notation 2.3. By definition, the geometric generic fiber of $P$ over $S$ is normal. In particular it is smooth in codimension 1 points. Therefore, by [Jou83, Théorème I.6.10.2], the geometric generic fiber of $\mathcal{Z}_{P,d,G} \to G_{P,d}$ is smooth.

By [Art69, Theorem 6.1], there exists an algebraic space $\Pi$ separated and locally of finite type over $G_{P,d}$, and a universal morphism

$$\sigma : \Pi \times_{G_{P,d}} \mathcal{Z}_{P,d,G} \to X_P$$

whose composition with projection to $P$ equals the composition $\mathrm{pr}_P \circ \mathrm{pr}_{\mathcal{Z}_{P,d,G}}$. Since the geometric generic fiber of $X_R$ is separably rationally connected, also the geometric generic fiber of $X_P \to P$ is separably rationally connected. Therefore, by [dJS03], the geometric generic fiber of $\Pi \to G_{P,d}$ is nonempty. Therefore some irreducible component $\Pi_i$ of $\Pi$ dominates $G_{P,d}$.

By [Knu71, Corollary 5.20], there exists a dense open subspace $U$ of $\Pi_i$ which is an affine scheme. There exists a dense open immersion of $U$ in a projective $G_{P,d}$-scheme $\overline{U}$. The morphism $\sigma$ induces a rational transformation

$$\sigma_U : \overline{U} \times_{G_{P,d}} \mathcal{Z}_{P,d,G} \dashrightarrow X_P.$$

Denote by $\overline{V}$ the normalization of $\overline{U} \times_{G_{P,d}} \mathcal{Z}_{P,d,G}$. Denote by $V$ the maximal open subscheme of $\overline{V}$ over which $\sigma_U$ extends to a regular morphism. By the valuative criterion of properness, the complement of $V$ has codimension 2. In particular, some irreducible component of $V_{\mathfrak{s}}$ dominates $(G_{P,d})_{\mathfrak{s}}$.

Since $k$ contains $\overline{\kappa(\mathfrak{s})}$ and the function field $\kappa(D)$ of $D$, it contains the function field $\kappa(I)$ of one of the irreducible components $I$ of $D \otimes_{\kappa(\mathfrak{s})} \overline{\kappa(\mathfrak{s})}$. Let $H$ denote an irreducible component of $V_{\mathfrak{s}} \otimes_{\kappa(\mathfrak{s})} \overline{\kappa(\mathfrak{s})}$ dominating $(G_{P,d})_{\mathfrak{s}} \otimes_{\kappa(\mathfrak{s})} \overline{\kappa(\mathfrak{s})}$. Let $G$, $\mathcal{Z}$, etc., be as in Lemma 2.4. Denote by $\mathcal{Z}_k$ the base change $\mathcal{Z} \otimes_{\mathcal{O}_I} k$. Note that the base change $X_I \otimes_{\mathcal{O}_I} k$ equals $X_k$ by definition of a finite type model.

The transformation $\sigma_U$ determines a morphism

$$\sigma_H : \mathcal{Z}_H \to X_I.$$

The base change to $k$ is a morphism of $k$-schemes,

$$\sigma_k : \mathcal{Z}_k \to X_k.$$

Denote the closure of the image by $Y$. By Lemma 2.4, $\mathcal{Z}_H$ is geometrically irreducible and the geometric generic fiber of $\mathcal{Z}_H \to I$ is irreducible. Thus $\mathcal{Z}_k \otimes_k \overline{k}$ is irreducible. Therefore $Y$ is a closed subscheme of $X_k$ such that $Y \otimes_k \overline{k}$ is irreducible. $\square$

*Proof of Theorem 1.1.* If $k$ is a perfect PAC field, then every geometrically irreducible $k$-scheme has a $k$-point. In particular, $Y$ has a $k$-point. Therefore $X_k$ has a $k$-point. $\square$

Every field $k$ is the closed fiber of a DVR $R$ whose generic fiber has characteristic 0. Every complete intersection in $\mathbb{P}^n_k$ is the closed fiber of a complete intersection in $\mathbb{P}^n_R$ whose generic fiber is smooth. If the complete intersection satisfies the $C_1$ inequality, the generic fiber is a Fano manifold. By [KMM92], [Cam92], a Fano manifold in characteristic 0 is rationally connected. Therefore Theorem 1.1 implies the complete intersection in $\mathbb{P}^n_k$ has a $k$-point if $k$ is a perfect PAC field containing the algebraic closure of its prime subfield. In other words, every perfect PAC field containing an algebraically closed field is $C_1$, cf. [FJ05, Theorem 21.3.6(a)].

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