Brauer groups and Galois cohomology of function fields of varieties

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Contents

1. Acknowledgments

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2. Introduction

These are the lecture notes for a minicourse presented in the XX Escola de Algebra at the Instituto Nacional de Matemática Pura e Aplicada in Rio de Janeiro, Brazil, in August of 2008. The minicourse begins with an introduction to Brauer groups and Galois cohomology followed by the proofs of the Chevalley-Warning and Tsen-Lang theorems and their applications to Brauer groups and Galois cohomology. The minicourse concludes with some more recent theorems about Brauer groups and Galois cohomology of function fields of varieties over an algebraically closed field: de Jong's Period-Index theorem and the split case of Serre's "Conjecture II" over function fields of surfaces due to de Jong, He and myself and following a strategy proposed by Philippe Gille. The split case completes the proof of the general case of Serre's "Conjecture II" over function fields of surfaces following the work of Merkurjev, Suslin, Bayer-Fluckiger, Parimala, Colliot-Thélène, and Gille.

In addition to introducing students and post-docs to the classical subjects of Brauer groups and Galois cohomology, a second goal of the minicourse is to show by example how results from geometry – specifically the geometry of rationally connected varieties and the geometry of spaces of curves – can be used to prove results in algebra – namely the Period-Index theorem and the split case of Serre's "Conjecture II".

Here is a short description of the contents of the minicourse.

Chapter 1: Brauer groups and Galois cohomology. Definition of division algebras and central simple algebras. Definition of the Brauer group. Short sketch of Galois cohomology and its relation to the Brauer group.

Chapter 2: The Chevalley-Warning and Tsen-Lang theorems. The statement and proofs of Chevalley's theorem and Tsen's theorem which prove that the Brauer group of a finite field, resp. the function field of a curve over an algebraically closed field, is trivial. Statement of the generalized results of Chevalley-Warning and Tsen-Lang with applications to Brauer groups and Galois cohomology.

Chapter 3: Rationally connected fibrations. A sketch of rationally connected varieties. The statement of the conjecture of Kollár-Miyaoka-Mori that every rationally connected fibration over a curve has a section. A sketch of the proof of this conjecture by Graber, Harris and myself.

Chapter 4: The Period-Index theorem of de Jong Explanation of de Jong's Period-Index theorem: for every division algebra over the function field of a surface, the period equals the index. Sketch of a second proof, by de Jong and myself, using rational simple connectedness. Comparison to a third proof by Lieblich, which also proves generalizations of de Jong's theorem.

Chapter 5: Rational simple connectedness and Serre's "Conjecture II" Description of known results about Serre's "Conjecture II": theorems of Merkurjev and Suslin, Bayer-Fluckiger and Parimala, Colliot-Thélène, Gille and Parimala, and Gille. Sketch of the proof of the split case of Serre's "Conjecture II" over function fields of surfaces using rational simple connectedness.

CHAPTER 1

Brauer groups and Galois cohomology

1. Abelian Galois cohomology

The first lecture gives a quick survey of some results in Galois cohomology and Brauer groups. Most results are stated without proof or with only an outline of a proof. The main references for this lecture are [**Ser02**] and [**Ser79**]. For results about homological algebra, the reference is [**Wei94**].

Let L/K be a Galois extension of fields, e.g., a separable closure K^s of K. The Galois group $\mathrm{Gal}(L/K)$ is a profinite group, namely the inverse limit of the Galois groups $\mathrm{Gal}(L_i/K)$ of all *finite* Galois subextensions L_i, i.e., $K \subset L_i \subset L$.

Definition 1.1. An *Abelian Gal*(L/K)*-module* is an Abelian group A together with a left action of $\mathrm{Gal}(L/K)$ by group homomorphisms. The module is *discrete* if every element of A is stabilized by $\mathrm{Gal}(L_i/K)$ for some finite Galois subextension L_i/K, or equivalently, if the canonical map of Abelian $\mathrm{Gal}(L/K)$-modules

$$A \to \lim_{K\subset L_i \subset L} A_i, \quad A_i := A^{\mathrm{Gal}(L/L_i)},$$

is a bijection.

The Abelian category of all Abelian $\mathrm{Gal}(L/K)$-modules is denoted $\mathrm{Gal}(L/K)-$ mod. And the full Abelian subcategory of discrete Abelian $\mathrm{Gal}(L/K)$-modules is denoted $\mathrm{Gal}(L/K) - \mathrm{mod}_{\mathrm{discrete}}$.

Let L_i/K be a finite Galois subextension of L/K. There is a left-exact additive functor

$$H^0(\mathrm{Gal}(L_i/K), -) : \mathrm{Gal}(L_i/K) - \mathrm{mod} \to \mathbb{Z} - \mathrm{mod},$$
$$B \mapsto B^{\mathrm{Gal}(L_i/K)},$$

the functor of $\mathrm{Gal}(L_i/K)$-invariants.

Definition 1.2. The sequence of right derived functors of $H^0(\mathrm{Gal}(L_i/K), -)$ considered as a cohomological δ-functor is the *Galois cohomology* or *Gal*(L_i/K)*-cohomology*,

$$H^p(\mathrm{Gal}(L_i/K), -) : \mathrm{Gal}(L_i/K) - \mathrm{mod} \to \mathbb{Z} - \mathrm{mod},$$
$$H^p(\mathrm{Gal}(L_i/K), B) := R^p H^0(\mathrm{Gal}(L_i/K), B).$$

For a discrete Abelian $\mathrm{Gal}(L/K)$-module A, for every integer $p \geq 0$ the sequence of group cohomologies $(H^p(\mathrm{Gal}(L_i/K), A_i))_{L_i}$ form a directed system of Abelian groups. Similarly, for every exact sequence of discrete Abelian $\mathrm{Gal}(L/K)$-modules,

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0,$$

the collection of connecting maps

$$\delta^p_{L_i} : H^p(\mathrm{Gal}(L_i/K), A''_i) \to H^{p+1}(\mathrm{Gal}(L_i/K), A'_i)$$

is a morphism of compatible systems.

DEFINITION 1.3. The *Galois cohomology* or *Gal*(L/K)*-cohomology* is the cohomological δ-functor from $\mathrm{Gal}(L/K)-\mathrm{mod}_{\mathrm{discrete}}$ to $\mathbb{Z}-\mathrm{mod}$ defined by the colimits

$$H^p(\mathrm{Gal}(L/K), -) : \mathrm{Gal}(L/K) - \mathrm{mod}_{\mathrm{discrete}} \to \mathbb{Z} - \mathrm{mod},$$

$$H^p(\mathrm{Gal}(L/K), A) := \lim_{L_i} H^p(\mathrm{Gal}(L_i/K), A_i),$$

and

$$\delta^p : H^p(\mathrm{Gal}(L/K), A'') \to H^{p+1}(\mathrm{Gal}(L/K), A'), \quad \delta_p = \lim_{L_i} \delta^p_{L_i}.$$

This is a universal δ-functor. In fact the category $\mathrm{Gal}(L/K) - \mathrm{mod}_{\mathrm{discrete}}$ has enough injective objects, and one can prove that the usual sequence of right derived functors of $A^{\mathrm{Gal}(L/K)}$ is canonically isomorphic as a δ-functor to the one from Definition 1.3. Definition 1.3 reflects one technique of studying $\mathrm{Gal}(L/K)$-cohomology: reduction to the case of finite Galois extensions L_i/K.

For the reader who knows of such things, $\mathrm{Gal}(L/K)$-cohomology is nothing other than continuous group cohomology for the group $\mathrm{Gal}(L/K)$ with its profinite topology. This raises the question, how is the study of Galois cohomology different from the study of profinite group cohomology? The answer has to do mainly with the particular Galois modules of interest.

1.1. Low degree Galois cohomology. The low degree Galois cohomology groups have interpretations of special interest.

1.1.1. *The zeroth group.* Let L/K be a Galois extension and let A be a discrete $\mathrm{Gal}(L/K)$-module.

PROPOSITION 1.4. *There is a canonical bijection between* $H^0(Gal(L/K), A)$ *and the normal subgroup* $A^{Gal(L/K)}$ *of elements of* A *left invariant under* $Gal(L/K)$.

PROOF. By definition, for every finite, Galois subextension $K \subset L_i \subset L$

$$H^0(\mathrm{Gal}(L_i/K), A_i) = A_i^{\mathrm{Gal}(L_i/K)}.$$

Since A_i is defined to be $A^{\mathrm{Gal}(L/L_i)}$, this gives

$$H^0(\mathrm{Gal}(L_i/K), A_i) = A^{\mathrm{Gal}(L/K)}.$$

Thus also

$$H^0(\mathrm{Gal}(L/K), A) = A^{\mathrm{Gal}(L/K)},$$

i.e., $H^0(\mathrm{Gal}(L/K), A)$ equals the subgroup of elements of A left invariant under $\mathrm{Gal}(L/K)$. □

1.1.2. *The first group, Interpretation I.*. Let L/K be a Galois extension and let A be a discrete, Abelian $\mathrm{Gal}(L/K)$-module. A *crossed homomorphism* from $\mathrm{Gal}(L/K)$ into A is a set map

$$a : \mathrm{Gal}(L/K) \to A$$

such that for every $g, h \in \mathrm{Gal}(L/K)$,

$$a_{g \cdot h} = a_g \cdot {}^g a_h.$$

The crossed homomorphism is *continuous* if it factors through $\mathrm{Gal}(L_i/K)$ for some finite subextension $K \subset L_i \subset L$.

There is a distinguished continuous crossed homomorphism e, namely

$$e_g = e,$$

the identity element of A for every g in $\mathrm{Gal}(L_i/K)$. The set of continous crossed homomorphisms from $\mathrm{Gal}(L/K)$ to A is denoted $Z^1_{\mathrm{cts.}}(\mathrm{Gal}(L/K), A)$. There is a left action of A on $Z^1_{\mathrm{cts.}}(\mathrm{Gal}(L/K), A)$ associating to a crossed homomorphism a and an element b of A, the new crossed homomorphism a^b,

$$a^b_g := b^{-1} \cdot a_g \cdot {}^g b.$$

Proposition 1.5. *The standard description of group cohomology establishes a bijection between the set $Z^1_{cts.}(Gal(L/K), A)/A$ of A-orbits and the set $H^1(Gal(L/K), A)$. This bijection identifies the crossed homomorphism e with the additive identity of $H^1(Gal(L/K), A)$.*

Proof. It suffices to consider the case when $L = L_i$ is a finite Galois extension of K. The "standard description" of group cohomology arises from the standard resolution (or *unnormalized bar resolution* of the trivial (left) $\mathbb{Z}[\mathrm{Gal}(L_i/K)]$-module by free $\mathbb{Z}[\mathrm{Gal}(L_i/K)]$-modules,

$$\ldots \to \mathbb{Z}[\mathrm{Gal}(L_i/K)](\mathrm{Gal}(L_i/K)^p) \xrightarrow{d_p} \cdots \to \mathbb{Z}[\mathrm{Gal}(L_i/K)](\mathrm{Gal}(L_i/K)\times\mathrm{Gal}(L_i/K))$$

$$\xrightarrow{d_2} \mathbb{Z}[\mathrm{Gal}(L_i/K)](\mathrm{Gal}(L_i/K)) \xrightarrow{d_1} \mathbb{Z}[\mathrm{Gal}(L_i/K)] \to \mathbb{Z} \to 0,$$

$$d_p(\mathbf{e}_{(g_1,\ldots,g_p)}) = g_1\mathbf{e}_{g_2,\ldots,g_p} + \sum_{i=1}^{p-1}(-1)^p \mathbf{e}_{g_1,\ldots,g_{i-1},g_i\cdot g_{i+1},g_{i+2},\ldots,g_p} + (-1)^p\mathbf{e}_{g_1,\ldots,g_{p-1}}.$$

Applying $\mathrm{Hom}_{\mathbb{Z}[\mathrm{Gal}(L_i/K)]-\mathrm{mod}}(-, B)$ to this chain complex gives a cochain complex computing

$$H^p(\mathrm{Gal}(L_i/K), A) = \mathrm{Ext}^p_{\mathbb{Z}[\mathrm{Gal}(L_i/K)]-\mathrm{mod}}(\mathbb{Z}, A).$$

Writing out the codifferential dual to the differential above gives the "standard description" of group cohomology in terms of cocycles and coboundaries. The group of 1-cocycles is precisely the group $Z^1(\mathrm{Gal}(L_i/K), A)$ of crossed homomorphisms with e being the additive identity. And two crossed homomorphisms are cohomologous if and only if they are equivalent. □

1.1.3. *The first group, Interpretation II.*. There is a second interpretation of $H^1(\mathrm{Gal}(L/K), A)$ in terms of discrete $\mathrm{Gal}(L/K)$-equivariant A-torsors.

Definition 1.6. A right *A-torsor* is a nonempty set E together with a right action of A which is free and transitive. A *$Gal(L/K)$-equivariant* A-torsor is a right A-torsor together with a compatible left action of $\mathrm{Gal}(L/K)$ on E, i.e., for every g in $\mathrm{Gal}(L/K)$, for every a in A and for every m in E,

$${}^g(m \cdot a) = ({}^g m) \cdot ({}^g a).$$

The $\mathrm{Gal}(L/K)$-equivariant A-torsor is *discrete* if every element of E is stabilized by $\mathrm{Gal}(L/L_i)$ for some finite Galois subextension L_i/K. An *isomorphism* of $\mathrm{Gal}(L_i/K)$-equivariant A-torsors is a bijection which is simultaneously left $\mathrm{Gal}(L_i/K)$-equivariant and right A-equivariant.

Let E be a $\mathrm{Gal}(L_i/K)$-equivariant A-torsor. For every element m of E, there is a bijection of right A-torsors

$$R_m : A \to E, \quad R_m(a) = m \cdot a.$$

This is left $\mathrm{Gal}(L_i/K)$-equivariant if and only if m is $\mathrm{Gal}(L_i/K)$-invariant. The failure of m to be $\mathrm{Gal}(L_i/K)$-invariant is measured by the map

$$a : \mathrm{Gal}(L_i/K) \to A, \quad \forall g, {}^g m = m \cdot a_g.$$

It is straightforward to verify a is a crossed homomorphism which is continuous if E is discrete. Moreover, replacing m by $m \cdot b^{-1}$ replaces the crossed homomorphism a by a^b. Thus the B-orbit of a is independent of the choice of m.

Proposition 1.7. *Let L/K be a Galois extension and let A be a discrete $Gal(L/K)$-module. The rule associating to every discrete $Gal(L/K)$-equivariant A-torsor E the A-orbit of a in $Z^1_{cts.}(Gal(L/K), A)/A$ determines a bijection between the set of isomorphism classes of discrete $Gal(L/K)$-equivariant A-torsors and $Z^1_{cts.}(Gal(L/K), A)/A$.*

Proof. For every pair (E_1, m_1) and (E_2, m_2), there is a unique isomorphism of A-torsors

$$u : E_1 \to E_2$$

such that $u(m_1)$ equals $u(m_2)$. This isomorphism is $\mathrm{Gal}(L_i/K)$-equivariant if and only if the crossed homomorphisms associated to (E_1, m_1) and (E_2, m_2) are equal.

Moreover, for every crossed homomorphism a, there is a pair (E, m) giving rise to a defined as follows. Define (E, m) to equal (A, e) as a right A-torsor. But endow this with a new $\mathrm{Gal}(L/K)$-action

$$g *_a b := a_g \cdot {}^g b.$$

If A is discrete and a is continuous, the new action is also discrete. And the crossed homomorphism associated to (E, m) is precisely a.

This shows the rule associating to (E, m) the continuous crossed homomorphism a gives a bijection between the set of isomorphism classes of pairs (E, m) and the set $Z^1_{\mathrm{cts.}}(\mathrm{Gal}(L/K), A)$. Since changing m to $m \cdot b^{-1}$ changes a to a^b this rule also gives a canonical bijection between the set of isomorphism classes of discrete $\mathrm{Gal}(L/K)$-equivariant A-torsors and the set $Z^1_{\mathrm{cts.}}(\mathrm{Gal}(L/K), A)/A$. $\square$

1.1.4. *The second group.* Let L_i/K be a finite Galois extension and let A be an Abelian $\mathrm{Gal}(L_i/K)$-module. An *extension* of $\mathrm{Gal}(L_i/K)$ by A is an exact sequence of groups

$$0 \longrightarrow A \longrightarrow E \longrightarrow \mathrm{Gal}(L_i/K) \longrightarrow 1,$$

i.e., a group E together with an isomorphism of A with a normal subgroup of E and an isomorphism of $\mathrm{Gal}(L_i/K)$ with the quotient group E/A such that the induced conjugation action of E on A factors through the given action of $\mathrm{Gal}(L_i/K)$ on A. An isomorphism of extensions is a commutative diagram of short exact sequences which is the identity on A and $\mathrm{Gal}(L_i/K)$.

Proposition 1.8. *There is a canonical bijection between the set of equivalence classes of extensions of $Gal(L_i/K)$ by A and $H^2(Gal(L_i/K), A)$. The bijection associates the additive identity in $H^2(Gal(L_i/K), A)$ to the equivalence class of the semidirect product $A \rtimes Gal(L_i/K)$.*

This is best checked by using the standard resolution to describe $H^2(\mathrm{Gal}(L_i/K), A)$ in terms of cocycles, cf. [**Wei94**, Theorem 6.6.3, p. 183].

1.1.5. *Cohomological dimension.* Let L/K be a Galois extension. Let p be a prime number and let n be a nonnegative integer.

DEFINITION 1.9. The *p-cohomological dimension of Gal*(L/K), $\mathrm{cd}_p(L/K)$, is the smallest nonnegative integer n such that for every torsion, discrete, Abelian $\mathrm{Gal}(L/K)$-module A, the p-primary component of $H^q(\mathrm{Gal}(L/K), A)$ is $\{0\}$ for all $q > n$. If there is no such integer n, then $\mathrm{cd}_p(L/K)$ is defined to be ∞. The *cohomological dimension of Gal*(L/K), $\mathrm{cd}(L/K)$, is defined to be $\sup\{\mathrm{cd}_p(L/K) | p \text{ prime }\}$. When $L = K^s$, a separable closure of K, these are denoted simply by $\mathrm{cd}_p(K)$ and $\mathrm{cd}(K)$.

2. Non-Abelian Galois cohomology and the long exact sequence

Let L/K be a Galois extension.

DEFINITION 2.1. A *Gal*(L/K)*-module* is a (not necessarily Abelian) group A together with a left action of $\mathrm{Gal}(L/K)$ acting by group homomorphisms. The module is *discrete* if every element of A is stabilized by $\mathrm{Gal}(L/L_i)$ for some finite Galois subextension subextension L_i/K.

The category of $\mathrm{Gal}(L/K)$-modules is not Abelian, thus one cannot define Galois cohomology in the same way as above. There is a theory of non-Abelian cohomology, developed in [**Gir71**] for instance. However, for H^0 and H^1 it is simpler to define $H^0(\mathrm{Gal}(L/K), A)$ and $H^1(\mathrm{Gal}(L/K), A)$ so that Propositions 1.4 and 1.5 continue to hold.

DEFINITION 2.2. For every discrete $\mathrm{Gal}(L/K)$-module A, the *zeroth Galois cohomology* is the subgroup

$$H^0(\mathrm{Gal}(L/K), A) := A^{\mathrm{Gal}(L/K)},$$

of elements of A left invariant under $\mathrm{Gal}(L/K)$. And the *first Galois cohomology* is the set of right orbits,

$$H^1(\mathrm{Gal}(L/K), A) := Z^1(\mathrm{Gal}(L/K), A)/A.$$

Let $u : A \to B$ be a homomorphism of discrete $\mathrm{Gal}(L/K)$-modules. Since u is $\mathrm{Gal}(L/K)$-equivariant, u maps $A^{\mathrm{Gal}(L/K)}$ to $B^{\mathrm{Gal}(L/K)}$. And for every crossed homomorphism

$$a : \mathrm{Gal}(L/K) \to A,$$

the composition

$$u \circ a : \mathrm{Gal}(L/K) \to B$$

is a crossed homomorphism. Moreover, $u \circ (a^b)$ equals $(u \circ a)^{u(b)}$. Thus $u \circ a$ maps A-orbits into B-orbits. And $u \circ e_A$ equals e_B, so u preserves the distinguished points.

DEFINITION 2.3. For every homomorphism $u : A \to B$ of discrete $\mathrm{Gal}(L/K)$-modules, the group homomorphism

$$H^0(\mathrm{Gal}(L/K), u) : H^0(\mathrm{Gal}(L/K), A) \to H^0(\mathrm{Gal}(L/K), B)$$

and the map of pointed sets

$$H^1(\mathrm{Gal}(L/K), u) : H^1(\mathrm{Gal}(L/K), A) \to H^1(\mathrm{Gal}(L/K), B)$$

are as defined above.

These maps complete the definition of the functors,

$$H^0(\mathrm{Gal}(L/K), -) : \text{non-Abelian}\mathrm{Gal}(L/K) - \mathrm{mod}_{\mathrm{discrete}} \to \text{Groups},$$
$$H^1(\mathrm{Gal}(L/K), -) : \text{non-Abelian}\mathrm{Gal}(L/K) - \mathrm{mod}_{\mathrm{discrete}} \to \text{Pointed Sets}.$$

Remark 2.4. **(i)** Note that the proof of Proposition 1.7 is still valid. Thus there is a canonical bijection between $H^1(\mathrm{Gal}(L/K), A)$ and the pointed set of equivalence classes of discrete $\mathrm{Gal}(L/K)$-equivariant A-torsors. **(ii)** Note also that for a product $\mathrm{Gal}(L/K)$-module, $B = A \times C$, the functorialities of H^0 and H^1 establish and isomorphism

$$H^0(\mathrm{Gal}(L/K), A \times C) \cong H^0(\mathrm{Gal}(L/K), A) \times H^0(\mathrm{Gal}(L/K), C)$$

and a bijection of pointed sets,

$$H^0(\mathrm{Gal}(L/K), A \times C) \cong H^0(\mathrm{Gal}(L/K), A) \times H^0(\mathrm{Gal}(L/K), C).$$

2.1. The long exact sequence. Notice that $H^1(\mathrm{Gal}(L/K), A)$ typically has no natural structure of group, although it does have the distinguished element $[e]$. Thus it may seem meaningless to ask about a long exact sequence of non-Abelian Galois cohomology associated to a short exact sequence of groups. However the situation is not as bad as it might at first seem.

2.1.1. *Step I. Twisting.* The first ingredient in describing the "long exact sequence" is the notion of twisting. Let E be a $\mathrm{Gal}(L/K)$-equivariant A-torsor and let F be a set with a left $\mathrm{Gal}(L/K)$-action and a $\mathrm{Gal}(L/K)$-equivariant right action of A, i.e.,

$$^g(f \cdot a) = {}^g f \cdot {}^g a$$

for every g in $\mathrm{Gal}(L/K)$, for every a in A and for every f in F. There is a left $\mathrm{Gal}(L/K)$-action on $E \times F$ by $^g(e, f) := ({}^g e, {}^g f)$. And there is a $\mathrm{Gal}(L/K)$-equivariant right action of A on $E \times F$ by $(e, f) \cdot a := (e \cdot a, f \cdot a)$. The set of right A-orbits

$${}_E F := (E \times F)/A$$

has a well-defined left action of $\mathrm{Gal}(L/K)$ such that $[{}^g(e, f)]$ equals ${}^g[(e, f)]$.

Definition 2.5. The $\mathrm{Gal}(L/K)$-set ${}_E F$ is the *twist* of F by E. If E and F are both discrete, ${}_E F$ is also discrete.

Twisting is functorial in both E and F. Since pairs (E, m) are unique up to unique isomorphism, for every element a in $Z^1_{\mathrm{cts.}}(\mathrm{Gal}(L/F), A)$, the twist ${}_a F$ is also well-defined up to unique isomorphism. But since two representatives a and a' of the same A-orbit in $Z^1_{\mathrm{cts.}}(\mathrm{Gal}(L, F), A)$ may be conjugate under many elements of A, there is no canonical isomorphism of ${}_a F$ and ${}_{a'} F$.

If A is non-Abelian then typically ${}_E F$ has no natural structure of A-set. However, if the action of A on F preserves some "structure", then the twist ${}_E F$ typically also has this "structure". Here are two examples of this.

Example 2.6 (Translation by F). Let A and B be $\mathrm{Gal}(L/K)$-modules and let F be a $\mathrm{Gal}(L/K)$-set that has commuting $\mathrm{Gal}(L/K)$-equivariant right actions of both A and B. Then for every $\mathrm{Gal}(L/K)$-equivariant A-torsor E, ${}_E F$ also has a $\mathrm{Gal}(L/K)$-equivariant right B-action. In particular, if F is a B-torsor, then also ${}_E F$ is a B-torsor. Thus, associated to every $\mathrm{Gal}(L/K)$-equivariant B-torsor F with a commuting $\mathrm{Gal}(L/K)$-equivariant right A-action, there is an induced set map

$$*F : H^1(\mathrm{Gal}(L/K), A) \to H^1(\mathrm{Gal}(L/K), B), \quad [E] \mapsto [{}_E F].$$

This maps sends the trivial A-torsor to F, and thus can be considered a "translation by F".

The canonical example of this comes from a homomorphism $u : A \to B$ of $\mathrm{Gal}(L/K)$-modules. There is a right A-action on B via $b \cdot a := bu(a)$. And there is a commuting "right" B-action on B via $b \cdot \beta := \beta^{-1} b$. Thus, as above, there is a set map

$$u_* : H^1(\mathrm{Gal}(L/K), A) \to H^1(\mathrm{Gal}(L/K), B), \quad [E] \mapsto [{}_E B].$$

Via Proposition 1.7, the map u_* is precisely the same as the map $H^1(\mathrm{Gal}(L/K), u)$.

A second example comes from the case that A is Abelian. Let B be A. For every A-torsor F, let the B-torsor structure be precisely the same as the given A-torsor structure. The "two" actions commute because multiplication in A is commutative. And the induced map $*F$ literally is translation by $[F]$ in the Abelian group $H^1(\mathrm{Gal}(L/K), A)$.

If F is itself a $\mathrm{Gal}(L/K)$-module and if A acts by group homomorphisms, then ${}_E F$ is also a group, i.e., a $\mathrm{Gal}(L/K)$-module.

Example 2.7 (Inner twists). For instance, A acts on itself by inner automorphisms. Thus associated to every discrete $\mathrm{Gal}(L/K)$-equivariant A-torsor E, there is an associated $\mathrm{Gal}(L/K)$-module ${}_E A$,

$${}_E A := (E \times A)/\sim, \quad (m, a) \sim (m\gamma, \gamma^{-1} a \gamma).$$

To make this more explicit, observe there is a well-defined composition,

$$* : (E \times A) \times (E \times A) \to E \times A,$$

$$(m_1, a_1) * (m_2, a_2) = (m_1 \cdot (a_1 b), b^{-1} a_1 b_1 a_2), \quad m_2 = m_1 \cdot b.$$

Observe that

$$(m_1 \cdot \alpha, \alpha^{-1} a_1 \alpha) * (m_2 \cdot \beta, \beta^{-1} a_2 \beta) = ((m_1 \cdot (a_1 b)) \cdot \beta, \beta^{-1}(b^{-1} a_1 b a_2)\beta).$$

Thus the composition gives a well-defined composition on ${}_E A$. The identity element is the class of $E \times \{e_A\}$. And the inverse of the class (m, a) is $(m \cdot a^{-1}, a)$. It is important to note that the composition above gives a well-defined *left* action of ${}_E A$ on $E = E \times \{e_A\}$ by

$$[(m_1, a_1)] * m_2 := m_1 a_1 b, \quad m_2 = m_1 b.$$

Since ${}_E A$ is a group, we can change this into a "right" action by

$$m_2 * [(m_1, a_1)] := [(m_1 a_1^{-1}, a_1^{-1})] * m_2 = m_1 a^{-1} b.$$

With respect to this action, E is both a $\mathrm{Gal}(L/K)$-equivariant A-torsor and a $\mathrm{Gal}(L/K)$-equivariant ${}_E A$-torsor. Thus, by the mechanism in Example 2.6, there are well-defined set maps

$$*E : H^1(\mathrm{Gal}(L/K), A) \to H^1(\mathrm{Gal}(L/K), {}_E A),$$

$$*E : H^1(\mathrm{Gal}(L/K), {}_E A) \to H^1(\mathrm{Gal}(L/K), A).$$

Each of these maps sends the trivial torsor to E. Observe that if A is Abelian, then ${}_E A$ equals A and the map $*E$ agrees with the map from Example 2.6, i.e., it is simply translation by $[E]$ in the Abelian group $H^1(\mathrm{Gal}(L/K), A)$.

Proposition 2.8. *The maps*

$$*E : H^1(Gal(L/K), A) \to H^1(Gal(L/K), {}_EA),$$
$$*E : H^1(Gal(L/K), {}_EA) \to H^1(Gal(L/K), A).$$

are both bijections.

Proof. Let F be a $\mathrm{Gal}(L/K)$-equivariant A-torsor. Denote by I the set

$$I = \mathrm{Isom}_A(E, F) = \{\phi : E \to F | \forall e \in E, \forall a \in A, \phi(e \cdot a) = \phi(e) \cdot a\}.$$

Make this into a $\mathrm{Gal}(L/K)$-set by defining,

$$({}^g\phi)(e) := {}^g(\phi({}^{g^{-1}}e)), \text{ i.e., } ({}^g\phi)({}^ge) = {}^g(\phi(e)).$$

Make this into a right ${}_EA$-set by defining

$$(\phi \cdot \alpha)(e) := \phi(e \cdot \alpha^{-1}), \text{ i.e., } (\phi \cdot \alpha)(e \cdot \alpha) = \phi(e).$$

Putting these together,

$$({}^g(\phi \cdot \alpha))({}^g(e \cdot \alpha)) = \phi(e) = ({}^g\phi \cdot {}^g\alpha)({}^ge \cdot {}^g\alpha).$$

Since ${}^g(e \cdot \alpha)$ equals ${}^ge \cdot {}^g\alpha$, it follows that ${}^g(\phi \cdot \alpha)$ equals ${}^g\phi \cdot {}^g\alpha$, i.e., the ${}_EA$-action is $\mathrm{Gal}(L/K)$-equivariant.

On the underlying level of groups and group-sets ignoring the $\mathrm{Gal}(L/K)$-actions, E equals A as A-sets, ${}_EA$ equals A as groups, and F equals A as A-sets. By considering this case, it is clear that the right ${}_EA$-action on I makes I into an ${}_EA$-torsor. Therefore I is a $\mathrm{Gal}(L/K)$-equivariant ${}_EA$-torsor.

There is a canonical map

$$c : I \times E \to F, \quad (\phi, e) \mapsto \phi(e).$$

This map is ${}_EA$-invariant for the diagonal right action of ${}_EA$ on $I \times E$. Thus it factors through a map

$$c' : {}_IE \to F.$$

Since c is $\mathrm{Gal}(L/K)$-equivariant and A-equivariant for the right A-action on E, c' is a morphism of $\mathrm{Gal}(L/K)$-equivariant A-torsors. Thus it is an isomorphism. So $*E$ sends I to F.

Conversely, for every $\mathrm{Gal}(L/K)$-equivariant ${}_EA$-torsor J and for every morphism

$$d' : {}_JE \to F$$

of $\mathrm{Gal}(L/K)$-equivariant A-torsors, the composition

$$d : J \times E \twoheadrightarrow {}_JE \xrightarrow{d'} F$$

is A-equivariant for the right A-action on E, and thus induces a set map

$$d'' : J \to I.$$

Since d is $\mathrm{Gal}(L/K)$-equivariant and ${}_EA$-equivariant, so is d''. Thus d'' is a map of $\mathrm{Gal}(L/K)$-equivariant ${}_EA$-torsors, which is automatically an isomorphism. Therefore, up to isomorphism, I is the unique $\mathrm{Gal}(L/K)$-equivariant ${}_EA$-torsor such that ${}_IE$ is isomorphic to F, i.e., the map

$$*E : H^1(\mathrm{Gal}(L/K), {}_EA) \to H^1(\mathrm{Gal}(L/K), A)$$

is bijective. The argument that the other map $*E$ is bijective is similar. $\square$

Let $u : A \to B$ be a homomorphism of $\mathrm{Gal}(L/K)$-modules and let E be a discrete $\mathrm{Gal}(L/K)$-equivariant A-torsor. There is a well-defined homomorphism of $\mathrm{Gal}(L/K)$-modules,

$$_E u : {}_E A \to {}_{u_* E} B$$

associated to the map

$$(E \times A) \to (E \times B), \quad (m, a) \mapsto (m, b).$$

The proof of the following is a straightforward exercise for the reader.

LEMMA 2.9. *With respect to this map, both the diagram*

$$\begin{array}{ccc} H^1(Gal(L/K), A) & \xrightarrow{u_*} & H^1(Gal(L/K), B) \\ {\scriptstyle *E}\downarrow & & \downarrow{\scriptstyle *u_*E} \\ H^1(Gal(L/K), {}_E A) & \xrightarrow{{}_E u_*} & H^1(Gal(L/K), {}_{u_*E} B) \end{array} \tag{1}$$

and the diagram

$$\begin{array}{ccc} H^1(Gal(L/K), {}_E A) & \xrightarrow{{}_E u_*} & H^1(Gal(L/K), {}_{u_*E} B) \\ {\scriptstyle *E}\downarrow & & \downarrow{\scriptstyle *u_*E} \\ H^1(Gal(L/K), A) & \xrightarrow{u_*} & H^1(Gal(L/K), B) \end{array} \tag{2}$$

commute.

EXAMPLE 2.10. Next, consider an *Abelian extension* of discrete $\mathrm{Gal}(L/K)$-modules,

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 1,$$

i.e., A' is Abelian (although neither A nor A'' need be Abelian). The conjugation action of A on the normal subgroup A' determines a well-defined action of A'' on A'. Thus, associated to every $\mathrm{Gal}(L/K)$-equivariant A''-torsor E'' there is a discrete Abelian $\mathrm{Gal}(L/K)$-module ${}_{E''}A'$.

2.1.2. *Step II. The first terms.* This is enough to begin describing the long exact sequence. Let $u : A \to B$ be an injective homomorphism of discrete $\mathrm{Gal}(L/K)$-modules. Denote by $C = B/A$ the set of right A-cosets of B with its natural discrete structure of $\mathrm{Gal}(L/K)$-set. Denote by $v : B \to C$ the natural map, which is $\mathrm{Gal}(L/K)$-equivariant. The homomorphism u determines an injective homomorphism

$$H^0(\mathrm{Gal}(L/K), u) : H^0(\mathrm{Gal}(L/K), A) \to H^0(\mathrm{Gal}(L/K), B),$$

and in particular an action of $H^0(\mathrm{Gal}(L/K), A)$ on $H^0(\mathrm{Gal}(L/K), B)$. Because v is $\mathrm{Gal}(L/K)$-equivariant, it also determines a map

$$v_* : H^0(\mathrm{Gal}(L/K), B) \to C^{\mathrm{Gal}(L/K)}.$$

Clearly the fibers of v_* are precisely the $H^0(\mathrm{Gal}(L/K), A)$-orbits. In this sense, the sequence of sets

$$1 \longrightarrow H^0(\mathrm{Gal}(L/K), A) \longrightarrow H^0(\mathrm{Gal}(L/K), B) \longrightarrow C^{\mathrm{Gal}(L/K)}$$

is an exact sequence of sets.

2.1.3. *Step III. The connecting map.* The $\mathrm{Gal}(L/K)$-invariant elements of C are the same as the A-orbits E of B which are fixed as a set (but not necessarily pointwise) by the action of $\mathrm{Gal}(L/K)$. Such a $\mathrm{Gal}(L/K)$-invariant A-orbit E need not contain a $\mathrm{Gal}(L/K)$-invariant element, i.e., v_* need not be surjective. For every $\mathrm{Gal}(L/K)$-invariant A-orbit E, the restrictions to E of the $\mathrm{Gal}(L/K)$-action and the A-action make E into a $\mathrm{Gal}(L/K)$-invariant A-torsor. Forming the equivalence class of this torsor gives a well-defined set map

$$\delta : C^{\mathrm{Gal}(L/K)} \to H^1(\mathrm{Gal}(L/K), A).$$

Observe that for every $\mathrm{Gal}(L/K)$-invariant A-orbit E and every $\mathrm{Gal}(L/K)$-invariant element b of B, left multiplication by b defines an isomorphism of the $\mathrm{Gal}(L/K)$-equivariant A-torsor E and the $\mathrm{Gal}(L/K)$-equivariant A-torsor $b \cdot E$. Thus the map δ is constan on each $H^0(\mathrm{Gal}(L/K), B)$-orbits in $C^{\mathrm{Gal}(L/K)}$. Conversely, if E_1 and E_2 are $\mathrm{Gal}(L/K)$-invariant A-orbits in B and if $v : E_1 \to E_2$ is an isomorphism of $\mathrm{Gal}(L/K)$-invariant A-torsors, then there exists a unique element b of B such that $v(m) = b \cdot m$ for every m in E_1. Since v is $\mathrm{Gal}(L/K)$-equivariant, b is $\mathrm{Gal}(L/K)$-invariant. Therefore $\delta(E_1)$ equals $\delta(E_2)$ if and only if E_1 and E_2 differ by left multiplication by an element of $H^0(\mathrm{Gal}(L/K), B)$. In particular, the fiber of δ over $[A]$ is precisely the image of $H^0(\mathrm{Gal}(L/K), B)$.

In all of these senses, the set map

$$1 \to H^0(\mathrm{Gal}(L/K), A) \to H^0(\mathrm{Gal}(L/K), B) \to C^{\mathrm{Gal}(L/K)} \xrightarrow{\delta} H^1(\mathrm{Gal}(L/K), A)$$

is "exact". The first map is injective. The fibers of the second map are precisely the orbits for the action of $H^0(\mathrm{Gal}(L/K), A)$ on $H^0(\mathrm{Gal}(L/K), B)$. And the fibers of the third map are precisely the orbits for the action of $H^0(\mathrm{Gal}(L/K), B)$ on $(B/A)^{\mathrm{Gal}(L/K)}$.

2.1.4. *Step IV. The map of torsors.* Let E be a $\mathrm{Gal}(L/K)$-equivariant A-torsor and let u_*E be the associated $\mathrm{Gal}(L/K)$-equivariant B-torsor as constructed in Example 2.6, $u_*E = E \times B/A$. Taking inverse images, a $\mathrm{Gal}(L/K)$-invariant element of u_*E is the same as a $\mathrm{Gal}(L/K)$-invariant A-orbit F of $E \times B$. The projection $\mathrm{pr}_E : F \to E$ is an isomorphism of $\mathrm{Gal}(L/K)$-invariant A-torsors. Inverting this, F is the graph $F = \Gamma_v$ of a unique map $v : E \to B$ which is simultaneously $\mathrm{Gal}(L/K)$-equivariant and A-equivariant. But then E is isomorphic to $v(E)$, which is a $\mathrm{Gal}(L/K)$-equivariant left A-orbit in B. Thus u_*E is isomorphic to the distinguished element of $H^1(\mathrm{Gal}(L/K), B)$ if and only if E is isomorphic to an A-torsor in the image of δ. In summary, this gives the following.

Proposition 2.11. *Associated to an injective homomorphism of $Gal(L/K)$-modules, $u : A \to B$, and denoting by $v : B \to C$ the quotient map of $Gal(L/K)$-sets, $B \to B/A$, the sequence of set maps*

$$1 \to H^0(Gal(L/K), A) \xrightarrow{u_*} H^0(Gal(L/K), B) \xrightarrow{v_*} C^{Gal(L/K)}$$

$$\xrightarrow{\delta} H^1(Gal(L/K), A) \xrightarrow{u_*} H^1(Gal(L/K), B)$$

is "exact" in the following sense: the first map u_* is injective, the fibers of v_* equal the orbits of $Image(u_*)$, the fibers of δ equal the orbits of $H^0(Gal(L/K), B)$, and the fiber of u_* over the distinguished element equals the image of δ.

2.1.5. *Step V. Fibers of u_*.* Using the twists from Example 2.7, we can push this a bit further. Let E be a $\mathrm{Gal}(L/K)$-equivariant A-torsor and let

$$_Eu : {}_EA \to {}_{u_*E}B$$

be the associated morphism of inner twist $\mathrm{Gal}(L/K)$-modules as in Example 2.7. This gives rise to a sequence of sets,

$$1 \to H^0(\mathrm{Gal}(L/K), {}_EA) \xrightarrow{{}_Eu_*} H^0(\mathrm{Gal}(L/K), {}_{u_*E}B) \xrightarrow{{}_Ev_*} {}_EC^{\mathrm{Gal}(L/K)}$$
$$\xrightarrow{{}_E\delta} H^1(\mathrm{Gal}(L/K), {}_EA) \xrightarrow{{}_Eu_*} H^1(\mathrm{Gal}(L/K), {}_{u_*E}B)$$

By the diagram in Equation 2, the map $*E$ defines a map from the fiber of ${}_Eu_*$ containing the trivial ${}_EA$-torsor to the fiber of $u_* : H^1(\mathrm{Gal}(L/K), A) \to H^1(\mathrm{Gal}(L/K), B)$ containing $[E]$.

Proposition 2.12. *The fiber of*

$$u_* : H^1(Gal(L/K), A) \to H^1(Gal(L/K), B)$$

containing $[E]$ equals the image of the map

$$*E \circ {}_E\delta : ({}_EC)^{Gal(L/K)} \to H^1(Gal(L/K), A).$$

Proof. This follows from Lemma 2.9, Proposition 2.8 and Proposition 2.11 applied to ${}_Eu$ as discussed above. $\square$

2.1.6. *Step VI. The image of u_*.* Let F be a $\mathrm{Gal}(L/K)$-equivariant B-torsor.

Proposition 2.13. *There is $Gal(L/K)$-equivariant A-torsor E such that $[F]$ equals $u_*[E]$ if and only if $F/u(A)$ has a $Gal(L/K)$-invariant element.*

Proof. This is straightforward. Certainly ${}_EB = (E \times B)/\sim$ has a $\mathrm{Gal}(L/K)$-invariant $u(A)$-orbit, namely the image of $E \times \{e_B\}$. So if $[F]$ equals $u_*[E]$, then $F/u(A)$ has $\mathrm{Gal}(L/K)$-invariant element.

Conversely, if $F/u(A)$ has a $\mathrm{Gal}(L/K)$-invariant element, the inverse image in F is a $\mathrm{Gal}(L/K)$-invariant $u(A)$-orbit E. The restrictions of the $\mathrm{Gal}(L/K)$-structure and the A-structure on F to E make E into a $\mathrm{Gal}(L/K)$-equivariant A-torsor. And the inclusion $E \hookrightarrow F$ induces a morphism ${}_EB \to F$ of $\mathrm{Gal}(L/K)$-equivariant B-torsors. Thus $[F]$ equals $u_*[E]$. $\square$

Observe that $G/u(A)$ is the same as the twist ${}_GC$. In the sense that the twist ${}_GC$ should be considered "trivial" if it has a $\mathrm{Gal}(L/K)$-invariant element, this extends the long exact sequence of sets one more place to the right.

2.1.7. *Step VII. Quotient by a normal subgroup.* If A is a normal subgroup, then C is naturally a discrete $\mathrm{Gal}(L/K)$-module. In this case, for every $\mathrm{Gal}(L/K)$-equivariant A-torsor E, the quotient ${}_EC = {}_{u_*E}B/{}_EA$ is canonically isomorphic as a $\mathrm{Gal}(L/K)$-set to C. In particular, the rule associating to every element E the set map

$$*E \circ {}_E\delta : H^0(\mathrm{Gal}(L/K), C) \to H^1(\mathrm{Gal}(L/K), A)$$

is an action of $H^0(\mathrm{Gal}(L/K), C)$ on $H^1(\mathrm{Gal}(L/K), A)$. The previous step shows that the fibers of

$$u_* : H^1(\mathrm{Gal}(L/K), A) \to H^1(\mathrm{Gal}(L/K), B)$$

are precisely the orbits of $H^0(\mathrm{Gal}(L/K), C)$. Finally, for every B-torsor G, the twist ${}_GC$ from the previous paragraph still has a left C-action, and thus is a discrete $\mathrm{Gal}(L/K)$-equivariant C-torsor. As above, it is a trivial C-torsor if and only if it has a $\mathrm{Gal}(L/K)$-invariant element if and only if G is in the image of u_*. In summary, this gives the following.

PROPOSITION 2.14. *Associated to an injective homomorphism of $Gal(L/K)$-modules, $u : A \to B$ whose image is normal, and denoting by $v : B \to C$ the quotient map of $Gal(L/K)$-modules, $B \to B/A$, the sequence of set maps*

$$1 \to H^0(Gal(L/K), A) \xrightarrow{u_*} H^0(Gal(L/K), B) \xrightarrow{v_*} H^0(Gal(L/K), C)$$

$$\xrightarrow{\delta} H^1(Gal(L/K), A) \xrightarrow{u_*} H^1(Gal(L/K), B) \xrightarrow{v_*} H^1(Gal(L/K), C)$$

is "exact" in the following sense: the first map u_* is injective with normal image, the fibers of v_* equal the orbits of $Image(u_*)$, the fibers of δ equal the orbits of $H^0(Gal(L/K), B)$, the fibers of the second map u_* equal the orbits of $H^0(Gal(L/K), C)$, and the image of the second map u_* equals the fiber of the second map v_* containing the neutral element.

2.1.8. *Step VIII. Quotient by an Abelian normal subgroup.* The next step is when A is both normal and Abelian. Then the conjugation action of B on A descends to a well-defined action of C on A. Given c in C and a in A, we will denote this action by cac^{-1} as a reminder that the action is bab^{-1} for any representative b of the A-coset c in B.

Let

$$c : \mathrm{Gal}(L/K) \to C$$

be a continuous crossed homomorphism. This in turn determines a new discrete action of G on A, namely

$$g *_c a := c_g \cdot {}^g a \cdot c_g^{-1}.$$

The action of $\mathrm{Gal}(L/K)$ on B defines a semidirect product $B \rtimes \mathrm{Gal}(L/K)$. And c defines an action of $B \rtimes \mathrm{Gal}(L/K)$ on C by

$$(b, g) *_c d := c_g \cdot {}^g d \cdot v(b)^{-1}.$$

The stabilizer of the identity e_C is a subgroup denoted ${}_cH$. Of course ${}_cH$ contains $A \times \{1\}$ as a subgroup. The induced quotient

$$H \subset B \rtimes \mathrm{Gal}(L/K) \xrightarrow{\pi_2} \mathrm{Gal}(L/K)$$

is surjective and has kernel A. The conjugation action of H on A factors through an action of $\mathrm{Gal}(L/K)$ on A. And of course this is the new action defined above. Thus, altogether ${}_cH$ is an extension

$$0 \longrightarrow {}_cA \longrightarrow {}_cH \longrightarrow \mathrm{Gal}(L/K) \longrightarrow 0.$$

This gives a well-defined element

$$\Delta(c) \in H^2(\mathrm{Gal}(L/K), {}_cA).$$

For an element b of B, conjugation by b defines an automorphism of A intertwining the actions $*_c$ and $*_{c^{v(b)}}$. Similarly conjugation by $(b, 1) \in B \rtimes \mathrm{Gal}(L/K)$ gives an isomorphism of ${}_cH$ and ${}_{c^{v(b)}}H$ which gives a morphism of extensions

$$\begin{array}{ccccccccc}
0 & \longrightarrow & {}_cA & \longrightarrow & {}_cH & \longrightarrow & \mathrm{Gal}(L/K) & \longrightarrow & 0 \\
 & & \cong\big\downarrow & & \big\downarrow\cong & & \big\downarrow= & & \\
0 & \longrightarrow & {}_{c^{v(b)}}A & \longrightarrow & {}_{c^{v(b)}}H & \longrightarrow & \mathrm{Gal}(L/K) & \longrightarrow & 0.
\end{array}$$

In this sense, $\Delta(c)$ is independent of the choice of c.

Finally, every splitting of the extension comes from a lift of c to a crossed homomorphism $\mathrm{Gal}(L/K) \to B$. This proves the following.

Proposition 2.15. *For every element c in $Z^1_{cts.}(Gal(L/K), C)$, $\Delta(c)$ equals 0 in $H^2(Gal(L/K), A)$ if and only if $[c]$ is in the image of v_*.*

2.1.9. *Step IX. Central extensions.* The final step is when A is a central subgroup of B. Then $*_c$ is the usual action of $\mathrm{Gal}(L/K)$ on A. Therefore $\Delta(c)$ is an element of $H^2(\mathrm{Gal}(L/K), A)$. Similarly, conjugation by b on A is trivial, and thus conjugation by $(b, 1)$ defines an isomorphism of extensions of ${}_cH$ and ${}_{c^{v(b)}}H$. Therefore $\Delta(c)$ is independent of the representative of $[c]$, i.e., there is a well-defined set map

$$\Delta : H^1(\mathrm{Gal}(L/K), C) \to H^2(\mathrm{Gal}(L/K), A).$$

Finally, twisting A, B and C by an element c again gives a well-defined map

$${}_c\Delta : H^1(\mathrm{Gal}(L/K), {}_cC) \to H^1(\mathrm{Gal}(L/K), A).$$

For the same reason as in Proposition 2.12,

$$*c : H^1(\mathrm{Gal}(L/K), {}_cC) \to H^1(\mathrm{Gal}(L/K), C)$$

defines a bijection between the fiber of Δ containing $[c]$ and the fiber of ${}_c\Delta$ over 0. Therefore

$$*c \circ {}_cv_* : H^1(\mathrm{Gal}(L/K), {}_cB) \to H^1(\mathrm{Gal}(L/K), C)$$

maps surjectively onto the fiber of Δ containing $[c]$. In summary, this gives the following.

Proposition 2.16. *Associated to a central extension of $Gal(L/K)$-modules,*

$$0 \longrightarrow A \xrightarrow{u} B \xrightarrow{v} C \longrightarrow 1$$

the sequence of set maps

$$0 \to H^0(Gal(L/K), A) \to H^0(Gal(L/K), B) \to H^0(Gal(L/K), C) \xrightarrow{\delta}$$

$$H^1(Gal(L/K), A) \xrightarrow{u_*} H^1(Gal(L/K), B) \xrightarrow{v_*} H^1(Gal(L/K), C) \xrightarrow{\Delta} H^2(Gal(L/K), A)$$

is exact in the following sense: the first map u_* is injective with central image, the fibers of v_* equal the orbits of $Image(u_*)$, the fibers of δ equal the orbits of $H^0(Gal(L/K), B)$, the fibers of the second map u_* equal the orbits of $H^0(Gal(L/K), C)$, the fibers of the second map v_* equal the orbits of the Abelian group $H^1(Gal(L/K), A)$, and the fiber of Δ containing the element $[c]$ equals the image of the map

$$*c \circ {}_cv_* : H^1(Gal(L/K), {}_cB) \to H^1(Gal(L/K), C).$$

Remark 2.17. The long exact sequence is functorial in the short exact sequence. To be precise, given a commutative diagram of homomorphisms of $\mathrm{Gal}(L/K)$-modules

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ f\downarrow & & \downarrow g \\ A' & \xrightarrow{u'} & B' \end{array}$$

with u and u' injective, the diagram

$$\begin{array}{ccc} C^{\mathrm{Gal}(L/K)} & \xrightarrow{\delta_u} & H^1(\mathrm{Gal}(L/K), A) \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ (C')^{\mathrm{Gal}(L/K)} & \xrightarrow{\delta_v} & H^1(\mathrm{Gal}(L/K), A') \end{array}$$

is commutative, where $C = B/A$ and $C' = B'/A'$. Moreover, if A, resp. A', is a central subgroup of B, resp. of B', then also the diagram

$$\begin{array}{ccc} H^1(\mathrm{Gal}(L/K), C) & \xrightarrow{\Delta_u} & H^2(\mathrm{Gal}(L/K), A) \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ H^1(\mathrm{Gal}(L/K), C') & \xrightarrow{\Delta_v} & H^2(\mathrm{Gal}(L/K), A') \end{array}$$

is commutative. The proof of each of these assertions is a good exercise for the reader.

3. Galois cohomology of smooth group schemes

Most of the flavor of Galois cohomology comes from the specific Galois modules one studies. For instance, one could consider the action of $\mathrm{Gal}(K^s/K)$ on the étale cohomology groups of an algebraic variety defined over K. Or, when K is a number field, one could consider the action of $\mathrm{Gal}(L/K)$ on the ideal class group of the ring of integers in L. There is another source of Galois modules which are somewhat simpler than the previous 2 examples, but which still play an important role.

Let K be a field, let G be a quasi-compact, smooth group scheme over K, and let L/K be a Galois extension. Then the set $G(L)$ of L-points of G is a group with a natural action of $\mathrm{Gal}(L/K)$ by group homomorphisms. Since G is locally finitely presented over K, $G(L)$ is a discrete $\mathrm{Gal}(L/K)$-module. In fact, this is even better than a usual Galois module. For every extension field K'/K and for every Galois extension L'/K', the set of L'-points $G(L')$ is a discrete $\mathrm{Gal}(L'/K')$-module. The Galois cohomologies of these modules satisfy all the functorialities one would expect. It is a good exercise to formulate and prove these for oneself.

3.1. Galois descent of torsors. It is straightforward that $H^0(\mathrm{Gal}(L/K), G(L))$ equals $G(K)$. There is also an alternative description of $H^1(\mathrm{Gal}(L/K), G(L))$ whose proof uses *descent*.

DEFINITION 3.1. A right *G-torsor* over K is a nonempty, locally finitely presented K-scheme T together with a right action of G on T,

$$\mu : T \times_K G \to T$$

in the category of K-schemes such that the induced morphism

$$(\mathrm{pr}_T, \mu) : T \times_K G \to T \times_K T$$

is an isomorphism. A *morphism of G-torsors* is a morphism of K-schemes which is equivariant for the action of G (such a morphism is automatically an isomorphism).

Let T be a G-torsor over K. There is a bijective correspondence between $T(K)$ and the set of morphisms of G-torsors from the trivial G-torsor G to T via the rule associating to t in $T(K)$ the morphism

$$\mu_t : G \to T, \quad g \mapsto \mu(t, g).$$

In particular, T is trivial if and only if $T(K)$ is nonempty. By the same token, for every field extension L/K, the base change $T \otimes_K L$ is trivial as a $G \otimes_K L$-torsor if and only if $T(L)$ is nonempty.

Let L/K be a Galois extension and let T be a G-torsor such that $T(L)$ is nonempty. Then the set $T(L)$ is naturally a $\mathrm{Gal}(L/K)$-equivariant $G(L)$-torsor. Moreover, to every morphism

$$\phi : T \to T'$$

of G-torsors over K, the induced set map

$$\phi(L) : T(L) \to T'(L)$$

is a morphism of $\mathrm{Gal}(L/K)$-equivariant $G(L)$-torsors. This defines a set map

$$p^* : \mathrm{Hom}_{G-\mathrm{equiv.}}(T, T') \to \mathrm{Hom}_{\mathrm{Gal}(L/K)-G(L)-\mathrm{equiv.}}(T(L), T'(L)), \quad \phi \mapsto \phi(L)$$

from the set of morphisms of G-torsors to the set of morphisms of $\mathrm{Gal}(L/K)$-equivariant $G(L)$-torsors.

PROPOSITION 3.2 (Galois descent). *Let L/K be a finite Galois extension and let G be a quasi-compact, smooth group scheme over K.*

(0) *Every G-torsor T over K is a quasi-projective, smooth K-scheme.*
(i) *For every pair T, T' of G-torsors over K such that $T(L)$ and $T'(L)$ are nonempty, the set map p^* defined above is a bijection.*
(ii) *Every Gal(L/K)-equivariant $G(L)$-torsor is isomorphic to one associated to a G-torsor over K.*

PROOF. This follows by the technique of *descent*, cf. [**Wei56**, Section I], [**Gro62**, no. 190], [**BLR90**, §6.5]. □

When L/K is a Galois extension which is not necessarily finite, Proposition 3.2 still holds so long as we restrict attention to discrete torsors. In particular, the theorem establishes a bijection between $H^1(\mathrm{Gal}(K^s/K), G(L))$ and the set of isomorphism classes of G-torsors.

3.1.1. *Weil restriction of scalars.* Let L/K be a finite extension of fields (not necessarily separable). Let X be an L-scheme.

DEFINITION 3.3. A *Weil restriction* of X with respect to L/K is a pair $(\mathcal{R}_{L/K}(X), f)$ of a K-scheme $\mathcal{R}_{L/K}(X)$ together with a morphism of L-schemes

$$f : \mathcal{R}_{L/K}(X) \otimes_K L \to X$$

which is universal among all such pairs.

PROPOSITION 3.4 (Weil Restriction). *For every quasi-projective L-scheme X there exists a Weil restriction of X with respect to L/K. Moreover, $\mathcal{R}_{L/K}(X)$ is a quasi-projective K-scheme. And if X is L-smooth, then $\mathcal{R}_{L/K}(X)$ is K-smooth.*

PROOF. This follows, for instance, from [**Gro62**, Part IV.4.c, p. 221-19]. See also [**BLR90**, Theorem 4, p. 194]. The smoothness assertion is an easy application of the infinitesimal lifting criterion for smoothness, cf. [**Gro67**, Théorème 17.5.1] or [**Har77**, Exercise II.8.6]. □

For every quasi-compact, smooth group scheme G_L over L, the Weil restriction $\mathcal{R}_{L/K}(G_L)$ has a natural structure of a group scheme over K. For every G_L-torsor T_L over L, the Weil restriction $\mathcal{R}_{L/K}(T_L)$ has a natural structure of a $\mathcal{R}_{L/K}(G_L)$-torsor.

Also, for every K-scheme Y, denoting by Y_L the base change $Y \otimes_K L$, the identity morphism

$$Y \otimes_K L \to Y_L$$

determines a K-morphism

$$i_Y : Y \to \mathcal{R}_{L/K}(Y_L).$$

For every open affine $U = \text{Spec } R$ of Y, for every K-algebra A and for every pair of morphisms of K-algebras,

$$\phi, \phi' : R \to A,$$

if the associated $A \otimes_K L$-points,

$$\phi \otimes \text{Id}_L, \phi' \otimes \text{Id}_L : R \to A \otimes_K L$$

are equal, then ϕ and ϕ' are equal; indeed, the map of K-vector spaces

$$\text{Hom}_{K-\text{v. space}}(R, A) \to \text{Hom}_{K-\text{v. space}}(R, A \otimes_K L)$$

is injective since $A \to A \otimes_K L$ is injective. Thus, for a quasi-projective K-scheme Y, it follows that i_Y is an unramified monomorphism.

PROPOSITION 3.5. *Let L/K be a finite Galois extension.*

(i) *For every quasi-compact, smooth group scheme G over K,*

$$i_G : G \to \mathcal{R}_{L/K}(G_L)$$

is both a closed immersion and a homomorphism of quasi-compact, smooth group schemes over K.

(ii) *The connecting map in Galois cohomology,*

$$\delta : (\mathcal{R}_{L/K}(G_L)/G)(K) \to H^1(\mathit{Gal}(L/K), G(L))$$

is surjective, i.e., every G-torsor over K having an L-point is isomorphic to a G-orbit in $\mathcal{R}_{L/K}(G_L)$.

PROOF. **(i).** First of all, since G is a quasi-compact, smooth group scheme over K, it is quasi-projective, cf. [**BLR90**, Theorem 1, p. 153] for instance. Thus G_L is also a quasi-projective, smooth group scheme over L. Thus, by Proposition 3.4, $\mathcal{R}_{L/K}(G_L)$ is a quasi-projective, smooth group scheme over K. For every K-scheme T, the pullback map $G(T) \to G(T \otimes_K L)$ is a group homomorphism. Thus also $i_G(T)$ is a group homomorphism, i.e., i_G is a homomorphism of group schemes. And by the usual argument, the image of i_G is a closed subgroup scheme of $\mathcal{R}_{L/K}(G_L)$: the closure of $i_G(G)$ is a group and $i_G(G)$ is a dense, equivariant open in this group, and therefore the entire group. A homomorphism of group schemes which is an unramified monomorphism is automatically a closed immersion; the closure of the image is a group scheme which contains $i_G(G)$ is an equivariant, dense open, hence as all of the closure.

(ii). Let T be a G-torsor over K. If $T(L)$ is nonempty, then there is an isomorphism of G_L and T_L as G_L-torsors over L. Thus there is an isomorphism

$$u : \mathcal{R}_{L/K}(T_L) \xrightarrow{\cong} \mathcal{R}_{L/K}(G_L)$$

as $\mathcal{R}_{L/K}(G_L)$-torsors over K. The morphism

$$i_G : G \to \mathcal{R}_{L/K}(G_L)$$

is a homomorphism of group schemes over K. And the composition

$$u \circ i_T : T \to \mathcal{R}_{L/K}(G_L)$$

embeds T as an $i_G(G)$-orbit in $\mathcal{R}_{L/K}(G_L)$. $\square$

Part (ii) can also be proved by writing out explicitly the map

$$H^1(\mathrm{Gal}(L/K), G(L)) \xrightarrow{i_{G,*}} H^1(\mathrm{Gal}(L/K), \mathcal{R}_{L/K}(G_L)(L))$$

and verifying it is constant.

3.2. Examples.

3.2.1. *The multiplicative group.* One particularly important case is when $G = \mathbb{G}_{m,K}$, the multiplicative group scheme Spec $K[t, t^{-1}]$.

THEOREM 3.6 (Hilbert's Theorem 90). *For the multiplicative group, $H^0(Gal(K^s/K), \mathbb{G}_{m,K}(K^s))$ equals K^* and $H^1(Gal(K^s/K), \mathbb{G}_{m,K}(K^s))$ is $\{[\mathbb{G}_{m,K}(L)]\}$.*

PROOF. The first assertion is obvious. For the second assertion, it suffices to verify $H^1(\mathrm{Gal}(L/K), \mathbb{G}_{m,K}(L))$ is trivial for every finite Galois extension L/K, i.e., every $\mathbb{G}_{m,K}$-torsor having an L-point also has a K-point. By Proposition 3.5, every $\mathbb{G}_{m,K}$-torsor having an L-point is a $\mathbb{G}_{m,K}$-orbit in $\mathcal{R}_{L/K}(\mathbb{G}_{m,L})$. Of course $\mathbb{G}_{m,K}$ is an open subscheme of the affine line $\mathbb{A}^1_K$, and the immersion is equivariant. Thus $\mathcal{R}_{L/K}(\mathbb{G}_{m,L})$ is an equivariant open subscheme of $\mathcal{R}_{L/K}(\mathbb{A}^1_L)$. Up to choosing a basis $\{b_1, \dots, b_n\}$ for L as a K-vector space, $\mathcal{R}_{L/K}(\mathbb{A}^1_L)$ is isomorphic to $\mathbb{A}^n_K$ as a K-scheme. Moreover, the action of $\mathbb{G}_{m,K}$ on $\mathcal{R}_{L/K}(\mathbb{A}^1_L)$ induced by the homomorphism

$$i_{\mathbb{G}_{m,K}} : \mathbb{G}_{m,K} \to \mathcal{R}_{L/K}(\mathbb{G}_{m,L})$$

equals the standard action of $\mathbb{G}_{m,K}$ on $\mathbb{A}^n_K$ via this isomorphism. So the $\mathbb{G}_{m,K}$-orbits in $\mathcal{R}_{L/K}(\mathbb{A}^1_K)$ are the same as the $\mathbb{G}_{m,K}$-orbits in $\mathbb{A}^n_K$, i.e., the set of nonzero elements in 1-dimensional K-linear subspaces of $\mathbb{A}^n_K$. Therefore Hilbert's Theorem 90 reduces to the statement that every 1-dimensional K-linear subspace of $\mathbb{A}^n_K$ contains a nonzero vector, which is obvious. $\square$

Of course there is another proof which forms a Poincaré series out of a 1-cocycle and then uses linear independence of automorphisms to conclude this Poincaré series is nonzero. In fact the proof above uses the full machinery of descent, parts of which are proved by using arguments very similar to the Poincaré series argument.

Because of Hilbert's Theorem 90, the first nontrivial cohomology group is $H^2(\mathrm{Gal}(K^s/K), \mathbb{G}_{m,K}(K^s))$.

DEFINITION 3.7. For every Galois extension L/K, the *cohomological Brauer group of* L/K, denoted $\mathrm{Br}'(L/K)$, is $H^2(\mathrm{Gal}(L/K), \mathbb{G}_{m,K}(L))$. When L is a separable closure K^s of K, this is called the *cohomological Brauer group of* K, $\mathrm{Br}'(K)$.

3.2.2. *Groups of roots of unity.* Let n be a positive integer not divisible by $\mathrm{char}(K)$. Then the group scheme $\boldsymbol{\mu}_{n,K} :=$ Spec $K[t, t^{-1}]/\langle t^n - 1\rangle$ is a smooth group scheme over K. There is a short exact sequence of smooth, commutative group schemes over K, the *Kummer sequence*,

$$1 \longrightarrow \boldsymbol{\mu}_{n,K} \longrightarrow \mathbb{G}_{m,K} \xrightarrow{(-)^n} \mathbb{G}_{m,K} \longrightarrow 1.$$

PROPOSITION 3.8. *The Kummer sequence induces isomorphisms of Abelian groups,*

$$H^1(Gal(K^s/K), \boldsymbol{\mu}_{n,K}(K^s)) = K^*/(K^*)^n$$

and

$$H^2(Gal(K^2/K), \boldsymbol{\mu}_{n,K}(K^s)) = Br'(K)[n].$$

Proof. This follows immediately from the long exact sequence of (Abelian) Galois cohomology and Theorem 3.6. $\square$

3.2.3. *Additive and unipotent groups.* Another important case is when $G = \mathbb{G}_{a,K} = \text{Spec } K[t]$, the additive group.

Proposition 3.9. *For the additive group $\mathbb{G}_{a,K}$, $H^0(Gal(K^s/K), \mathbb{G}_{a,K}(K^s))$ equals K and for every $q > 0$, $H^q(Gal(K^s/K), \mathbb{G}_{a,K}(K^s))$ equals $\{0\}$.*

Proof. For every finite Galois extension L/K with Galois group Γ, the normal basis theorem gives an isomorphism between L and $K[\Gamma]$ as $K[\Gamma]$-modules. Since $K[\Gamma]$ is a free module over itself, it has no higher Galois cohomology. $\square$

If $\text{char}(K) = p$, then there is a short exact sequence of smooth, commutative group schemes, the *Artin-Schreier sequence*,

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{G}_{a,K} \xrightarrow{F} \mathbb{G}_{a,K} \longrightarrow 0$$

where $F(a) = a^p - a$.

Corollary 3.10. *If $char(K) = p$, then $H^0(Gal(K^s/K), \mathbb{Z}/p\mathbb{Z})$ equals $\mathbb{Z}/p\mathbb{Z}$, $H^1(Gal(K^s/K), \mathbb{Z}/p\mathbb{Z})$ equals $K/F(K)$, and for every $q > 1$, $H^q(Gal(K^s/K), \mathbb{Z}/p\mathbb{Z})$ equals $\{0\}$.*

Corollary 3.11. *If a quasi-compact, smooth group scheme U over K has a composition series by normal, closed subgroup schemes whose subquotients are each isomorphic to $\mathbb{G}_{a,K}$, then $H^1(Gal(K^s/K), U(K^s))$ equals $\{*\}$.*

The first corollary follows from Proposition 3.9 and the long exact sequence of Abelian Galois cohomology. The second corollary follows by inductively applying Proposition 2.14 to the terms in the composition series, using Proposition 3.9 for the base case of the induction.

Definition 3.12. A connected, quasi-compact, smooth group scheme U over K is *unipotent* if for some integer N, U is isomorphic to a closed subgroup scheme of $\mathbf{GL}_{N,K}$ whose K^s-points are all unipotent (i.e., 1 is the only eigenvalue).

Remark 3.13. If K is a *perfect* field, then every connected, unipotent group over K has a composition series whose subquotients are each isomorphic to $\mathbb{G}_{a,K}$. Thus $H^1(\text{Gal}(K^s/K), U(K^s))$ equals $\{*\}$ for every connected, unipotent group U over a perfect field K.

3.2.4. *Matrix groups.* For every integer n, the affine group scheme $\mathbf{GL}_{n,K}$ over K, resp. $\mathbf{SL}_{n,K}$, $\mathbf{PGL}_{n,K}$ over K, represents the functor

$$K-\text{algebras} \to \text{Groups}, \quad A \mapsto \text{Isom}_{A-\text{mod}}(A^{\oplus n}, A^{\oplus n}),$$

resp.

$$K-\text{algebras} \to \text{Groups}, \quad A \mapsto \{\phi \in \text{Isom}_{A-\text{mod}}(A^{\oplus n}, A^{\oplus n}) | \det(\phi) = 1\},$$

$$K-\text{algebras} \to \text{Groups}, \quad A \mapsto \text{Isom}_{A-\text{mod}}(A^{\oplus n}, A^{\oplus n})/\sim, \quad \phi \sim \lambda \cdot \phi, \forall \lambda \in A^*.$$

Basically the same, for every finite dimensional K-vector space V, the group schemes $\mathbf{GL}_{V,K}$, resp. $\mathbf{SL}_{V,K}$, $\mathbf{PGL}_{V,K}$, represents the functor

$$K-\text{algebras} \to \text{Groups}, \quad A \mapsto \text{Isom}_{A-\text{mod}}(A \otimes_K V, A \otimes_K V),$$

resp.

$$K-\text{algebras} \to \text{Groups}, \quad A \mapsto \{\phi \in \text{Isom}_{A-\text{mod}}(A \otimes_K V, A \otimes_K V) | \det(\phi) = 1\},$$

$$K-\text{algebras} \to \text{Groups}, \quad A \mapsto \text{Isom}_{A-\text{mod}}(A \otimes_K V, A \otimes_K V)/\sim, \quad \phi \sim \lambda \cdot \phi, \forall \lambda \in A^*.$$

Of course up to choosing a basis for V, these are the same groups.

THEOREM 3.14 (Speiser's Theorem). *For every integer $n \geq 1$, every $\mathbf{GL}_{n,K}$-torsor over K, resp. every $\mathbf{SL}_{n,K}$-torsor over K, is trivial.*

PROOF. This is almost the same as the proof of Theorem 3.6. By the same argument as there, for every finite Galois extension L/K, every $\mathbf{GL}_{n,K}$-torsor over K with an L-point is isomorphic as a $\mathbf{GL}_{n,K}$-torsor to a free (right) $\mathbf{GL}_{n,K}$-orbit in $\text{Hom}_{K-\text{v. space}}(K^{\oplus n}, L^{\oplus n})$. Every free orbit is of the form $\text{Isom}_{K-\text{v. space}}(K^{\oplus n}, V)$ where V is an n-dimensional K-linear subspace of $L^{\oplus n}$. Choosing a basis for V, such an orbit has a K-point.

Consider the short exact sequence of group schemes

$$1 \longrightarrow \mathbf{SL}_{n,K} \longrightarrow \mathbf{GL}_{n,K} \xrightarrow{\det} \mathbb{G}_{m,K} \longrightarrow 0.$$

This is "split" by the homomorphism

$$s : \mathbb{G}_{m,K} \to \mathbf{GL}_{n,K},$$

$$\lambda \mapsto \begin{bmatrix} \lambda & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

Thus the connecting map in the associated long exact sequence of Galois cohomology is constant. Therefore $H^1(\text{Gal}(K^s/K), \mathbf{SL}_{n,K}(K^s))$ is a subset of $H^1(\text{Gal}(K^s/K), \mathbf{GL}_{n,K}(K^s))$, which is a singleton set by the previous paragraph. $\square$

The long exact sequence of Galois cohomology associated to the central extension

$$1 \longrightarrow \mathbb{G}_{m,K} \xrightarrow{(-)\cdot \text{Id}_{n\times n}} \mathbf{GL}_{n,K} \xrightarrow{q} \mathbf{PGL}_{n,K} \longrightarrow 1$$

gives a connecting map

$$\Delta_n : H^1(\text{Gal}(L/K), \mathbf{PGL}_{n,K}(L)) \to H^2(\text{Gal}(L/K), \mathbb{G}_{m,K}(L)) = \text{Br}'(L/K).$$

THEOREM 3.15. ***(i)*** *The fiber of Δ_n over the neutral element is $\{*\}$.* ***(ii)*** *The image of Δ_n is contained in the n-torsion subgroup, $Br'(L/K)[n]$.*

PROOF. **(i).** This follows from Proposition 2.16 and Theorem 3.14.

(ii). By Remark 2.17, associated to the following commutative diagram of central extensions of smooth, linear group schemes,

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{G}_{m,K} & \xrightarrow{(-)\cdot \text{Id}_{n\times n}} & \mathbf{GL}_{n,K} & \xrightarrow{q} & \mathbf{PGL}_{n,K} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle (-)^n} & & \downarrow{\scriptstyle (\det, q)} & & \downarrow{\scriptstyle =} & & \\ 1 & \longrightarrow & \mathbb{G}_{m,K} & \xrightarrow{e_1} & \mathbb{G}_{m,K} \times \mathbf{PGL}_{n,K} & \xrightarrow{\pi_2} & \mathbf{PGL}_{n,K} & \longrightarrow & 1 \end{array}$$

there is a commutative diagram

$$\begin{array}{ccc} H^1(\mathrm{Gal}(L/K), \mathbf{PGL}_{n,K}(L)) & \xrightarrow{\Delta_n} & H^2(\mathrm{Gal}(L/K), \mathbb{G}_{m,K}(L)) \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle (-)^n} \\ H^1(\mathrm{Gal}(L/K), \mathbf{PGL}_{n,K}(L)) & \xrightarrow{\Delta_{e_1}} & H^2(\mathrm{Gal}(L/K), \mathbb{G}_{m,K}(L)) \end{array}.$$

Since e_1 is split, Δ_{e_1} is the constant map. Thus the image of Δ_n is in the kernel of the raising to the n^{th} power map, i.e., it is in $\mathrm{Br}'(L/K)[n]$. $\square$

For two finite dimensional K-vector spaces V and W, there is a morphism of smooth group schemes,

$$\otimes : \mathbf{GL}_{V,K} \times \mathbf{GL}_{W,K} \to \mathbf{GL}_{V\otimes_K W,K}, \quad (\phi, \psi) \mapsto \phi \otimes \psi.$$

This induces morphisms of smooth group schemes,

$$\otimes : \mathbf{SL}_{V,K} \times \mathbf{SL}_{W,K} \to \mathbf{SL}_{V\otimes_K W,K},$$

and

$$\otimes : \mathbf{PGL}_{V,K} \times \mathbf{PGL}_{W,K} \to \mathbf{PGL}_{V\otimes_K W,K}.$$

In particular, by the functoriality of Galois cohomology and by Remark 2.4, this induces a map of pointed sets,

$$\otimes : H^1(\mathrm{Gal}(L/K), \mathbf{PGL}_{V,K}(L)) \times H^1(\mathrm{Gal}(L/K), \mathbf{PGL}_{W,K}(L)) \to H^1(\mathrm{Gal}(L/K), \mathbf{PGL}_{V\otimes_K W,K}(L)).$$

Up to choosing bases, this is the same as a map

$$\otimes : H^1(\mathrm{Gal}(L/K), \mathbf{PGL}_{l,K}(L)) \times H^1(\mathrm{Gal}(L/K), \mathbf{PGL}_{n,K}(L)) \to H^1(\mathrm{Gal}(L/K), \mathbf{PGL}_{ln,K}(L)).$$

LEMMA 3.16. *For every $\mathbf{PGL}_{l,K}$-torsor E, resp. $\mathbf{PGL}_{n,K}$-torsor F, with an L-point, $\Delta_{ln}(E \otimes F)$ equals $\Delta_l(E) + \Delta_n(F)$.*

PROOF. By Remark 2.17, associated to the following commutative diagram of central extensions of smooth, linear group schemes,

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{G}_{m,K} \times \mathbb{G}_{m,K} & \longrightarrow & \mathbf{GL}_{l,K} \times \mathbf{GL}_{n,K} & \longrightarrow & \mathbf{PGL}_{l,K} \times \mathbf{PGL}_{n,K} & \longrightarrow & 1 \\ & & {\scriptstyle \times}\downarrow & & \downarrow{\scriptstyle \otimes} & & \downarrow{\scriptstyle \otimes} & & \\ 1 & \longrightarrow & \mathbb{G}_{m,K} & \longrightarrow & \mathbf{GL}_{lm,K} & \longrightarrow & \mathbf{PGL}_{lm,K} & \longrightarrow & 1 \end{array},$$

there is a commutative diagram

$$\begin{array}{ccc} H^1(\mathrm{Gal}(L/K), \mathbf{PGL}_{l,K}(L)) \times H^1(\mathrm{Gal}(L/K), \mathbf{PGL}_{n,K}(L)) & \xrightarrow{\Delta_l \times \Delta_n} & \mathrm{Br}'(L/K) \times \mathrm{Br}'(L/K) \\ {\scriptstyle \otimes}\downarrow & & \downarrow{\scriptstyle \times} \\ H^1(\mathrm{Gal}(L/K), \mathbf{PGL}_{ln,K}(L)) & \xrightarrow{\Delta_{ln}} & \mathrm{Br}'(L/K) \end{array}.$$

The lemma is equivalent to the commutativity of this diagram. $\square$

4. The Brauer group

Definition 4.1. Let L/K be a Galois extension. A *central simple algebra* over L/K is a finite dimensional K-algebra A (so K is contained in the center of A) such that $A \otimes_K L$ is isomorphic as an L-algebra to the matrix algebra $\mathrm{Mat}_{n\times n,L}$ for some integer n. When L is a separable closure K^s, this is simply called a central simple algebra over K.

In particular, matrix algebras $\mathrm{Mat}_{n\times n,K}$ are central simple algebras over K. If A is a central simple algebra over K, then so is the *opposite algebra* A^{opp} defined to be A but with opposite multiplication $a*b := ba$. And if A and A' are central simple algebras over K, so is $A\otimes_K A'$. Thus the set of isomorphism classes of central simple algebras is an associative, commutative semigroup with identity element $[K]$. But it is better than this.

Definition 4.2. Two central simple algebras A and A' are *Morita equivalent* if there exist positive integers n and n' such that $\mathrm{Mat}_{n\times n,K} \otimes_K A$ is isomorphic to $\mathrm{Mat}_{n'\times n',K} \otimes_K A'$.

Proposition 4.3. *Let A, A' and B be central simple algebras over L/K. If A and A' are Morita equivalent, so are A^{opp} and $(A')^{opp}$, and so are $A \otimes_K B$ and $A' \otimes_K B$. Also, the natural map*

$$A \otimes_K A^{opp} \to Hom_{K-v.\ space}(A, A),$$
$$(a, b) \mapsto (c \mapsto acb)$$

is an isomorphism of central simple algebras. In fact the composition with the trace,

$$A \otimes_K A^{opp} \to Hom_{K-v.\ space}(A, A) \xrightarrow{Trace} K$$

is a perfect pairing of K-vector spaces.

Proof. The assertions about Morita equivalence are straightforward. The last two statements say that a certain maps of K-vector spaces are isomorphisms. These may be checked after base change from K to K^s, and then A is isomorpic to a matrix algebra. The assertions are easy to verify for matrix algebras (using the bases of elementary matrices, for instance). □

Because of the proposition, the set of Morita equivalence classes is an Abelian group under $[A] + [B] = [A \otimes_K B]$ and $-[A] = [A^{\mathrm{opp}}]$. The identity element is $0 = [\mathrm{Mat}_{n,K}]$ for any integer $n \geq 1$.

Definition 4.4. Let L/K be a Galois extension. The group of Morita equivalence classes of central simple algebras over L/K is the *Brauer group of L/K*, denote $\mathrm{Br}(L/K)$. When L is a separable closure K^s of K, this is referred to as the *Brauer group of K*, $\mathrm{Br}(K)$.

4.1. Relation to PGL-torsors.

Let n be a positive integer. The group $\mathbf{PGL}_{n,K}$ acts by algebra automorphisms on $\mathrm{Mat}_{n\times n,K}$ via conjugation. Thus, for every $\mathbf{PGL}_{n,K}$-torsor E over K, the twist

$$A_E := {}_E\mathrm{Mat}_{n\times n,K} = (E \times \mathrm{Mat}_{n\times n,K})/\mathbf{PGL}_{n,K}$$

is a central simple algebra of dimension n^2. For a field extension L/K, if E has an L-point, then $A_E \otimes_K L$ is isomorphic to $\mathrm{Mat}_{n\times n,L}$. This defines a set map,

$$A_- : H^1(\mathrm{Gal}(L/K), \mathbf{PGL}_{n,K}(L)) \to \mathcal{A}_n(L/K)$$

where $\mathcal{A}_n(L/K)$ is the set of isomorphism classes of central simple algebras A over K such that $A \otimes_K L \cong \mathrm{Mat}_{n\times n,L}$.

Proposition 4.5 (Skolem-Noether). *The set map above is a bijection. Moreover, for $\mathbf{PGL}_{n,K}$-torsors E and F, the induced map from the set of morphisms in the category of torsors to the set of morphisms in the category of central simple algebras,*

$$A_{-} : Hom_{\mathbf{PGL}_{n,K}}(E,F) \to Hom_{K-CSA}(A_E, A_F),$$

is a bijection.

Proof. Let A be a central simple algebra in $\mathcal{A}_n(L/K)$. Define E to be the affine K-scheme representing the functor

$$K-\text{algebras} \to \text{Sets}, \quad R \mapsto \mathrm{Isom}_{R-\text{algebra}}(\mathrm{Mat}_{n\times n,R}, A \otimes_K R).$$

There is a right action of $\mathbf{PGL}_{n,K}$ on E by precomposing isomorphisms with automorphisms of $\mathrm{Mat}_{n\times n,K}$. The Skolem-Noether theorem says that this action makes I a $\mathbf{PGL}_{n,K}$-torsor. In fact this can be checked after base change from K to K^s, so it reduces to the classical formulation of the Skolem-Noether theorem: every K-algebra endomorphism of $\mathrm{Mat}_{n\times n,K^s}$ comes from $\mathbf{PGL}_{n,K}(K^s)$. The natural map

$$E \times \mathrm{Mat}_{n\times n,K} \to A$$

is invariant for the natural $\mathbf{PGL}_{n,K}$-action and thus factors through a K-algebra homomorphism

$$\phi : A_E \to A.$$

Since this is a K-algebra homomorphism of central simple algebras, it is automatically an isomorphism (this can be checked after base change to K^s where it follows from Skolem-Noether).

For every $\mathbf{PGL}_{n,K}$-torsor F and every K-algebra homomorphism

$$\psi : A_F \to A,$$

the composition

$$F \times \mathrm{Mat}_{n\times n,K} \to A$$

induces a morphism of schemes from $\tau : F \to E$. It is straightforward to verify this is an isomorphism of $\mathbf{PGL}_{n,K}$-torsors, and is the unique isomorphism such that $\psi = \phi \circ A_\tau$. The second part of the proposition follows. $\square$

4.2. The cohomological Brauer group equals the Brauer group. Because of Proposition 4.5, the map from Theorem 3.15 factors through a well-defined set map,

$$\Delta_n : \mathcal{A}_n(L/K) \to \mathrm{Br}'(L/K), \quad \Delta_n(A_E) := \Delta_n(E).$$

Lemma 4.6. *For every $[A]$ in $\mathcal{A}_l(L/K)$ and for every $[B]$ in $\mathcal{A}_n(L/K)$, $\Delta_{ln}(A \otimes_K B)$ equals $\Delta_l(A) + \Delta_n(B)$. In particular, if A and B are Morita equivalent, then $\Delta_l(A)$ equals $\Delta_n(B)$. Thus there is a well-defined group homomorphism,*

$$\Delta : Br(L/K) \to Br'(L/K), \quad \Delta([A]) = \Delta_l(A).$$

This group homomorphism is injective.

Proof. For a $\mathbf{PGL}_{l,K}$-torsor E and a $\mathbf{PGL}_{n,K}$-torsor F, $A_{E\otimes F}$ equals $A_E \otimes_K A_F$ as central simple algebras (this is easiest to verify using cocycles). Thus Lemma 3.16 implies that $\Delta_{ln}(A_E \otimes_K A_F)$ equals $\Delta_l(A_E) + \Delta_n(A_F)$. Together with Proposition 4.5, this implies the first assertion.

Since $\Delta_n(E)$ equals 0 for the trivial $\mathbf{PGL}_{n,K}$-torsor, also $\Delta_n(\mathrm{Mat}_{n\times n,K})$ equals 0. Thus, by the previous paragraph, $\Delta_{ln}(A\otimes_K \mathrm{Mat}_{n\times n,K})$ equals $\Delta_l(A)$. Therefore $\Delta_l(A)$ equals $\Delta_n(B)$ if A and B are Morita equivalent. Thus there is a well-defined map

$$\Delta : \mathrm{Br}(L/K) \to \mathrm{Br}'(L/K)$$

such that for every integer $l \geq 1$ and every A in $\mathcal{A}_l(L/K)$, $\Delta([A])$ equals $\Delta_l(A)$.

Finally, assume $\Delta([A])$ equals $\Delta([B])$. Then $A^{\mathrm{opp}} \otimes_K B$ and $\mathrm{Mat}_{ln\times ln,K}$ are elements of $\mathcal{A}_{ln}(L/K)$ with the same image under Δ_{ln}. By Theorem 3.15(i) and Proposition 4.5, this implies that $A^{\mathrm{opp}} \otimes_K B$ is isomorphic to $\mathrm{Mat}_{ln\times \mathrm{ln},K}$ as central simple algebras, i.e., $[B] - [A]$ equals 0 in $\mathrm{Br}(L/K)$. Therefore $[A]$ equals $[B]$ in $\mathrm{Br}(L/K)$. So Δ is injective. $\square$

Theorem 4.7. *Let L/K be a finite Galois extension of degree n. Then*

$$\Delta_n : \mathcal{A}_n(L/K) \to Br'(L/K)$$

is surjective. Thus for every Galois extension L/K (not necessarily finite), the homomorphism

$$\Delta : Br(L/K) \to Br'(L/K)$$

is surjective, and hence bijective. In other words, the Brauer group and the cohomological Brauer group are canonically isomorphic.

Proof. Using Proposition 1.8, every element in $\mathrm{Br}'(L/K)$ comes from an extension

$$1 \longrightarrow L^* \xrightarrow{\ u\ } M \xrightarrow{\ v\ } \mathrm{Gal}(L/K) \longrightarrow 1.$$

The group algebra $K[M]$ is an associative K-algebra which is a free K-vector space with basis $\{\mathbf{e}_m | m \in M\}$. There is an action of M on L by K-algebra homomorphisms through the quotient v. This defines a multiplication on the tensor product

$$L[M] := L \otimes_K K[M], \quad (a \otimes \mathbf{e}_m) \cdot (b \otimes \mathbf{e}_n) := (a_{v(m)} b) \otimes \mathbf{e}_{mn}$$

for elements $a, b \in L$ and $m, n \in M$. It is straightforward to verify this multiplication is K-linear and associative with $1 \otimes \mathbf{1_N}$ as an identity element.

Define I to be the 2-sided ideal in $L[M]$ defined by the generators,

$$I := \langle \lambda \otimes \mathbf{e}_{1_M} - 1 \otimes \mathbf{e}_{u(\lambda)} | \lambda \in L^* \rangle.$$

Denote the quotient algebra by $A_M := L[M]/I$. As a (left) L-vector space, it is straightforward to check that A_M has dimension equal to $\#\mathrm{Gal}(L/K) = n$. This algebra is usually denoted as the *cross-product algebra* associated to the extension M.

It is a somewhat involved exercise to check that A_M is a central simple algebra and that $\Delta_n(A_M)$ equals $[M]$. This is usually done by choosing a set map splitting $s : \mathrm{Gal}(L/K) \to M$ of v and using this to translate everything into cocycles. Since the details do not seem very illuminating, they are left to the interested reader, cf. [**Ser79**, Exercises 1 and 2, p.159]. $\square$

4.3. Division algebras.

Definition 4.8. A *division algebra* over L/K is a central simple algebra A over L/K such that every nonzero element of A is invertible. Equivalently, it is a central simple algebra whose only right ideals are $\{0\}$ and A (and the same also holds with left ideals in place of right ideals). When L is a separable closure K^s of K, these are referred to simply as *division algebras* over K.

Lemma 4.9. *Let A be a division algebra over K Every finitely generated left (or right) A-module is free.*

Proof. Let M be a left A-module. If M equals $\{0\}$ then the assertion is vacuous, $M = A^{\oplus 0}$. Thus assume M is nonzero. Consider a surjective homomorphism of D-modules,

$$q : A^{\oplus n} \twoheadrightarrow M$$

for which the integer n is minimal. Denote by N the kernel of q. If N is nonempty, then for some integer $1 \leq i \leq n$, the projection

$$\pi_i : K \to A$$

is nonzero. Thus $\pi_i(N)$ is a nonzero left A-submodule of A, i.e., a left ideal in A. Since A is a division algebra, $\pi_i(N)$ equals A. But this means that, modulo N, the i^{th} basis element of $A^{\oplus n}$ is congruent to a linear combination of the other basis elements. So the restriction of q to the free submodules generated by the other basis elements is still surjective, contradicting the minimality of n. Therefore N equals $\{0\}$, i.e., q is an isomorphism of A-modules. $\square$

In order to prove the next result, it is useful to describe the right ideals in $\mathrm{Mat}_{n\times n,K}$. The proof is left as an exercise for the reader.

Lemma 4.10. *For a nonzero, finite dimensional K-vector space V, the right ideals of $\mathrm{Mat}_{V,k}$ are in 1-to-1 correspondence with the K-linear subspaces W of V via the rule,*

$$W \mapsto Hom_{K-v.\ space}(V,W).$$

Lemma 4.11 (Schur's Lemma). *Let A be a central simple algebra over K. Assume that A is not a division algebra, i.e., A has a proper, nonzero right ideal I. Let I be a proper, nonzero right ideal which is minimal, i.e., has minimal dimension as a K-vector space. Then every nonzero left A-module homomorphism,*

$$\phi : I \to I$$

is an isomorphism and the K-algebra of A-module homomorphisms,

$$D := End_{A-mod}(I,I)$$

is a division algebra.

Proof. The image $\phi(I)$ is a left ideal in A. Since I is minimal, if ϕ is nonzero then $\phi(I)$ equals I, i.e., ϕ is surjective. A surjective K-linear endomorphism of a finite dimension K-vector space is injective, thus ϕ is an isomorphism. Thus every nonzero element of D is nonzero. After base change from K to L, $A\otimes_K L$ is isomorphic to $\mathrm{Mat}_{V,L}$ for some nonzero, finite dimensional L-vector space V. By Lemma 4.10, $I \otimes_K L$ is isomorphic to $\mathrm{Hom}_{L-\text{v. space}}(V,W)$ for some nonzero, proper L-subspace W of V. It is straightforward to check that

$$\mathrm{End}_{\mathrm{Mat}_{V,L}}(\mathrm{Hom}_{L-\text{v. space}}(V,W)) = \mathrm{Mat}_{W,L}.$$

Thus $D \otimes_K L$ is isomorphic to $\mathrm{Mat}_{W,L}$, i.e., D is a central simple algebra over K. Therefore D is a division algebra. $\square$

There is an induced left action of D on I which commutes with the right action of A on I. Thus there is an induced K-algebra homomorphism,

$$A \to \mathrm{End}_D(I).$$

THEOREM 4.12 (Artin-Wedderburn Theorem). *Every central simple algebra A over L/K is isomorphic to $D \otimes_K \mathrm{Mat}_{n\times n,K}$ for a unique division algebra D over L/K.*

PROOF. By base change from K to L, one can check that the K-linear map

$$A \to \mathrm{End}_D(I)$$

above is an isomorphism. By Lemma 4.9, I is a free D-module, say $I \cong D^{\oplus n}$. Therefore A is isomorphic to $D \otimes_K \mathrm{Mat}_{n\times n,K}$.

Let D be a division algebra and let A be $D \otimes_K \mathrm{Mat}_{n\times n,K}$. Then, up to isomorphism, the unique (nonzero) simple A-module is $M := D^{\oplus n}$ with its natural A-action. And the natural map

$$D \to \mathrm{End}_{A-\mathrm{mod}}(M)$$

is an isomorphism of algebras. Therefore D is uniquely determined by A. $\square$

4.4. The period-index problem.

DEFINITION 4.13. Let A be a central simple algebra over K. The *index* of A, denote index(A), is the unique integer n such that $\dim_K(D) = n^2$ where D is a division algebra Morita equivalent to A. The *period* or *exponent* of A, denoted period(A), is the order of $[A]$ in $\mathrm{Br}(K)$. A *splitting field* for A is a separable field extension L/K such that $A \otimes_K L$ is isomorphic to $\mathrm{Mat}_{m\times m,L}$ for some integer m.

It turns out that index(A) equals both the minimal degree $[L : K]$ of a splitting field of A as well as the greatest common divisor of the degrees $[L : K]$ of all splitting fields.

PROPOSITION 4.14 (R. Brauer). *For every central simple algebra A over K, period(A) divides index(A), and both integers have the same prime factors (but often with different multiplicities).*

PROOF. See [**Ser79**, Exercise 3, p. 159]. $\square$

In light of this proposition, it is natural to ask about the precise relation between the period and the index.

PROBLEM 4.15 (The Period-Index Problem). Let K be a field. If there exists a nonnegative integer r such that index(A) divides period$(A)^r$ for every central simple algebra A, the least such integer is the *power* of K, denoted $r(K)$; otherwise $r(K)$ is defined to be ∞. What is the power of K? For a given integer r, what "conditions" on K guarantee that $r(K) \leq r$?

Of course this problem is ill-posed since the meaning of "condition" is not specified. In Chapter 4 we will explain some answers which hopefully give a clearer meaning to this problem.

5. The universal cover sequence

There is a generalization of the map Δ_n from Theorem 3.15 to other linear algebraic group schemes. The generalization uses some of the structure theory for linear algebraic group schemes. Unipotent groups were defined in Definition 3.12.

Definition 5.1. A connected, quasi-compact, smooth group scheme T is *multiplicative* or *of multiplicative type* if for some integer $n \geq 0$, $T \otimes_K K^s$ is isomorphic to $\mathbb{G}^n_{m,K^s}$ as a group scheme over K^s.

For a quasi-compact, smooth group scheme G over K, the *unipotent radical* of G, $R_u(G)$, is the maximal connected, normal subgroup scheme of G which is unipotent. If $R_u(G)$ is trivial, then G is *reductive*.

For a connected, smooth, linear algebraic group scheme G over K, the *multiplicative quotient* of G, $T(G)$, is the maximal quotient group of G (by a closed, normal subgroup scheme) which is multiplicative. If $R_u(G)$ and $T(G)$ are trivial, then G is *semisimple*. If $T(G)$ is the quotient of G by $R_u(G)$, then G is *solvable*. The maximal connected, normal, solvable subgroup of G is the *solvable radical* of G, or simply the *radical* of G, denoted $R(G)$.

Let G be a connected, smooth, linear algebraic group scheme over K. Of course $R_u(G)$ is the unipotent radical of $R(G)$. And $R_u(G)$ is contained in the kernel of $G \to T(G)$. Thus there is an induced homomorphism of group schemes over K,

$$R(G)/R_u(G) \to T(G).$$

This is a finite and faithfully flat morphism, i.e., an *isogeny* of group schemes. But it is not necessarily an isomorphism. (Notice however that $T(G)$ is trivial if and only if $R(G)$ equals $R_u(G)$, so that G is semisimple if and only if $R(G)$ is trivial.) Because $T(G)$ and $R(G)/R_u(G)$ are different, there are 2 different decompositions that arise naturally in the study of group schemes.

Notation 5.2. Let G be a connected, smooth, linear algebraic group scheme over K. The *first associated semisimple group* of G is the group

$$L(G) = \mathrm{Ker}(G \to T(G))/R_u(G).$$

The *second associated semisimple group* of G is the group

$$L'(G) = G/R(G).$$

As the definition of $L'(G)$ is simpler, it is the one that arises most often. But $L(G)$ is slightly better behaved with respect to universal covers and fundamental groups. Both of these definitions should be considered as decompositions of G into a unipotent group, $R_u(G)$, a multiplicative group, $T(G)$, resp. $R(G)/R_u(G)$, and a semisimple group, $L(G)$, resp., $L'(G)$.

Definition 5.3. A connected, semisimple algebraic group L is *simply connected* if for every connected, semisimple algebraic group M, every isogeny

$$M \to L$$

is an isomorphism. For every connected, semisimple algebraic group L, there exists a connected, simply connected, semisimple algebraic group $\widetilde{L}$ and an isogeny

$$\widetilde{L} \to L,$$

called the *universal cover* of L. The kernel of the universal cover, denote $\pi_1 L$, is the *fundamental group scheme* of L.

There is a central extension of group schemes,

$$0 \longrightarrow \pi_1 L \longrightarrow \widetilde{L} \longrightarrow L \longrightarrow 1$$

Note that $\pi_1 L$ is sometimes not smooth, e.g., $\pi_1 \mathbf{PGL}_{n,K}$ is the non-smooth group scheme $\boldsymbol{\mu}_{n,K}$ when $\mathrm{char}(K)$ divides n. There is a generalization of Galois cohomology, cohomology for the fppf topology, $H^1_{\mathrm{fppf}}(K; -)$. This agrees with Galois cohomology for smooth group schemes. This cohomology theory also has a long exact sequence, leading to a sequence

$$0 \to H^0_{\mathrm{fppf}}(K; \pi_1 G) \to H^0(\mathrm{Gal}(K^s/K), \widetilde{G}(K^s)) \to H^0(\mathrm{Gal}(K^s/K), G(K^s)) \xrightarrow{\delta}$$

$$H^1_{\mathrm{fppf}}(K, \pi_1 G) \xrightarrow{u_*} H^1(\mathrm{Gal}(K^s/K), \widetilde{G}(K^s)) \xrightarrow{v_*} H^1(\mathrm{Gal}(K^s/K), G(K^s)) \xrightarrow{\Delta} H^2_{\mathrm{fppf}}(K, \pi_1 G).$$

This sequence satisfies the same conditions as in Proposition 2.16.

Serre posed two conjectures about this sequence. The first conjecture was proved by R. Steinberg.

THEOREM 5.4 (Steinberg's Theorem = Serre's "Conjecture I"). [**Ste65**] *If K has cohomological dimension ≤ 1, then for every connected, semisimple group scheme G over K, $H^1(\mathit{Gal}(K^s/K), G(K^s))$ is $\{*\}$, and the converse holds. More generally, if K has cohomological dimension ≤ 1, then $H^1(\mathit{Gal}(K^s/K), G(K^s))$ equals $\{*\}$ for every connected, reductive group scheme G over K. If K is perfect, it also holds for every connected, linear algebraic group scheme G over K.*

If $H^1(\mathrm{Gal}(K^s/K), \mathbf{PGL}_{n,K}(K^s))$ equals $\{*\}$ for every n, then $\mathrm{Br}'(K)$ equals $\{[K]\}$ by Theorem 3.15. Using Proposition 3.8 and similar arguments, this implies $\mathrm{cd}(K) \leq 1$, cf. also Theorem 3.3. For a field of cohomological dimension ≤ 1, $H^1(\mathrm{Gal}(K^s/K), T)$ equals $\{*\}$ for every multiplicative group T, cf. [**Ser79**, Application, p. 162]. Thus Steinberg's theorem reduces to the case that G is connected and semisimple. There is a further reduction to the case that G is simply connected and *quasi-split*, i.e., there exists a closed subgroup B of G such that $B \otimes_K K^s$ is a maximal solvable subgroup (i.e., a *Borel subgroup*) of $G \otimes_K K^s$. Steinberg proves that for every field K, for every simply connected, semisimple group G with a Borel subgroup defined over K, every G-torsor arises as $i_* E$ where $i : T \to G$ is the inclusion of a multiplicative subgroup and where E is a T-torsor. When K has cohomological dimension ≤ 1, E is a trivial T-torsor, and thus also $i_* E$ is a trivial G-torsor.

CONJECTURE 5.5 (Serre's "Conjecture II"). [**Ser02**, §3] If K has cohomological dimension 2 and if K is perfect, then for every connected, simply connected, semisimple group scheme $\widetilde{G}$ over K, $H^1(\mathrm{Gal}(K^s/K), \widetilde{G}(K^s))$ is $\{*\}$. Equivalently, for every connected, semisimple group scheme G over K, the connecting map

$$\Delta : H^1(\mathrm{Gal}(K^s/K), G(K^s)) \to H^2_{\mathrm{fppf}}(K; \pi_1 G)$$

is injective.

REMARK 5.6. In [**Ser95**], Serre explains that the hypothesis that K is perfect is too strong in these conjectures. They should also hold if the perfect hypothesis is replaced by the hypothesis that $[K : K^p] \leq p^2$ and $H^3_p(K')$ is 0 for all finite, separable extensions K' of K. In particular, this new hypothesis holds when K is a function field of a surface over an algebraically closed field k, i.e., K is a finitely generated k-extension and $\mathrm{tr.deg.}(K/k)$ equals 2.

This conjecture remains open in general, although much is known, cf. [**Ser02**, §3], [**Ser95**], [**CTGP04**]. At least when K is perfect, a converse theorem is known, cf. Theorem 3.3(ii). We will return to this conjecture in Chapter 5.

CHAPTER 2

The Chevalley-Warning and Tsen-Lang theorems

1. The Chevalley-Warning Theorem

A motivating problem in both arithmetic and geometry is the following.

PROBLEM 1.1. Given a field K and a K-variety X find sufficient, resp. necessary, conditions for existence of a K-point of X.

The problem depends dramatically on the type of K: number field, finite field, p-adic field, function field over a finite field, or function field over an algebraically closed field. In arithmetic the number field case is most exciting. However the geometric case, i.e., the case of a function field over an algebraically closed field, is typically easier and may suggest approaches and conjectures in the arithmetic case.

Two results, the Chevalley-Warning theorem and Tsen's theorem, deduce a sufficient condition for existence of K-points by "counting". More generally, counting leads to a relative result: the Tsen-Lang theorem that a strong property about existence of k-points for a field k propagates to a weaker property about K-points for certain field extensions K/k. The prototype result, both historically and logically, is a theorem of Chevalley and its generalization by Warning. The counting result at the heart of the proof is Lagrange's theorem together with the observation that a nonzero single-variable polynomial of degree $\leq q-1$ cannot have q distinct zeroes.

LEMMA 1.2. *For a finite field K with q elements, the polynomial $1 - x^{q-1}$ vanishes on K^* and $x^q - x$ vanishes on all of K. For every integer $n \geq 0$, for the K-algebra homomorphism*

$$ev_n : K[X_0, \dots, X_n] \to Hom_{Sets}(K^{n+1}, K),$$

$$ev_n(p(X_0, \dots, X_n)) = ((a_0, \dots, a_n) \mapsto p(a_0, \dots, a_n)),$$

the kernel equals the ideal

$$I_n = \langle X_0^q - X_0, \dots, X_n^q - X_n \rangle.$$

Finally, the collection $(X_i^q - X_i)_{i=0,\dots,n}$ is a Gröbner basis with respect to every monomial order refining the grading of monomials by total order. In particular, for every p in I_n some term of p of highest degree is in the ideal $\langle X_0^q, \dots, X_n^q \rangle$.

PROOF. Because K^* is a group of order $q-1$, Lagrange's theorem implies $a^{q-1} = 1$ for every element a of K^*, i.e., $1 - x^{q-1}$ vanishes on K^*. Multiplying by x shows that $x^q - x$ vanishes on K. Thus the ideal I_n is at least contained in the kernel of ev_n.

Modulo $X_n^q - X_n$, every element of $K[X_0, \dots, X_n]$ is congruent to one of the form

$$p(X_0, \dots, X_n) = p_{q-1}X_n^{q-1} + \dots + p_0 X_n^0, \quad p_0, \dots, p_{q-1} \in K[X_0, \dots, X_{n-1}].$$

(Of course K^n is defined to be $\{0\}$ and $K[X_0,\dots,X_{n-1}]$ is defined to be K if n equals 0.) Since K has q elements and since a nonzero polynomial of degree $\leq q-1$ can have at most $q-1$ distinct zeroes, for every $(a_0,\dots,a_{n-1}) \in K^n$ the polynomial $p(a_0,\dots,a_{n-1},X_n)$ is zero on K if and only if

$$p_0(a_0,\dots,a_{n-1}) = \cdots = p_{q-1}(a_0,\dots,a_{n-1}) = 0.$$

Thus $\mathrm{ev}_n(p)$ equals 0 if and only if each $\mathrm{ev}_{n-1}(p_i)$ equals 0. In that case, by the induction hypothesis, each p_i is in I_{n-1} (in case $n=0$, each p_i equals 0). Then, since $I_{n-1}K[X_0,\dots,X_n]$ is in I_n, p is in I_n. Therefore, by induction on n, the kernel of ev_n is precisely I_n.

Finally, Buchberger's algorithm applied to the set $(X_0^q - X_0,\dots,X_n^q - X_n)$ produces S-polynomials

$$S_{i,j} = X_j^q(X_i^q - X_i) - X_i^q(X_j^q - X_j) = X_j(X_i^q - X_i) - X_i(X_j^q - X_j)$$

which have remainder 0. Therefore this set is a Gröbner basis by Buchberger's criterion. □

THEOREM 1.3. [**Che35**],[**War35**] *Let K be a finite field. Let n and r be positive integers. Let $F_1,\dots,F_r$ be nonconstant, homogeneous polynomials in $K[X_0,\dots,X_n]$. If*

$$deg(F_1) + \cdots + deg(F_r) \leq n$$

then there exists $(a_0,\dots,a_n) \in K^{n+1} - \{0\}$ such that for every $i = 1,\dots,r$, $F_i(a_0,\dots,a_n)$ equals 0. Stated differently, the projective scheme $\mathbb{V}(F_1,\dots,F_r) \subset \mathbb{P}^n_K$ has a K-point.

PROOF. Denote by q the number of elements in K. The polynomial

$$G(X_0,\dots,X_n) = 1 - \prod_{i=0}^{n}(1 - X_i^{q-1})$$

equals 0 on $\{0\}$ and equals 1 on $K^{n+1} - \{0\}$. For the same reason, the polynomial

$$H(X_0,\dots,X_n) = 1 - \prod_{j=1}^{r}(1 - F_j(X_0,\dots,X_n)^{q-1})$$

equals 0 on

$$\{(a_0,\dots,a_n) \in K^{n+1} | F_1(a_0,\dots,a_n) = \cdots = F_r(a_0,\dots,a_n) = 0\}$$

and equals 1 on the complement of this set in K^{n+1}. Since each F_i is homogeneous, 0 is a common zero of $F_1,\dots,F_r$. Thus the difference $G-H$ equals 1 on

$$\{(a_0,\dots,a_n) \in K^{n+1} - \{0\} | F_1(a_0,\dots,a_n) = \cdots = F_r(a_0,\dots,a_n) = 0\}$$

and equals 0 on the complement of this set in K^{n+1}. Thus, to prove that $F_1,\dots,F_r$ have a nontrivial common zero, it suffices to prove the polynomial $G-H$ does not lie in the ideal I_n.

Since

$$\deg(F_1) + \cdots + \deg(F_r) \leq n,$$

H has strictly smaller degree than G. Thus the leading term of $G-H$ equals the leading term of G. There is only one term of G of degree $\deg(G)$. Thus, for every monomial ordering refining the grading by total degree, the leading term of G equals

$$(-1)^{n+1}X_0^{q-1}X_1^{q-1}\cdots X_n^{q-1}.$$

This is clearly divisible by none of X_i^q for $i = 0, \ldots, n$, i.e., the leading term of $G - H$ is not in the ideal $\langle X_0^q, \ldots, X_n^q \rangle$. Because $(X_0^q - X_0, \ldots, X_n^q - X_n)$ is a Gröbner basis for I_n with respect to the monomial order, $G - H$ is not in I_n. $\square$

2. The Tsen-Lang Theorem

On the geometric side, an analogue of Chevalley's theorem was proved by Tsen, cf. [**Tse33**]. This was later generalized independently by Tsen and Lang, cf. [**Tse36**], [**Lan52**]. Lang introduced a definition which simplifies the argument.

Definition 2.1. [**Lan52**] Let m be a nonnegative integer. A field K is called C_m, or said to have *property* C_m, if it satisfies the following. For every positive integer n and every sequence of positive integers $(d_1, \ldots, d_r)$ satisfying

$$d_1^m + \cdots + d_r^m \leq n,$$

every sequence $(F_1, \ldots, F_r)$ of homogeneous polynomials $F_i \in K[X_0, \ldots, X_n]$ with $\deg(F_i) = d_i$ has a common zero in $K^{n+1} - \{0\}$.

Remark 2.2. In fact the definition in [**Lan52**] is a little bit different than this. For fields having normic forms, Lang proves the definition above is equivalent to his definition. And the definition above works best with the following results.

With this definition, the statement of the Chevalley-Warning theorem is quite simple: every finite field has property C_1. The next result proves that property C_m is preserved by algebraic extension.

Lemma 2.3. *For every nonnegative integer m, every algebraic extension of a field with property C_m has property C_m.*

Proof. Let K be a field with property C_m and let L'/K be an algebraic extension. For every sequence of polynomials $(F_1, \ldots, F_r)$ as in the definition, the coefficients generate a finitely generated subextension L/K of L'/K. Thus clearly it suffices to prove the lemma for finitely generated, algebraic extensions L/K.

Denote by e the finite dimension $\dim_K(L)$. Because multiplication on L is K-bilinear, each homogeneous, degree d_i, polynomial map of L-vector spaces,

$$F_i : L^{\oplus(n+1)} \to L,$$

is also a homogeneous, degree d_i, polynomial map of K-vector spaces. Choosing a K-basis for L and decomposing F_i accordingly, F_i is equivalent to e distinct homogeneous, degree d_i, polynomial maps of K-vector spaces,

$$F_{i,j} : L^{\oplus(n+1)} \to K, \quad j = 1, \ldots, e.$$

The set of common zeroes of the collection of homogeneous polynomial maps $(F_i | i = 1, \ldots, r)$ equals the set of common zeroes of the collection of homogeneous polynomial functions $(F_{i,j} | i = 1, \ldots, r, j = 1, \ldots, e)$. Thus it suffices to prove there is a nontrivial common zero of all the functions $F_{i,j}$.

By hypothesis,

$$\sum_{i=1}^{r} \deg(F_i)^m \text{ is no greater than } n.$$

Thus, also

$$\sum_{i=1}^{r} \sum_{j=1}^{e} \deg(F_{i,j})^m = e \sum_{i=1}^{r} \deg(F_i)^m \text{ is no greater than } en.$$

Since K has property C_m and since

$$\dim_K(L^{\oplus(n+1)}), \text{ i.e., } (n+1)\dim_K(L) = e(n+1),$$

is larger than en, the collection of homogeneous polynomials $F_{i,j}$ has a common zero in $L^{\oplus(n+1)} - \{0\}$. $\square$

The heart of the Tsen-Lang theorem is the following proposition.

PROPOSITION 2.4. *Let K/k be a function field of a curve, i.e., a finitely generated, separable field extension of transcendence degree* 1*. If k has property C_m then K has property C_{m+1}.*

This is proved in a series of steps. Let n, r and $d_1, \dots, d_r$ be positive integers such that

$$d_1^{m+1} + \cdots + d_r^{m+1} \leq n.$$

For every collection of homogeneous polynomials

$$F_1, \dots, F_r \in K[X_0, \dots, X_n], \quad \deg(F_i) = d_i,$$

the goal is to prove that the collection of homogeneous, degree d_i, polynomial maps of K-vector spaces

$$F_1, \dots, F_r : K^{\oplus(n+1)} \to K$$

has a common zero. Of course, as in the proof of Lemma 2.3, this is also a collection of homogeneous polynomial maps of k-vector spaces. Unfortunately both of these k-vector spaces are infinite dimensional. However, using geometry, these polynomial maps can be realized as the colimits of polynomial maps of finite dimensional k-vector spaces. For these maps there is an analogue of the Chevalley-Warning argument replacing the counting argument by a *parameter counting argument* which ultimately follows from the Riemann-Roch theorem for curves. The first step is to give a *projective model* of K/k.

LEMMA 2.5. *For every separable, finitely generated field extension K/k of transcendence degree* 1*, there exists a smooth, projective, connected curve C over k and an isomorphism of k-extensions $K \cong k(C)$. Moreover the pair $(C, K \cong k(C))$ is unique up to unique isomorphism.*

PROOF. This is essentially the *Zariski-Riemann surface* of the extension K/k. For a proof in the case that k is algebraically closed, see [**Har77**, Theorem I.6.9]. The proof in the general case is similar. $\square$

The isomorphism $K \cong k(C)$ is useful because the infinite dimensional k-vector space $k(C)$ has a plethora of naturally-defined finite dimensional subspaces. For every Cartier divisor D on C, denote by V_D the subspace

$$V_D := H^0(C, \mathcal{O}_C(D)) = \{f \in k(C) | \text{div}(f) + D \geq 0\}.$$

The collection of all Cartier divisors D on C is a partially ordered set where

$$D' \geq D \text{ if and only if } D' - D \text{ is effective.}$$

The system of subspaces V_D of $k(C)$ is compatible for this partial order, i.e., if $D' \geq D$ then $V_{D'} \supset V_D$. And K is the union of all the subspaces V_D, i.e., it is the colimit of this compatible system of finite dimensional k-vector spaces. Thus for all k-multilinear algebra operations which commute with colimits, the operation on $k(C)$ can be understood in terms of its restrictions to the finite dimensional subspaces $k(C)$. The next lemma makes this more concrete for the polynomial map F.

LEMMA 2.6. *Let C be a smooth, projective, connected curve over a field k and let*

$$F_i \in k(C)[X_0, \dots, X_n]_{d_i}, \quad i = 1, \dots, r$$

be a collection of polynomials in the spaces $k(C)[X_0, \dots, X_n]_{d_i}$ of homogeneous, degree d_i polynomials. There exists an effective, Cartier divisor P on C and for every $i = 1, \dots, r$ there exists a global section $F_{C,i}$ of the coherent sheaf $\mathcal{O}_C(P)[X_0, \dots, X_n]_{d_i}$ such that for every $i = 1, \dots, r$ the germ of $F_{C,i}$ at the generic point of C equals F_i.

REMARK 2.7. In particular, for every Cartier divisor D on C and for every $i = 1, \dots, r$ there is a homogeneous, degree d, polynomial map of k-vector spaces

$$F_{C,D,i} : V_D^{\oplus(n+1)} \to W_{d_i,P,D}, \quad W_{d_i,P,D} := V_{d_iD+P},$$

such that for every $i = 1, \dots, r$ the restriction of F_i to $V_D^{\oplus(n+1)}$ equals $F_{C,D,i}$ considered as a map with target K (rather than the subspace V_{dD+P}).

PROOF. The coefficients of each F_i are rational functions on C. Each such function has a polar divisor. Since there are only finitely many coefficients of the finitely many polynomials $F_1, \dots, F_r$, there exists a single effective, Cartier divisor P on C such that every coefficient is a global section of $\mathcal{O}_C(P)$. $\square$

Because of Lemma 2.6, the original polynomial maps $F_1, \dots, F_r$ can be understood in terms of their restrictions to the subspaces V_D. The dimensions of these subspaces are determined by the Riemann-Roch theorem.

THEOREM 2.8 (Riemann-Roch for smooth, projective curves). *Let k be a field. Let C be a smooth, projective, connected curve over k. Denote by $\omega_{C/k}$ the sheaf of relative differentials of C over k and denote by $g(C) = genus(C)$ the unique integer such that $deg(\omega_{C/k}) = 2g(C) - 2$. For every invertible sheaf $\mathcal{L}$ on C,*

$$h^0(C, \mathcal{L}) - h^0(C, \omega_C \otimes_{\mathcal{O}_C} \mathcal{L}^\vee) = deg(\mathcal{L}) + 1 - g(C).$$

REMARK 2.9. In particular, if $\deg(\mathcal{L}) > \deg(\omega_C) = 2g(C) - 2$ so that $\omega_C \otimes_{\mathcal{O}_C} \mathcal{L}^\vee$ has negative degree, then $h^0(C, \omega_C \otimes_{\mathcal{O}_C} \mathcal{L}^\vee)$ equals zero. And then

$$h^0(C, \mathcal{L}) = \deg(\mathcal{L}) + 1 - g(C).$$

For a Cartier divisor D satisfying

$$\deg(D) > 2g(C) - 2 \text{ and for each } i = 1, \dots, r, d_i\deg(D) + \deg(P) > 2g(C) - 2,$$

the Riemann-Roch theorem gives that $V_D^{\oplus(n+1)}$ and $W_{d_i,P,D}$ are finite dimensional k-vector spaces of respective dimensions,

$$\dim_k(V_D^{\oplus(n+1)}) = (n+1)h^0(C, \mathcal{O}_C(D)) = (n+1)(\deg(D) + 1 - g)$$

and

$$\dim_k(W_{d_i,P,D}) = \dim(V_{d_iD+P}) = d_i\deg(D) + \deg(P) + 1 - g.$$

In this case, choosing a basis for $W_{d_i,P,D}$ and decomposing

$$F_{C,D,i} : V_D^{\oplus(n+1)} \to W_{d_i,P,D}$$

into its associated components, there exist $\dim_k(W_{d_i,P,D})$ homogeneous, degree d, polynomial functions

$$(F_{C,D,i})_j : V_D^{\oplus(n+1)} \to k, \quad j = 1, \ldots, \dim_k(W_{d_i,P,D})$$

such that a zero of $F_{C,D,i}$ is precisely the same as a common zero of all the functions $(F_{C,D,i})_j$.

Proof of Proposition 2.4. By hypothesis, each d_i and $n + 1 - \sum_{i=1}^r d_i^{m+1}$ are nonzero so that the fractions

$$\frac{2g(C) - 2 - \deg(P)}{d_i} \text{ for each } i = 1, \ldots, r, \; \frac{(n + 1 - \sum_{i=1}^r d_i^m)(g - 1) + \sum_{i=1}^r d_i^m \deg(P)}{n + 1 - \sum_{i=1}^r d_i^{m+1}}$$

are all defined. Let D be an effective, Cartier divisor on C such that

$$\deg(D) > 2g(C) - 2, \; \deg(D) > \frac{2g(C) - 2 - \deg(P)}{d_i}, i = 1, \ldots, r, \text{ and}$$

$$\deg(D) > \frac{(n + 1 - \sum_{i=1}^r d_i^m)(g - 1) + \sum_{i=1}^r d_i^m \deg(P)}{n + 1 - \sum_{i=1}^r d_i^{m+1}}.$$

Because $\deg(D) > 2g(C) - 2$, the Riemann-Roch theorem states that

$$\dim_k(V_D^{\oplus(n+1)}) = (n + 1)\dim_k(V_D) = (n + 1)(\deg(D) + 1 - g).$$

For every $i = 1, \ldots, r$, because d_i is positive and because $\deg(D) > (2g(C) - 2 - \deg(P))/d_i$, also

$$\deg(d_iD + P) = d_i\deg(D) + \deg(P) \text{ is greater than } 2g(C) - 2.$$

Thus the Riemann-Roch theorem states that

$$\dim_k(W_{d_i,P,D}) = \dim_k(V_{d_iD+P}) = d_i\deg(D) + \deg(P) + 1 - g(C).$$

Thus for the collection of polynomial functions $(F_{C,D,i})_j$,

$$\dim_k(V_D^{\oplus(n+1)}) - \sum_{i=1}^r \sum_j \deg((F_{C,D,i})_j)^m$$

equals

$$(n + 1)(\deg(D) + 1 - g) - \sum_{i=1}^r (d_i\deg(D) + \deg(P) + 1 - g(C))d_i^m =$$

$$(n + 1 - \sum_{i=1}^r d_i^{m+1})\deg(D) - [(n + 1 - \sum_{i=1}^r d_i^m)(g - 1) + \sum_{i=1}^r d_i^m \deg(P)].$$

Because

$$\deg(D) > \frac{(n + 1 - \sum_{i=1}^r d_i^m)(g - 1) + \sum_{i=1}^r d_i^m \deg(P)}{n + 1 - \sum_{i=1}^r d_i^{m+1}}$$

and because $n + 1 - \sum_{i=1}^r d_i^{m+1}$ is positive, also

$$(n + 1 - \sum_{i=1}^r d_i^{m+1})\deg(D) > [(n + 1 - \sum_{i=1}^r d_i^m)(g - 1) + \sum_{i=1}^r d_i^m \deg(P)].$$

Therefore

$$\dim_k(V_D^{\oplus(n+1)}) \text{ is greater than } \sum_{i=1}^r \sum_j \deg((F_{i,C,D})_j)^m.$$

Because of the inequality above, and because k has property C_m, there is a nontrivial common zero of the collection of homogeneous polynomial functions $(F_{C,D,i})_j$, $i = 1, \dots, r$, $j = 1, \dots, \dim_k(W_{d_i,P_i,D})$. Therefore there is a nontrivial common zero of the collection of homogeneous polynomial maps $F_{C,D,i}$, $i = 1, \dots, r$. By Lemma 2.6, the image of this nonzero element in $K^{\oplus(n+1)}$ is a nonzero element which is a common zero of the polynomials $F_1, \dots, F_r$. □

Proposition 2.4 is the main step in the proof of the Tsen-Lang theorem.

THEOREM 2.10 (The Tsen-Lang Theorem). [**Lan52**] *Let K/k be a field extension with finite transcendence degree, tr.deg.$(K/k) = t$. If k has property C_m then K has property C_{m+t}.*

PROOF. The proof of the theorem is by induction on t. When $t = 0$, i.e., when K/k is algebraic, the result follows from Lemma 2.3. Thus assume $t > 0$ and the result is known for $t - 1$. Let $(b_1, \dots, b_t)$ be a transcendence basis for K/k. Let E_t, resp. E_{t-1}, denote the subfield of K generated by k and $b_1, \dots, b_t$, resp. generated by k and $b_1, \dots, b_{t-1}$. Since E_{t-1}/k has transcendence degree $t-1$, by the induction hypothesis E_{t-1} has property C_{m+t-1}. Now E_t/E_{t-1} is a purely transcendental extension of transcendence degree 1. In particular, it is finitely generated and separable. Since E_{t-1} has property C_{m+t-1}, by Proposition 2.4 E_t has property C_{m+t}. Finally by Lemma 2.3 again, since K/E_t is algebraic and E_t has property C_{m+t}, also K has property C_{m+t}. □

The homogeneous version of the Nullstellensatz implies a field k has property C_0 if and only if k is algebraically closed. Thus one corollary of Theorem 2.10 is the following.

COROLLARY 2.11. *Let k be an algebraically closed field and let K/k be a field extension of finite transcendence degree t. The field K has property C_t.*

In particular, the case $t = 1$ is historically the first result in this direction.

COROLLARY 2.12 (Tsen's theorem). [**Tse36**] *The function field of a curve over an algebraically closed field has property C_1.*

3. Applications to Brauer groups

Chevalley and Tsen recognized that property C_1, which they called *quasi-algebraic closure*, has an important consequence for division algebras. Lang recognized that property C_2 also has an important consequence for division algebras, cf. [**Lan52**, Theorem 13].

Let A be a central simple algebra over K with $\dim_K(A) = n^2$, and let E be the corresponding $\mathbf{PGL}_{n,K}$-torsor. Because the determinant polynomial det on $\mathrm{Mat}_{n\times n,K}$ is invariant for the inner action of $\mathbf{PGL}_{n,K}$, det determines a well-defined polynomial map on A.

DEFINITION 3.1. For a central simple algebra A over K with $\dim_K(A) = n^2$, the *reduced norm* is the unique, degree n polynomial map

$$\mathrm{Nrm}_{A/k} : A \to K$$

such that the induced degree n polynomial map

$$\mathrm{Nrm}_{A/K} \otimes \mathrm{Id} : A \otimes_K K^s \to K^s$$

agrees with the determinant map

$$\det : \mathrm{Mat}_{n\times n, K^s} \to K^s$$

via one, and hence every, K^s-algebra isomorphism of A and $\mathrm{Mat}_{n\times n,K^s}$.

Since det is multiplicative, so is $\mathrm{Nrm}_{A/K}$, i.e.,

$$\forall a, b \in A, \quad \mathrm{Nrm}_{A/K}(ab) = \mathrm{Nrm}_{A/K}(a)\mathrm{Nrm}_{A/K}(b).$$

And the restriction to the center K is the polynomial map $\lambda \mapsto \lambda^n$. These properties characterize the reduced norm. By the same type of Galois invariance argument as above, and using Cramer's rule, an element a of A has a (left and right) inverse if and only if $\mathrm{Nrm}_{A/K}(a)$ is nonzero. In particular, if D is a division algebra the only zero of $\mathrm{Nrm}_{A/K}$ is $a = 0$.

PROPOSITION 3.2. *Let K be a field*

(i) *If K has property C_1, then the only division algebra with center K is K itself. Thus $Br(K)$ equals $\{[K]\}$.*
(ii) *If K has property C_2 then for every division algebra D with center K the reduced norm map*

$$Nrm_{D/K} : D \to K$$

is surjective.

PROOF. **(i).** Let D be a division algebra with center K. Denote by n the index of D. Because $\mathrm{Mat}_{n\times n}(\overline{K})$ has dimension n^2 as a $\overline{K}$-vector space, also D has dimension n^2 as a K-vector space. If K has property C_1, then since the homogeneous polynomial map $\mathrm{Nrm}_{D/K}$ has only the trivial zero,

$$n = \deg(\mathrm{Nrm}_{D/K}) \geq \dim_K(D) = n^2,$$

i.e., $n = 1$. Thus for a field K with property C_1, the only finite dimensional, division algebra with center K has dimension 1, i.e., D equals K.

(ii). Next suppose that K has property C_2. Clearly $\mathrm{Nrm}_{D/k}(0)$ equals 0. Thus to prove that

$$\mathrm{Nrm}_{D/K} : D \to K$$

is surjective, it suffices to prove that for every nonzero $c \in K$ there exists b in D with $\mathrm{Nrm}_{D/K}(b) = c$. Consider the homogeneous, degree n, polynomial map

$$F_c : D \oplus K \to K, \quad (a, \lambda) \mapsto \mathrm{Nrm}_{D/K}(a) - c\lambda^n.$$

Since

$$\dim_K(D \oplus K) = n^2 + 1 > \deg(F_c)^2,$$

by property C_2 the map F_c has a zero $(a, \lambda) \neq (0, 0)$, i.e., $\mathrm{Nrm}_{D/k}(a) = c\lambda^n$. In particular, λ must be nonzero since otherwise a is a nonzero element of D with $\mathrm{Nrm}_{D/K}(a) = 0$. But then $b = (1/\lambda)a$ is an element of D with $\mathrm{Nrm}_{D/k}(b) = c$. $\square$

It was later recognized, particularly through the work of Merkurjev and Suslin, that these properties of division algebras are equivalent to properties of Galois cohomology.

THEOREM 3.3. [**Ser02**, Proposition 5, §I.3.1], [**Sus84**, Corollary 24.9] *Let K be a field.*

(i) *The cohomological dimension of K is ≤ 1 if and only if for every finite extension L/K, the only division algebra with center L is L itself.*

(ii) *If K is perfect, the cohomological dimension of K is ≤ 2 if and only if for every finite extension L/K, for every division algebra D with center L, the reduced norm map $Nrm_{D/L}$ is surjective.*

CHAPTER 3

Rationally connected fibrations over curves

1. Rationally connected varieties

The theorems of Chevalley-Warning and Tsen-Lang are positive answers to Problem 1.1 for certain classes of fields. It is natural to ask whether these theorems can be generalized for such fields.

PROBLEM 1.1. Let r be a nonnegative integer. Give sufficient geometric conditions on a variety such that for every C_r field K (or perhaps every C_r field satisfying some additional hypotheses) and for every K-variety satisfying the conditions, X has a K-point.

As with Problem 1.1, this problem is quite vague. Nonetheless there are important partial answers. One such answer is the following.

THEOREM 1.2. [**Man86**] [**CT87**] *Let K be a C_1 field and let X be a projective K-variety. If $X \otimes_K \overline{K}$ is birational to $\mathbb{P}^2_{\overline{K}}$ then X has a K-point.*

This begs the question: What (if anything) is the common feature of rational surfaces and of the varieties occurring in the Chevalley-Warning and Tsen-Lang theorems, i.e., complete intersections in $\mathbb{P}^n$ of hypersurfaces of degrees $d_1, \dots, d_r$ with $d_1 + \cdots + d_r \leq n$? One answer is *rational connectedness*. This is a property that was studied by Kollár-Miyaoka-Mori and Campana, cf. [**Kol96**].

DEFINITION 1.3. Let k be an algebraically closed field. An integral (thus nonempty), separated, finite type, k-scheme X is *rationally connected*, resp. *separably rationally connected*, if there exists an integral, finite type k-scheme M and a morphism of k-schemes

$$u : M \times_k \mathbb{P}^1_k \to X, \quad (m, t) \mapsto u(m, t)$$

such that the induced morphism of k-schemes

$$u^{(2)} : M \times_k \mathbb{P}^1_k \times_k \mathbb{P}^1_k \to X \times_k X, \quad (m, t_1, t_1) \mapsto (u(m, t_1), u(m, t_2))$$

is surjective, resp. surjective and generically smooth.

In a similar way, X is *rationally chain connected*, resp. *separably rationally chain connected*, if for some integer $m \geq 1$, the analogous property holds after replacing $\mathbb{P}^1_k$ by the proper, connected, nodal, reducible curve C_m which is a chain of m smooth rational curves.

In other references, the condition on $u^{(2)}$ is that it be dominant rather than surjective. In the following arguments, it is preferrable to demand that $u^{(2)}$ is surjective. When X is proper and $\text{char}(k) = 0$, the definitions turn out to be equivalent.

Figure 1. Every pair of points in a rationally connected variety lie in an image of the projective line

Figure 1 shows a rationally connected variety, where every pair of points is contained in an image $u(\mathbb{P}^1)$ of the projective line.

The definition of rational connectedness, resp. rational chain connectedness, mentions a particular parameter space M. However, using the general theory of Hilbert schemes, it suffices to check that every pair (x_1, x_2) of K-points of $X \otimes_k K$ is contained in some rational K-curve, resp. a chain of rational K-curves, (not necessarily from a fixed parameter space) for one *sufficiently large*, algebraically closed, field extension K/k, i.e., for an algebraically closed extension K/k such that for every countable collection of proper closed subvarieties $Y_i \subsetneq X$, there exists a K-point of X contained in none of the sets Y_i. For instance, K/k is sufficiently large if K is uncountable or if K/k contains the fraction field $k(X)/k$ as a subextension.

A very closely related property is the existence of a very free rational curve. For a d-dimensional variety X, a *very free rational curve* is a morphism

$$f : \mathbb{P}^1_k \to X_{\text{smooth}}$$

into the smooth locus of X such that f^*T_X is ample, i.e.,

$$f^*T_X \cong \mathcal{O}_{\mathbb{P}^1_k}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_k}(a_d), \ a_1, \ldots, a_d > 0.$$

Definition 1.4. Let k be a field and let X be a quasi-projective k-scheme. Denote by X_{smooth} the smooth locus of X. The *very free locus* $X_{\text{v.f.}}$ of X is the union of the images in X_{smooth} of all very free rational curves to $X_{\text{smooth}} \otimes_k K$

as K/k varies over all algebraically closed extensions. More generally, for a flat, quasi-projective morphism of schemes,

$$\pi : X \to B,$$

denoting by $X_{\pi,\text{smooth}}$ the smooth locus of the morphism π, the *very free locus* $X_{\pi,\text{v.f.}}$ is the union in $X_{\pi,\text{smooth}}$ of the images of every very free rational curve in every geometric fiber of $X_{\pi,\text{smooth}}$ over B.

The next theorem explains the relation of these different properties.

THEOREM 1.5. [**Kol96**, §IV.3], [**HT06**], *Unless stated otherwise, all varieties below are d-dimensional, reduced, irreducible, quasi-projective schemes over an algebraically closed field k.*

(0) *In characteristic* 0*, every rationally connected variety is separably rationally connected.*

(i) *For every flat, proper morphism $\pi : X \to B$ (not necessarily of quasi-projective varieties over a field), the subset of B parameterizing points whose geometric fiber is rationally chain connected is stable under specialization. (If one bounds the degree of the chains with respect to a relatively ample invertible $\mathcal{O}_X$-module, then it is a closed subset.)*

(ii) *The very free locus of a quasi-projective variety is open. More generally, for every flat, quasi-projective morphism, $\pi : X \to B$, the subset $X_{\pi,v.f.}$ of $X_{\pi,smooth}$ is an open subset.*

(iii) *The very free locus $X_{v.f.}$ of a quasi-projective variety is (separably) rationally connected in the following strong sense. For every positive integer N, for every positive integer m, and for every positive integer a, for every collection of distinct closed points $t_1, \ldots, t_N \in \mathbb{P}^1_k$, for every collection of closed points $x_1, \ldots, x_N \in X_{v.f.}$, and for every specification of an m-jet of a smooth curve in X at each point x_i, there exists a morphism*

$$f : \mathbb{P}^1_k \to X_{v.f.}$$

such that for every $i = 1, \ldots, N$, f is unramified at t_i, $f(t_i)$ equals x_i and the m-jet of t_i in $\mathbb{P}^1_k$ maps isomorphically to the specified m-jet at x_i, and

$$f^*T_X \cong \mathcal{O}_{\mathbb{P}^1_k}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_k}(a_d), \ a_1, \ldots, a_d \geq a.$$

(iv) *Every rational curve in X_{smooth} intersecting $X_{v.f.}$ is contained in $X_{v.f.}$. Thus for every smooth, rationally chain connected variety, if X contains a very free rational curve then $X_{v.f.}$ equals all of X.*

(v) *A proper variety X is rationally chain connected if it is* generically *rationally chain connected, i.e., if there exists a morphism u as in the definition such that $u^{(2)}$ is dominant (but not necessarily surjective).*

(vi) *For the morphism $u : M \times_k \mathbb{P}^1_k \to X$, let l be a closed point of M such that $u_l : \mathbb{P}^1 \to X$ has image in X_{smooth} and such that $u^{(2)}$ is smooth at (l, t_1, t_2) for some $t_1, t_2 \in \mathbb{P}^1_k$. Then the morphism u_l is very free. Thus an irreducible, quasi-projective variety X contains a very free curve if and only if there is a separably rationally connected open subset of X_{smooth}. Also, a smooth, quasi-projective variety X in characteristic* 0 *which is generically rationally connected contains a very free morphism.*

(vii) *For a surjective morphism $f : X \to Y$ of varieties over an algebraically closed field, if X is rationally connected, resp. rationally chain connected, then also Y is rationally connected, resp. rationally chain connected.*
(viii) *For a birational morphism $f : X \to Y$ of proper varieties over an algebraically closed field, if Y is rationally connected then X is rationally chain connected. If the characteristic is zero, then X is rationally connected.*

REMARK 1.6. Item (ii) is proved in Proposition 3.6. The *generic case* of Item (iii), which is all we will need, is proved in Proposition 3.7. The complete result was proved by Hassett and Tschinkel, [**HT06**]. Item (iv) follows from Corollary 3.8. The remaining items are not proved, nor are they used in the proof of the main theorem. For the most part they are proved by similar arguments; complete proofs are in [**Kol96**, §IV.3].

Rational connectedness is analogous to path connectedness in topology, and satisfies the analogues of many properties of path connectedness. One property of path connectedness is this: for a fibration of CW complexes, if the base space and the fibers are path connected, then also the total space is path connected. This led to two conjectures by Kollár, Miyaoka and Mori.

CONJECTURE 1.7. [**Kol96**, Conjecture IV.5.6] Let $\pi : X \to B$ be a surjective morphism of smooth, projective schemes over an algebraically closed field of characteristic 0. If both B and a general fiber of π are rationally connected, then X is also rationally connected.

Conjecture 1.7 is implied by the following conjecture about rationally connected fibrations over curves.

CONJECTURE 1.8. [**Kol96**, Conjecture IV.6.1.1] Let $\pi : X \to B$ be a surjective morphism of projective schemes over an algebraically closed field of characteristic 0. If B is a smooth curve and if a general fiber of f is rationally connected, then there exists a morphism $s : B \to X$ such that $\pi \circ s$ equals Id_B, i.e., s is a section of π.

Our next goal is to prove the following result.

THEOREM 1.9. [**GHS03**] *Conjecture 1.8 of Kollár-Miyaoka-Mori is true. Precisely, let k be an algebraically closed field of characteristic 0 and let $\pi : X \to B$ be a surjective morphism from a normal, projective k-scheme X to a smooth, projective, connected k-curve B. If the geometric generic fiber $X_{\overline{\eta}_B}$ is a normal, integral scheme whose smooth locus contains a very free curve, then there exists a morphism $s : B \to X$ such that $\pi \circ s$ equals Id_B.*

This was generalized by A. J. de Jong to the case that k is algebraically closed of arbitrary characteristic, [**dJS03**]. The key difference has to do with extensions of valuation rings in characteristic 0 and in positive characteristic. Given a flat morphism of smooth schemes in characteristic 0, $\pi : U \to B$, and given codimension 1 points η_D of U and η_Δ of B with $\pi(\eta_D) = \eta_\Delta$, the induced local homomorphism of stalks $\pi_U^* : \widehat{\mathcal{O}}_{B,\eta_\Delta} \to \widehat{\mathcal{O}}_{U,\eta_D}$, is equivalent to

$$k(\Delta)\,[\![t]\!] \to k(D)\,[\![r]\!]\,, \quad t \mapsto ur^m$$

for a unit u and a positive integer m, cf. the proof of Lemma 4.3 below. In particular, it is *rigid* in the sense that $t \mapsto ur^m + vr^{m+1} + \dots$ is equivalent to

$t \mapsto ur^m$. However, extensions of positive characteristic valuation rings are not rigid, e.g., $t \mapsto r^p + v_1 r^{p+1}$ is equivalent to $t \mapsto r^p + v_2 r^{p+1}$ only if $v_1 = v_2$. But there is a weak rigidity of local homomorphisms, Krasner's lemma in the theory of non-Archimedean valuations. This is a key step in the generalization to positive characteristic.

Of course when k has characteristic 0, then since X is normal the fiber $X_{\overline{\eta}_B}$ is automatically normal. If X is also smooth (which can be achieved thanks to resolution of singularities in characteristic 0), then also $X_{\overline{\eta}_B}$ is smooth. Then the hypothesis on $X_{\overline{\eta}_B}$ is equivalent to rational connectedness.

2. Outline of the proof

The proof that follows is based on a proof by T. Graber, J. Harris and myself (not quite the version we chose to publish) together with several major simplifications due to A. J. de Jong. The basic idea is to choose a smooth curve $C \subset X$ such that $\pi|_C : C \to B$ is finite, and then try to deform C as a curve in X until it specializes to a reducible curve in X, one component of which is the image of a section s of π. Here are some definitions that make this precise.

Definition 2.1. Let $\pi_C : C \to B$ be a finite morphism of smooth, projective k-curves. A *linked curve with handle* C is a reduced, connected, projective curve C_{link} with irreducible components

$$C_{\text{link}} = C \cup L_1 \cup \cdots \cup L_m$$

together with a morphism

$$\pi_{C,\text{link}} : C_{\text{link}} \to B$$

such that

(i) $\pi_{C,\text{link}}$ restricts to π_C on the component C,
(ii) the restriction of $\pi_{C,\text{link}}$ to each *link* component L_i is a constant morphism with image b_i, where $b_1, \dots, b_m$ are distinct closed points of B,
(iii) and each link L_i is a smooth, rational curve intersecting C in a finite number of nodes of C_{link}.

If every link L_i intersects C in a single node of C_{link}, then $(C_{\text{link}}, \pi_{C,\text{link}})$ is called a *comb* and the links L_i are called *teeth.* For combs we will use the notation C_{comb} rather than C_{link}.

A *one-parameter deformation* of a linked curve $(C_{\text{link}}, \pi_{C,\text{link}})$ is a datum of a smooth, connected, pointed curve $(\Pi, 0)$ and a projective morphism

$$(\rho, \pi_{\mathcal{C}}) : \mathcal{C} \to \Pi \times_k B$$

such that ρ is flat and such that $\mathcal{C}_0 := \rho^{-1}(0)$ together with the restriction of $\pi_{\mathcal{C}}$ equals the linked curve $(C_{\text{link}}, \pi_{C,\text{link}})$.

A one-parameter deformation *specializes to a section curve* if there exists a closed point $\infty \in \Pi$ and an irreducible component B_i of $\mathcal{C}_\infty := \rho^{-1}(\infty)$ such that

(i) $\mathcal{C}_\infty$ is reduced at the generic point of B_i
(ii) and the restriction of $\pi_{\mathcal{C}}$ to B_i is an isomorphism

$$\pi_{\mathcal{C}}|_{B_i} : B_i \xrightarrow{\cong} B.$$

FIGURE 2. A linked curve with handle C and some links intersecting C in more than 1 point

Given a linked curve, a one-parameter deformation of the linked curve and a B-morphism $j : C_{\text{link}} \to X$, an *extension* of j is an open neighborhood of 0, $0 \in N \subset B$ and a B-morphism

$$j_N : \mathcal{C}_N \to X, \quad \mathcal{C}_N := \rho^{-1}(N)$$

restricting to j on $\mathcal{C}_0 = C_{\text{link}}$.

Figure 2 shows a linked curve with some links intersecting the handle in more than 1 point. And Figure 3 shows a comb, where every tooth intersects the handle precisely once.

For the purposes of producing a section, the particular parameter space $(\Pi, 0)$ of the one-parameter deformation is irrelevant. Thus, it is allowed to replace the one-parameter deformation by the new one-parameter deformation obtained from a finite base change $(\Pi', 0') \to (\Pi, 0)$. The following lemma is straightforward.

LEMMA 2.2. *Let* $(\Pi, 0, \infty)$ *together with* $(\rho, \pi_{\mathcal{C}}) : \mathcal{C} \to \Pi \times_k B$ *be a one-parameter deformation of* $(C_{link}, \pi_{C,link})$ *specializing to a section curve* B_i. *For every morphism of* 2*-pointed, smooth, connected curves*

$$(\Pi', 0', \infty') \to (\Pi, 0, \infty),$$

the base change morphism

$$\Pi' \times_\Pi \mathcal{C} \to \Pi' \times_k B$$

is also a one-parameter deformation of $(C_{link}, \pi_{C,link})$ *specializing to the section curve* B_i.

FIGURE 3. A comb is a linked curve where every link, or tooth, intersects the handle precisely once

The usefulness of these definitions is the following simple consequence of the valuative criterion of properness.

LEMMA 2.3. *Let (C_{link}, π_{link}) be a linked curve together with a B-morphism $j : C_{link} \to X$. If there exists a one-parameter deformation of the linked curve specializing to a section curve and if there exists an extension of j, then there exists a section $s : B \to X$ of π.*

PROOF. Let R denote the stalk $\mathcal{O}_{\mathcal{C},\eta_{B_i}}$ of $\mathcal{O}_{\mathcal{C}}$ at the generic point η_{B_i} of B_i. By the hypotheses on $\mathcal{C}$ and B_i, R is a discrete valuation ring with residue field $\kappa = k(B_i)$ and fraction field $K = k(\mathcal{C})$. The restriction of j_N to the generic point of $\mathcal{C}$ is a B-morphism

$$j_K : \text{Spec } K \to X.$$

Because $\pi : X \to B$ is proper, by the valuative criterion of properness the B-morphism j_K extends to a B-morphism

$$j_R : \text{Spec } R \to X,$$

which in turn gives a B-morphism from the residue field Spec κ to X, i.e., a rational B-map

$$j_{B_i} : B_i \supset U \to X, \quad U \subset B_i \text{ a dense, Zariski open.}$$

Finally, because B_i is a smooth curve, the valuative criterion applies once more and this rational transformation extends to a B-morphism

$$j_{B_i} : B_i \to X.$$

Because $\pi_{\mathcal{C}}|_{B_i} : B_i \to B$ is an isomorphism, there exists a unique B-morphism

$$s : B \to X$$

such that $j_{B_i} = s \circ \pi_{\mathcal{C}}|_{B_i}$. The morphism s is a section of π. $\square$

Thus the proof of the theorem breaks into three parts:

(i) find a "good" linked curve $j : C_{\text{link}} \to X$,
(ii) find a one-parameter deformation of the linked curve specializing to a section curve,
(iii) and find an extension of j to the one-parameter deformation.

The first step in finding $j : C_{\text{link}} \to X$ is to form a curve C_{init} which is an intersection of X with $\dim(X) - 1$ general hyperplanes in projective space. By Bertini's theorem, if the hyperplanes are sufficiently general, then C_{init} will satisfy any reasonable transversality property. Moreover, there is a technique due to Kollár-Miyaoka-Mori – the smoothing combs technique – for improving C_{init} to another curve $C \subset X$ still satisfying the transversality property and also satisfying a positivity property with respect to the vertical tangent bundle of $\pi : X \to B$.

Unfortunately, even after such an improvement, there may be no one-parameter deformation of $\pi|_C : C \to B$ specializing to a section curve. However, after attaching sufficiently many link components over general closed points of B, there does exist a one-parameter deformation of C_{link} specializing to a section curve. This is one aspect of the well-known theorem that for a fixed base curve B and for a fixed degree d, if the number β of branch points is sufficiently large the Hurwitz scheme of degree d covers of B with β branch points is irreducible. (This was proved by Hurwitz when $g(B) = 0$, [**Hur91**], proved by Richard Hamilton for arbitrary genus in his thesis, and periodically reproved ever since, cf. [**GHS02**].) Because the general fibers of $\pi : X \to B$ are rationally connected, the inclusion $C \subset X$ extends to a B-morphism $j : C_{\text{link}} \to X$.

Finally the positivity property mentioned above implies j extends to the one-parameter deformation, at least after base change by a morphism $\Pi' \to \Pi$.

3. Hilbert schemes and smoothing combs

The smoothing combs technique of Kollár-Miyaoka-Mori depends on a result from the deformation theory of Hilbert schemes. Here is the setup for this result. Let $Y \to S$ be a flat, quasi-projective morphism and let

$$(\rho_{\text{Hilb}} : \text{Hilb}(Y/S) \to S, \text{Univ}(Y/S) \subset \text{Hilb}(Y/S) \times_S T)$$

be universal among pairs $(\rho : T \to S, Z \subset T \times_S Y)$ of an S-scheme T and a closed subscheme $Z \subset T \times_S Y$ such that $Z \to T$ is proper, flat and finitely presented. In other words, $\text{Hilb}(Y/S)$ is the *relative Hilbert scheme* of Y over S.

In particular, for every field K the K-valued points of $\text{Hilb}(Y/S)$ are naturally in bijection with pairs (s, Z) of a K-valued point s of S and a closed subscheme Z of $Y_s := \{s\} \times_S Y$. The closed immersion $Z \to Y_s$ is a *regular embedding* if at every point of Z the stalk of the ideal sheaf $\mathcal{I}_{Z/Y_s}$ is generated by a regular sequence of elements in the stalk of $\mathcal{O}_{Y_s}$. In this case the *conormal sheaf* $\mathcal{I}_{Z/Y_s}/\mathcal{I}^2_{Z/Y_s}$ is a locally free $\mathcal{O}_Z$-module, and hence also the *normal sheaf*

$$\mathcal{N}_{Z/Y_s} := \mathit{Hom}_{\mathcal{O}_Z}(\mathcal{I}_{Z/Y_s}/\mathcal{I}^2_{Z/Y_s}, \mathcal{O}_S)$$

is a locally free $\mathcal{O}_Z$-module. The regular embeddings which arise in the proof of Theorem 1.9 are precisely closed immersions of at-worst-nodal curves in a smooth variety.

PROPOSITION 3.1. [**Kol96**, Theorem I.2.10, Lemma I.2.12.1, Proposition I.2.14.2] *If $Z \subset Y_s$ is a regular embedding and if $h^1(Z, \mathcal{N}_{Z/Y_s})$ equals 0, then $Hilb(Y/S)$ is smooth over S at (s, Z).*

There is a variation of this proposition which is also useful. There is a *flag Hilbert scheme* of Y over S, i.e., a universal pair

$$(\rho_{\mathrm{fHilb}} : \mathrm{fHilb}(Y/S) \to S, \mathrm{Univ}_1(Y/S) \subset \mathrm{Univ}_2(Y/S) \subset \mathrm{Hilb}(Y/S) \times_S T)$$

among all pairs $(\rho : T \to S, Z_1 \subset Z_2 \subset T \times_S Y)$ of an S-scheme T and a nested pair of closed subschemes $Z_1 \subset Z_2 \subset T \times_S Y$ such that for $i = 1, 2$, the projection $Z_i \to T$ is proper, flat and finitely presented. There are obvious forgetful morphisms

$$F_i : \mathrm{fHilb}(Y/S) \to \mathrm{Hilb}(Y/S), \quad F_i(s, Z_1, Z_2) = (s, Z_i).$$

PROPOSITION 3.2. *Let K be a field and let (s, Z_1, Z_2) be a K-point of $fHilb(Y/S)$. If each closed immersion $Z_1 \subset Z_2$ and $Z_2 \subset Y_s$ is a regular embedding and if*

$$h^1(Z_2, \mathcal{N}_{Z_2/Y_s}) = 0, h^1(Z_1, \mathcal{N}_{Z_1/Z_2}) = 0, \text{ and if } h^i(Z_2, \mathcal{I}_{Z_1/Z_2} \cdot \mathcal{N}_{Z_2/Y_2}) = 0 \text{ for } i = 1, 2,$$

then $fHilb(Y/S)$ is smooth over S at (s, Z_1, Z_2), for each $i = 1, 2$, $Hilb(Y/S)$ is smooth over S at (s, Z_i), and each forgetful morphism $F_i : fHilb(Y/S) \to Hilb(Y/S)$ is smooth at (s, Z_1, Z_2).

PROOF. Since $h^1(Z_2, \mathcal{N}_{Z_2/Y_s})$ equals 0, $\mathrm{Hilb}(Y/S)$ is smooth at (s, Z_2) by Proposition 3.1. It is easy to see that the forgetful morphism F_2 is equivalent to the relative Hilbert scheme $\mathrm{Hilb}(\mathrm{Univ}(Y/S)/\mathrm{Hilb}(Y/S))$ over $\mathrm{Hilb}(Y/S)$. Thus, applying Proposition 3.1 to this Hilbert scheme, the vanishing of $h^1(Z_1, \mathcal{N}_{Z_1/Z_2})$ implies F_2 is smooth at (s, Z_1, Z_2). Since a composition of smooth morphisms is smooth, also $\mathrm{fHilb}(Y/S)$ is smooth over S at (s, Z_1, Z_2). The long exact sequence of cohomology associated to the short exact sequence

$$0 \longrightarrow \mathcal{I}_{Z_1/Z_2} \cdot \mathcal{N}_{Z_2/Y_s} \longrightarrow \mathcal{N}_{Z_2/Y_s} \longrightarrow \mathcal{N}_{Z_2/Y_s}|_{Z_1} \longrightarrow 0$$

implies that $h^1(Z_1, \mathcal{N}_{Z_2/Y_s}|_{Z_1})$ equals $h^1(Z_2, \mathcal{N}_{Z_2/Y_s})$, which is 0. Thus, the long exact sequence of cohomology associated to

$$0 \longrightarrow \mathcal{N}_{Z_1/Z_2} \longrightarrow \mathcal{N}_{Z_1/Y_s} \longrightarrow \mathcal{N}_{Z_2/Y_s}|_{Z_1} \longrightarrow 0$$

implies that $h^1(Z_2, \mathcal{N}_{Z_1/Y_2})$ equals 0. So again by Proposition 3.1, $\mathrm{Hilb}(Y/S)$ is smooth over S at (s, Z_1). Finally, F_1 is a morphism of smooth S-schemes at (s, Z_1, Z_2). Thus, to prove F_1 is smooth, it suffices to prove it is surjective on Zariski tangent vector spaces. This follows from the vanishing of $h^1(Z_2, \mathcal{I}_{Z_1/Z_2} \cdot \mathcal{N}_{Z_2/Y_s})$. □

Another ingredient in the smoothing combs technique is a a simple result about *elementary transforms* of locally free sheaves on a curve: the higher cohomology of the sheaf becomes zero after applying elementary transforms at sufficiently many points.

LEMMA 3.3. *Let C be a projective, at-worst-nodal, connected curve over a field k and let $\mathcal{E}$ be a locally free $\mathcal{O}_C$-module.*

(i) *There exists a short exact sequence of coherent sheaves,*

$$0 \to \mathcal{F}^\vee \to \mathcal{E}^\vee \to \mathcal{T} \to 0$$

such that $\mathcal{T}$ is a torsion sheaf with support in C_{smooth} and such that $h^1(C, \mathcal{F})$ equals 0.

(ii) *Inside the parameter space of torsion quotients $q : \mathcal{E}^\vee \twoheadrightarrow \mathcal{T}$ with support in C_{smooth}, denoting*

$$\mathcal{F}^\vee := Ker(\mathcal{E}^\vee \to \mathcal{T}) \text{ and } \mathcal{F} := Hom_{\mathcal{O}_C}(\mathcal{F}^\vee, \mathcal{O}_C),$$

the subset parameterizing quotients for which $h^1(C, \mathcal{F}) = 0$ is an open subset.

(iii) *If $h^1(C, \mathcal{F})$ equals 0, then for every short exact sequence of coherent sheaves*

$$0 \to \mathcal{G}^\vee \to \mathcal{E}^\vee \xrightarrow{q'} \mathcal{S} \to 0$$

admitting a morphism $r : \mathcal{S} \to \mathcal{T}$ of torsion sheaves with support in C_{smooth} for which $q = r \circ q'$, $h^1(C, \mathcal{G})$ equals 0.

Proof. **(i)** By Serre's vanishing theorem, there exists an effective, ample divisor D in the smooth locus of C such that $h^1(C, \mathcal{E}(D))$ equals 0. Define $\mathcal{F} = \mathcal{E}(D)$, define $\mathcal{E} \to \mathcal{F}$ to be the obvious morphism $\mathcal{E} \to \mathcal{E}(D)$, and define $\mathcal{T}$ to be the cokernel of $\mathcal{F}^\vee \to \mathcal{E}^\vee$.

(ii) This follows immediately from the semicontinuity theorem, cf. [**Har77**, §III.12].

(iii) There exists an injective morphism of coherent sheaves $\mathcal{F} \to \mathcal{G}$ with torsion cokernel. Because $h^1(C, \mathcal{F})$ equals 0 and because h^1 of every torsion sheaf is zero, the long exact sequence of cohomology implies that also $h^1(C, \mathcal{G})$ equals 0. □

It is worth noting one interpretation of the sheaf $\mathcal{F}$ associated to a torsion quotient $\mathcal{T}$. Assume that $\mathcal{T}$ is isomorphic to a direct sum of skyscraper sheaves at n distinct points $c_1, \dots, c_n$ of C_{smooth}. (Inside the parameter space of torsion quotients, those with this property form a dense, open subset.) For each point c_i, the linear functional $\mathcal{E}^\vee|_{c_i} \twoheadrightarrow \mathcal{T}|_{c_i}$ gives a one-dimensional subspace $\text{Hom}_k(\mathcal{T}|_{c_i}, k) \hookrightarrow \mathcal{E}|_{c_i}$. The sheaf $\mathcal{F}$ is precisely the sheaf of rational sections of $\mathcal{E}$ having at worst a simple pole at each point c_i in the direction of this one-dimensional subspace of $\mathcal{E}|_{c_i}$. This is often called an *elementary transform up* of $\mathcal{E}$ at the point c_i in the specified direction. So Lemma 3.3 says that h^1 becomes zero after sufficiently many elementary transforms up at general points in general directions.

This interpretation is useful because the normal sheaf of a reducible curve can be understood in terms of elementary transforms up. To be precise, let Y be a k-scheme, let C be a proper, nodal curve, let C_0 be a closed subcurve (i.e., a union of irreducible components of C), and let $j : C \to Y$ be a regular embedding such that Y is smooth at every node $p_1, \dots, p_n$ of C which is contained in C_0 and which is not a node of C_0. Then $j_0 : C_0 \to Y$ is also a regular embedding and both $\mathcal{N}_{C/Y}|_{C_0}$ and $\mathcal{N}_{C_0/Y}$ are locally free sheaves on C_0. For each i, there is a branch C_i of C at p_i other than C_0. Denote by T_{C_i,p_i} the tangent direction of this branch in T_{Y,p_i}.

Lemma 3.4. [**GHS03**, Lemma 2.6] *The restriction $\mathcal{N}_{C/Y}|_{C_0}$ equals the sheaf of rational sections of $\mathcal{N}_{C_0/Y}$ having at most a simple pole at each point p_i in the normal direction determined by T_{C_i,p_i}.*

Proof. The restrictions of $\mathcal{N}_{C/Y}|_{C_0}$ and $\mathcal{N}_{C_0/Y}$ to the complement of $\{p_1, \dots, p_n\}$ are canonically isomorphic. The lemma states that this canonical isomorphism is the restriction of an injection $\mathcal{N}_{C_0/Y} \hookrightarrow \mathcal{N}_{C/Y}|_{C_0}$ which identifies $\mathcal{N}_{C/Y}|_{C_0}$ with the sheaf of rational sections, etc. This local assertion can be verified in a formal neighborhood of each node p_i.

Locally near p_i, $C \to Y$ is formally isomorphic to the union of the two axes inside a 2-plane inside an n-plane, i.e., the subscheme of $\mathbb{A}^n_k$ with ideal $I_{C/Y} = \langle x_1x_2, x_3, \dots, x_n\rangle$. The branch C_0 corresponds to just one of the axes, e.g., the subscheme of $\mathbb{A}^n_k$ with ideal $I_{C_0/Y} = \langle x_2, x_3, \dots, x_n\rangle$. The tangent direction of the other branch C_i is spanned by $(0, 1, 0, \dots, 0)$. Thus it is clear that $I_{C/Y}/I_{C_0/Y} \cdot I_{C/Y}$ is the submodule of $I_{C_0/Y}/I^2_{C_0/Y}$ of elements whose fiber at 0 is contained in the annihilator of T_{C_i,p_i}. Dualizing gives the lemma. $\square$

The final bit of deformation theory needed has to do with deforming nodes. Let C be a proper, nodal curve and let $j : C \to Y$ be a regular embedding. Let p be a node of C and assume that Y is smooth at p. There are two branches C_1 and C_2 of C at p (possibly contained in the same irreducible component of C). The sheaf

$$\mathcal{T} := \mathit{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)$$

is a skyscraper sheaf supported at p and with fiber canonically identified to

$$\mathcal{T}|_p = T_{C_1,p} \otimes_k T_{C_2,p}.$$

The following lemma is as much definition as lemma.

Lemma 3.5. *There exists a quotient of coherent sheaves*

$$\mathcal{N}_{C/Y} \twoheadrightarrow \mathcal{T}$$

such that for both $i = 1, 2$ the quotient $\mathcal{N}_{C/Y}|_{C_i}/\mathcal{N}_{C_i/Y}$ equals $\mathcal{T}$. A first-order deformation of $C \subset Y$, i.e., a global section of $\mathcal{N}_{C/Y}$ is said to smooth the node p to first-order *if the image of the section in $T_{C_1,p} \otimes_k T_{C_2,p}$ is nonzero. For a deformation*

$$\mathcal{C} \subset \Pi \times_k Y$$

of $C \subset Y$ over a smooth pointed curve $(\Pi, 0)$ (i.e., $\mathcal{C}_0 = C$), if the associated first-order deformation of $C \subset Y$ smooths the node p to first-order, then p is not contained in the closure of the singular locus of the projection,

$$(\Pi - \{0\}) \times_\Pi \mathcal{C} \to (\Pi - \{0\})$$

i.e., a general fiber $\mathcal{C}_t$ of the deformation smooths the node.

This is a well-known result. A good reference for this result, and many other results about deformations of singularities, is [**Art76**], particularly §I.6. Here is a brief remark on the proof. Because $C \subset Y$ is a regular embedding, the conormal sequence is exact on the left, i.e.,

$$0 \longrightarrow \mathcal{I}_{C/Y}/\mathcal{I}^2_{C/Y} \longrightarrow \Omega_Y|_C \longrightarrow \Omega_C \longrightarrow 0$$

is a short exact sequence. Applying global Ext, there is a connecting map

$$\delta : H^0(C, \mathcal{N}_{C/Y}) \to \mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C).$$

There is also a local-to-global sequence for global Ext inducing a map

$$\mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C) \to H^0(C, \mathit{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)) = H^0(C, \mathcal{T}) = T_{C_1,p} \otimes_k T_{C_2,p}.$$

The composition of these two maps is precisely the map on global sections associated to $\mathcal{N}_{C/Y} \to \mathcal{T}$. The global Ext group is identified with the first-order deformations of C as an abstract scheme, and the Ext term is identified with the first-order deformations of the node. It is worth noting that even if the first-order deformation does not smooth the node, the full deformation $\mathcal{C} \subset \Pi \times_k Y$ may smooth the node if the total space $\mathcal{C}$ is singular at $(0, p)$.

The first result using the smoothing combs technique is the following.

PROPOSITION 3.6. *Let Y be a quasi-projective scheme over an algebraically closed field k. The very free locus $Y_{v.f.}$ is an open subset of Y. More generally, for a flat, quasi-projective morphism $\pi : Y \to B$, the relative very free locus $Y_{\pi,v.f.}$ is an open subset of Y.*

Let Y be an irreducible, quasi-projective scheme over an algebraically closed field k. Denote by t_1, resp. t_2, the closed point of $\mathbb{P}^1_k$, $t_1 = 0$, resp. $t_2 = \infty$. Let y_1 and y_2 be closed points of $Y_{v.f.}$, let a and k be nonnegative integers, and let there be given curvilinear k-jets in Y at each of y_1 and y_2. If the given k-jets are general among all curvilinear k-jets at y_1 and y_2, then there exists a morphism

$$f : (\mathbb{P}^1_k, t_1, t_2) \to (Y_{v.f.}, y_1, y_2)$$

mapping the k-jet of $\mathbb{P}^1$ at t_i isomorphically to the given k-jet at y_i for $i = 1, 2$ and such that

$$f^*T_Y \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n), \ a_1, \ldots, a_n \geq a.$$

PROOF. In the absolute case, resp. relative case, the very free locus $Y_{\text{v.f.}}$, resp. $Y_{\pi,\text{v.f.}}$, is defined to be the same as the very free locus of the smooth locus Y_{smooth}, resp. $Y_{\pi,\text{smooth}}$. Since the smooth locus is open in Y, and since an open subset of an open subset is an open subset, it suffices to prove the very free locus is open under the additional hypothesis that Y is smooth, resp. that π is smooth.

By the definition of $Y_{\text{v.f.}}$, for each $i = 1, 2$ there exists a very free morphism

$$f_i : (\mathbb{P}^1, 0) \to (Y_{\text{v.f.}}, y_i), \quad f_i^*T_Y \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n), a_1, \ldots, a_n \geq 1.$$

In particular, for each $i = 1, 2$, $h^1(\mathbb{P}^1, f_i^*T_Y(-\underline{0} - \underline{\infty}))$ equals 0, where $\underline{0}$, resp. $\underline{\infty}$, is the Cartier divisor of the point 0, resp. ∞, in $\mathbb{P}^1$. Since the normal sheaf of f_i is a quotient of $f_i^*T_Y$, also $h^1(\mathbb{P}^1, \mathcal{N}_{f_i}(-\underline{0} - \underline{\infty}))$ equals 0. Thus, applying Proposition 3.2 where $Z_1 = \{0, \infty\}$ and $Z_2 = \mathbb{P}^1$, there exist deformations of the morphism f_i such that $f_i(0)$ equals y_i and $f_i(\infty)$ is any point in a nonempty Zariski open subset of Y. The same argument holds in the relative case.

Next assume that Y is irreducible and quasi-projective. Then the smooth locus Y_{smooth} is also irreducible (or empty). Thus, by the same argument as above, a proof of the second result for smooth varieties implies the second result in general. Thus assume Y is also smooth.

Since Y is irreducible, the open for $i = 1$ intersects the open for $i = 2$. Thus there exist very free morphisms f_1 and f_2 such that $f_1(\infty) = f_2(\infty)$. Let C be the nodal curve with two irreducible components C_1 and C_2 each isomorphic to $\mathbb{P}^1$ and with a single node which, when considered as a point in either C_1 or C_2, corresponds to ∞ in $\mathbb{P}^1$. Let $f : C \to Y$ be the unique morphism whose restriction to each component C_i equals f_i. Denote by

$$0 \longrightarrow \mathcal{N}'_{C/Y} \longrightarrow \mathcal{N}_{C/Y} \longrightarrow \mathcal{T} \longrightarrow 0$$

the short exact sequence coming from Lemma 3.5. Using Lemma 3.4, there is an exact sequence

$$0 \longrightarrow \mathcal{N}_{C/Y}|_{C_1}(-\underline{0} - \underline{\infty}) \longrightarrow \mathcal{N}'_{C/Y}(-y_1 - y_2) \longrightarrow \mathcal{N}_{C_2/Y}(-\underline{0}) \longrightarrow 0$$

and an exact sequence

$$0 \longrightarrow \mathcal{N}_{C_1/Y}(-\underline{0} - \underline{\infty}) \longrightarrow \mathcal{N}_{C/Y}|_{C_1}(-\underline{0} - \underline{\infty}) \longrightarrow \kappa_\infty \longrightarrow 0$$

where κ_∞ is the skyscraper sheaf on C_1 supported at ∞. Applying the long exact sequence of cohomology, using that $h^1(C_i, \mathcal{N}_{C_i/Y}(-\underline{0} - \underline{\infty}))$ equals 0 for $i = 1, 2$, and chasing diagrams, this finally gives that $h^1(C, \mathcal{N}'_{C/Y}(-y_1 - y_2))$ also equals 0.

This has two consequences. First, this implies $h^1(C, \mathcal{N}_{C/Y}(-y_1 - y_2))$ equals 0, and thus the space of deformations of C containing y_1 and y_2 is smooth by Proposition 3.2. And second, the map

$$H^0(C, \mathcal{N}_{C/Y}(-y_1 - y_2)) \to T_{C_1,\infty} \otimes T_{C_2,\infty}$$

is surjective. Thus there exist first-order deformations of C containing y_1 and y_2 and smoothing the node at ∞. Since the space of deformations containing y_1 and y_2 is smooth, this first-order deformation is the one associated to a one-parameter deformation

$$\mathcal{C} \subset \Pi \times_k Y$$

of $[C]$ over a smooth, pointed curve $(\Pi, 0)$ (e.g., choose Π to be a general complete intersection curve in the smooth deformation space containing the given Zariski tangent vector). By Lemma 3.5, for a general point t of Π, $\mathcal{C}_t$ is a smooth, connected curve containing y_1 and y_2. Since the arithmetic genus of C is 0, the arithmetic genus of $\mathcal{C}_t$ is also 0, i.e., $\mathcal{C}_t \cong \mathbb{P}^1_k$. Let

$$f_1 : \mathbb{P}^1_k \to \mathcal{C}_t$$

be an isomorphism with $f_1(t_i) = y_i$ for $i = 1, 2$. Because $h^1(C, \mathcal{N}_{C/Y}(-y_1 - y_2))$ equals 0, by the semicontinuity theorem also $h^1(\mathcal{C}_t, \mathcal{N}_{\mathcal{C}_t/Y}(-y_1 - y_2))$ equals 0. This implies that

$$f_1^* T_Y \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n), \text{ for integers } a_1, \ldots, a_n \geq 1.$$

Next, for every integer a, let $g_a : (\mathbb{P}^1 \to \mathbb{P}^1$ be the morphism $z \mapsto z^a$. Then the composition $f_a = f_1 \circ g_1$ is a morphism

$$f_a : (\mathbb{P}^1_k, t_1, t_2) \to (Y_{\text{v.f.}}, y_1, y_2)$$

with

$$f_a^* T_Y = g_a^*(f_1^* T_Y) \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n) \text{ for integers } a_1, \ldots, a_n \geq a,$$

namely the new integer $a_i(f_a)$ equals $a \cdot a_i(f_1)$. Next, choosing $a \geq 2k + 1$, this implies that

$$h^1(\mathbb{P}^1, f_a^* T_Y(-(k+1)(t_1 + t_2))) = 0.$$

Applying Proposition 3.2 with $\mathbb{P}^1 \times_k Y$ in the place of Y, with the graph of f_a in the place of Z_2 and with $Z_1 = (k+1)(t_1 + t_2)$ in the place of Z, deformations of f_a map the k-jet of $\mathbb{P}^1$ at t_1, resp. at t_2, isomorphically to a general k-jet at y_1, resp. at y_2. □

The following proposition is the strongest generalization of Proposition 3.6 we will need. It is stated as a theorem about finding new sections of a rationally connected fibration under the hypothesis that one such section exists. In this sense it may seem premature (and dangerously close to circular logic), since Theorem 1.9 is not yet proved. In fact the proposition is used in the proof of Theorem 1.9 not for the original fibration, but only for a *constant* fibration

$$\mathrm{pr}_{\mathbb{P}^1} : \mathbb{P}^1_k \times_k Y \to \mathbb{P}^1_k$$

which obviously admits sections (constant sections). So there is nothing circular in the application of the proposition to the proof of Theorem 1.9.

PROPOSITION 3.7 (Generic weak approximation). [**KMM92**], [**HT06**] *Let B be a smooth, connected, projective curve over an algebraically closed field k. Let $\pi : U \to B$ be a smooth, quasi-projective morphism having irreducible geometric fibers. Assume there exists a section $s : B \to U$ mapping the generic point of B into the very free locus of the generic fiber of π. Let $(b_1, \dots, b_M, b'_1, \dots, b'_{M'})$ be distinct closed points of B such that $s(b_i)$ is in the very free locus $U_{b_i,v.f.}$ of the fiber U_{b_i} for each $i = 1, \dots, M$. Let k and a be nonnegative integers. For each i, let x_i be a closed point of $U_{b_i,v.f.}$ and let there be given a curvilinear k-jet in U at x_i. Assuming each of these k-jets is a general k-jet at x_i, there exists a section $\sigma : B \to U$ such that*

- (i) *for each $i = 1, \dots, M$, $\sigma(b_i)$ equals x_i,*
- (ii) *for each $i = 1, \dots, M'$, $\sigma(b'_i)$ equals $s(b'_i)$,*
- (iii) *for every invertible $\mathcal{O}_B$-module $\mathcal{L}$ of degree $\leq a$, $h^1(B, \mathcal{N}_{\sigma(B)/U} \otimes_{\mathcal{O}_B} \mathcal{L}^\vee)$ equals 0,*
- (iv) *and σ maps the k-jet of b_i in B isomorphically to the given k-jet at x_i for each i.*

In fact Hassett and Tschinkel proved much more: the result holds for *arbitrary* k-jets transverse to the fibers of π (i.e., for k-jets whose associated Zariski tangent vector is not contained in a fiber of π). In what follows we only need the "generic" result, which is all we prove.

PROOF. Denote by Ω_π the locally free sheaf of relative differentials of π, and denote by T_π the dual locally free sheaf. Choose a large integer N and enlarge the set of pairs $((b_i, x_i))|_{i=1,\dots,M}$ to a set $((b_i, x_i))_{i=1,\dots,N}$ having the same properties above and such that the collection $(b_i)_{i=M+1,\dots,N}$ is a general collection of $N - M$ points in B (this is possible because for all but finitely many closed points of B, $s(b)$ is contained in $U_{b,\text{v.f.}}$). By Proposition 3.6, applied with $k = 1$, i.e., in the case that k-jets are simply tangent directions, for every $i = 1, \dots, N$ there exists a morphism

$$f_i : (\mathbb{P}^1, 0, \infty) \to (U_{b_i,\text{v.f.}}, s(b_i), x_i)$$

such that

$$f_i^* T_\pi \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n), \quad a_1, \dots, a_n \geq 1$$

and the tangent direction of $f_i(\mathbb{P}^1)$ at $s(b_i)$ is a general tangent direction in $T_{U_{b_i}, s(b_i)}$. But of course the tangent space $T_{U_{b_i}, s(b_i)}$ equals the normal space $N_{s(B)/U}|_{s(b_i)}$. Thus the tangent direction of $f_i(\mathbb{P}^1)$ at $s(b_i)$ gives a general normal direction to $s(B)$ in U at $s(b_i)$.

Form the comb $j : C_{\text{comb}} \to U$ with handle $s(B)$ and with each morphism f_i being a tooth L_i attached at $s(b_i)$. By Lemma 3.4, $\mathcal{N}_{C_{\text{comb}}/U}|_{s(B)}$ equals the sheaf of rational sections of $\mathcal{N}_{s(B)/U}$ having at most a simple pole at each point $s(b_i)$ in a general normal direction at $s(b_i)$. Assuming the integer N is sufficiently large, Lemma 3.3 then implies that $h^1(B, s^*\mathcal{N}_{C_{\text{comb}}/U})$ equals 0. Moreover, fixing an auxiliary invertible sheaf $\mathcal{M}$ on B of degree $g(B)+1$ and applying Lemma 3.3 to $s^*\mathcal{N}_{s(B)/U}(-(b'_1+\cdots+b'_{M'}))\otimes_{\mathcal{O}_B}\mathcal{M}^\vee$, for N sufficiently large also $h^1(B, s^*\mathcal{N}_{C_{\text{comb}}/U}(-(b'_1+\cdots+b'_{M'}))\otimes_{\mathcal{O}_B}\mathcal{M}^\vee)$ equals 0.

For every i, there is a short exact sequence

$$0 \longrightarrow f_i^*\mathcal{N}_{L_i/U_{b_i}} \longrightarrow f_i^*\mathcal{N}_{L_i/U} \longrightarrow f_i^*\mathcal{N}_{U_{b_i}/U} \longrightarrow 0.$$

Of course the normal sheaf $\mathcal{N}_{U_{b_i}/U}$ is just $\mathcal{O}_{U_{b_i}}$ since U_{b_i} is a smooth fiber of a morphism to a curve. Also the tangent direction of $s(B)$ at $s(b_i)$ surjects onto the fiber of $\mathcal{N}_{U_{b_i}/U}$ at $s(b_i)$. Thus the elementary transform up of $\mathcal{N}_{L_i/U}$ at $s(b_i)$ in this direction surjects onto the elementary transform up of $\mathcal{O}^1_{\mathbb{P}}$ at ∞, i.e., it surjects onto $\mathcal{O}_{\mathbb{P}^1}(1)$. Thus, by Lemma 3.4, there is a short exact sequence

$$0 \longrightarrow f_i^*\mathcal{N}_{L_i/U_{b_i}} \longrightarrow f_i^*\mathcal{N}_{C_{\text{comb}}/U} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow 0.$$

Twisting by $\mathcal{O}_{\mathbb{P}^1}(-2)$ and applying the long exact sequence of cohomology associated to the short exact sequence, $h^1(\mathbb{P}^1, f_i^*\mathcal{N}_{C_{\text{comb}}/U}(-\underline{0}-\underline{\infty}))$ equals 0. Combined with the result of the previous paragraph and joining the two types of normal sheaf via the short exact sequence

$$0 \to \oplus_{i=1}^N \mathcal{N}_{C_{\text{comb}}/U}|_{L_i}(-x_i - s(b_i)) \to \mathcal{N}_{C_{\text{comb}}/U}(-(x_1+\cdots+x_N)-(b'_1+\cdots+b'_{M'}))$$
$$\to \mathcal{N}_{C_{\text{comb}}/U}|_{s(B)}(-(b'_1+\cdots+b'_{M'})) \to 0,$$

the long exact sequence of cohomology implies both that

$$h^1(C_{\text{comb}}, \mathcal{N}_{C_{\text{comb}}/U}(-(x_1+\cdots+x_N)-(b'_1+\cdots+b'_{M'}))) \text{ equals } 0,$$

and that the map

$$H^0(C_{\text{comb}}, \mathcal{N}_{C_{\text{comb}}/U}(-(x_1+\cdots+x_N)-(b'_1+\cdots+b'_{M'}))) \to H^0(B, s^*\mathcal{N}_{C_{\text{comb}}/U}(-(b'_1+\cdots+b'_{M'})))$$

is surjective.

Thus, by Proposition 3.2, the space of deformations of C_{comb} containing $x_1,\dots,x_N$ and $b'_1,\dots,b'_{M'}$ is smooth. And, by Lemma 3.5, to prove there exists a deformation smoothing every node of C_{comb}, it suffices to prove for every i there exists a section of $s^*\mathcal{N}_{C_{\text{comb}}/U}(-(b'_1+\cdots+b'_{M'}))$ whose image in $T_{s(B),s(b_i)}\otimes_k T_{L_i,s(b_i)}$ is nonzero. Of course this skyscraper sheaf $\mathcal{T}_{s(b_i)}$ is a quotient of the fiber of $s^*\mathcal{N}_{C_{\text{comb}}/U}(-(b'_1+\cdots+b'_{M'}))$ at b_i. Thus it suffices to prove for every i that $h^1(B, s^*\mathcal{N}_{C_{\text{comb}}/U}(-b_i-(b'_1+\cdots+b'_{M'})))$ equals 0. Recall the auxiliary invertible sheaf $\mathcal{M}$ of degree $g(B)+1$. Because the invertible sheaf $\mathcal{M}(-b_i)$ has degree $g(B)$, it is effective, say $\mathcal{O}_B(\Delta_i)$. Thus there exists an injective $\mathcal{O}_B$-module homomorphism

$$s^*\mathcal{N}_{C_{\text{comb}}/U}(-(b'_1+\cdots+b'_{M'}))\otimes_{\mathcal{O}_B}\mathcal{M}^\vee \hookrightarrow$$

$$s^*\mathcal{N}_{C_{\text{comb}}/U}(-(b'_1+\dots b'_{M'}))\otimes_{\mathcal{O}_B}\mathcal{M}^\vee(\Delta_i) = s^*\mathcal{N}_{C_{\text{comb}}/U}(-b_i-(b'_1+\cdots+b'_{M'}))$$

with torsion cokernel. Since $h^1(B, s^*\mathcal{N}_{C_{\text{comb}}/U}(-(b'_1+\cdots+b'_{M'}))\otimes_{\mathcal{O}_B}\mathcal{M}^\vee)$ equals 0, and since every torsion sheaf has h^1 equal to 0, also $h^1(B, s^*\mathcal{N}_{C_{\text{comb}}/U}(-b_i-(b'_1+\cdots+b'_{M'})))$ equals 0 for every i. Therefore there exist a one-parameter family of deformations $(\mathcal{C}_t)_{t\in\Pi}$ of C_{comb} containing each of $x_1,\dots,x_M$, containing each of $s(b'_1),\dots,s(b'_{M'})$ and smoothing every node of C_{comb}, i.e., for t general, $\mathcal{C}_t$ is smooth.

Because π_U maps $s(B)$ to B with degree 1, also π_U maps $\mathcal{C}_t$ to B with degree 1. Because $\mathcal{C}_t$ is smooth, this means the projection $\mathcal{C}_t \to B$ is an isomorphism. Therefore there exists a section $\sigma_t : B \to U$ of π_U with image $\mathcal{C}_t$. In particular, $\sigma_t(b_i) = x_i$ for every $i = 1, \dots, M$ and $\sigma_t(b'_i) = s(b'_i)$ for every $i = 1, \dots, M'$. Because $h^1(C_{\mathrm{comb}}, \mathcal{N}_{C_{\mathrm{comb}}/U}(-(x_1 + \dots + x_N)))$ equals 0, by semicontinuity also $h^1(B, \sigma_t^*\mathcal{N}_{\sigma_t(B)/U}(-(x_1 + \dots + x_N)))$ equals 0 for t general. In particular, if $N \geq a + g(B)$, then for every invertible sheaf $\mathcal{L}$ of degree $\leq a$, $\mathcal{L}^\vee(x_1 + \dots + x_N)$ has degree $\geq g(B)$ and thus is effective, say $\mathcal{O}_B(\Delta)$. Therefore there exists an injective sheaf homomorphism

$$\sigma_t^*\mathcal{N}_{\sigma_t(B)/U}(-(x_1+\dots+x_N)) \hookrightarrow \sigma_t^*\mathcal{N}_{\sigma_t(B)/U}(-(x_1+\dots+x_N)+\Delta) = \sigma_t^*\mathcal{N}_{\sigma_t(B)/U} \otimes_{\mathcal{O}_B} \mathcal{L}^\vee$$

with torsion cokernel. So, by the same type of argument as above, $h^1(B, \sigma_t^*\mathcal{N}_{\sigma_t(B)/U} \otimes_{\mathcal{O}_B} \mathcal{L}^\vee)$ equals 0 for every invertible sheaf $\mathcal{L}$ of degree $\leq a$.

Finally, applying the last result when $a = (k+1)(M+M')$ and $\mathcal{L} = \mathcal{O}_B((k+1)(b_1 + \dots + b_M + b'_1 + \dots + b'_{M'}))$, there exists a section $\sigma : B \to U$ of π_U as above and satisfying $h^1(B, \sigma^*\mathcal{N}_{\sigma(B)/U}(-(k+1)(b_1 + \dots + b_M + b'_1 + \dots + b'_{M'})))$ equals 0. Therefore, by Proposition 3.2 once more, for a general deformation of $\sigma(B)$ containing $x_1, \dots, x_M$ and $s(b'_1), \dots, s(b'_{M'})$, the k-jet of the curve at each point x_i and $s(b'_i)$ is a general curvilinear k-jet in U at that point. $\square$

The main application is to the case when U equals $\mathbb{P}^1 \times_k Y$ where Y is a smooth, irreducible, quasi-projective k-scheme whose very free locus $Y_{\text{v.f.}}$ is nonempty.

Corollary 3.8. *Every rational curve in Y intersecting $Y_{v.f.}$ is contained in $Y_{v.f.}$. For every integer k, for every integer a, for every collection of distinct, closed points $b_1, \dots, b_M$ of $\mathbb{P}^1$, for every collection of closed points $y_1, \dots, y_M$ of $Y_{v.f.}$ (not necessarily distinct), and for every choice of a curvilinear k-jet in Y at each point y_i, if each k-jet is general among curvilinear k-jets at y_i, then there exists a morphism*

$$f : (\mathbb{P}^1, b_1, \dots, b_M) \to (Y, y_1, \dots, y_M)$$

mapping the k-jet of $\mathbb{P}^1$ at b_i isomorphically onto the given k-jet at y_i and such that

$$f^*T_Y \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n),\ a_1, \dots, a_n \geq a.$$

Proof. Let $B = \mathbb{P}^1$, let $U = B \times_k Y$ and let π_B be the obvious projection. The sections of π_B are precisely the graphs of morphisms $f : \mathbb{P}^1 \to Y$. In particular, if f is a morphism whose image intersects $Y_{\text{v.f.}}$, then the section $s = (\mathrm{Id}_B, f)$ satisfies the hypotheses of Proposition 3.7. Thus, for every point $b' = b'_1$ of $\mathbb{P}^1$, there exists a section $\sigma = (\mathrm{Id}_{\mathbb{P}^1}, \phi)$ with $\sigma(b') = s(b')$ and with $h^1(B, \sigma^*\mathcal{N}_{\sigma(B)/U}(-2))$ equal to 0. In other words, $\phi : \mathbb{P}^1_k \to Y$ is a morphism with $\phi(b') = f(b')$ and with $h^1(\mathbb{P}^1, \phi^*T_Y(-2))$ equal to 0. Thus ϕ is a very free morphism whose image contains $f(b')$. Therefore every point in the image of f is contained in the very free locus, i.e., every rational curve in Y intersecting $Y_{\text{v.f.}}$ is contained in $Y_{\text{v.f.}}$.

The rest of the corollary is just a straightforward translation of Proposition 3.7 to this context. $\square$

There is one more result in this direction which is useful. The proof is similar to the arguments above.

Lemma 3.9. [**Kol96**, Lemma II.7.10.1] *Let C_{comb} be a comb with handle C and teeth $L_1, \dots, L_n$. Let $\rho : \mathcal{C} \to \Pi$ be a one-parameter deformation of C_{comb} over a pointed curve $(\Pi, 0)$ whose general fiber $\mathcal{C}_t$ is smooth. Let $\mathcal{E}$ be a locally free sheaf* *on $\mathcal{C}$. If $\mathcal{E}|_{L_i}$ is ample for every i and if $h^1(C, (\mathcal{E}|_C) \otimes_{\mathcal{O}_C} \mathcal{M})$ equals 0 for every invertible $\mathcal{O}_C$-module $\mathcal{M}$ of degree $\geq n$, then $h^1(\mathcal{C}_t, \mathcal{E}|_{\mathcal{C}_t})$ equals 0 for general t in Π.*

4. Ramification issues

The argument sketched in Section 2 and the powerful smoothing combs technique from Section 3 form the core of the proof of Theorem 1.9. However there is a technical issue complicating matters. There may be codimension 1 points of X at which the morphism $\pi : X \to B$ is not smooth. In other words, finitely many scheme-theoretic fibers of π may have irreducible components occurring with multiplicity ≥ 1. This is a well-known issue when working with fibrations. Although there are sophisticated ways to deal with this (using log structures or Deligne-Mumford stacks), for the purposes of this proof it suffices to deal with this in a more naive manner.

In fact there may be codimension 0 points of X at which π is not smooth, at least if k has positive characteristic. The hypotheses in Theorem 1.9 prevent this, but something slightly weaker suffices. Let B be a smooth k-curve, let X be a reduced, finite type k-scheme and let $\pi : X \to B$ be a flat morphism. From here on, we assume the following hypothesis.

HYPOTHESIS 4.1. The geometric generic fiber of π is reduced. Equivalently, π is smooth at every generic point of X, cf. [**Gro67**, Proposition 4.6.1]. This hypothesis is automatic if $\mathrm{char}(k)$ equals 0.

DEFINITION 4.2. The *good locus* of π is the maximal open subscheme U of X such that U is smooth and such that for every point b of B the reduced scheme of the fiber $\pi^{-1}(b) \cap U$ is smooth. Denote the restriction of π to U by π_U. The morphism π is *good* if the good locus equals all of X. The *log divisor* of π is the Cartier divisor $D_{\pi,\log}$ of U given by

$$D_{\pi,\log} := \sum_{b \in B(k)} \pi_U^*(b) - \pi_U^*(b)_{\mathrm{red}},$$

where $\pi_U^*(b)_{\mathrm{red}}$ is the reduced Cartier divisor.

Since the geometric generic fiber of π is reduced, so is the geometric generic fiber of π_U (or else it is empty if U is empty). Thus the sum in the definition of the log divisor reduces to a sum over those finitely many closed points b of B for which $\pi_U^*(b)$ is nonreduced.

LEMMA 4.3. *The complement of U in X has codimension ≥ 2. If $\mathrm{char}(k)$ equals 0, then the pullback map on relative differentials*

$$\pi_U^* : \pi_U^* \Omega_{B/k} \to \Omega_{U/k}$$

factors uniquely through the inclusion

$$\pi_U^* \Omega_{B/k} \hookrightarrow \pi_U^* \Omega_{B/k}(D_{\pi,log})$$

and the cokernel

$$\Omega_{\pi,log} := \mathit{Coker}(\pi_U^* \Omega_{B/k}(D_{\pi,log}) \to \Omega_{U/k})$$

is locally free.

Proof. To construct U, first remove the closure of the singular locus of the geometric generic fiber of π and next remove the singular locus from the reduced scheme of the finitely many singular fibers. Both of these sets have codimension 2 in X (the first by Hypothesis 4.1).

The proof of the second part uses that $\text{char}(k) = 0$. It can be checked formally locally near every closed point x of U. Denote by b the image $\pi(x)$ in B and denote by D the reduced structure on the irreducible component of $\pi^{-1}(b)$ containing x. Since x is in U, D is a smooth Cartier divisor in U. Let r be a defining equation for D in U and let t be a defining equation for b in B. Near x, $\pi^*(b) = mD+$ other terms. Thus, in $\widehat{\mathcal{O}}_{U,x}$,

$$\pi^* t = a_m r^m + a_{m+1} r^{m+1} + \dots$$

where a_m is a unit. Because $\text{char}(k) = 0$, the power series

$$u = \sqrt[m]{a_m + a_{m+1} r + \dots}$$

is a well-defined unit in $\widehat{\mathcal{O}}_{U,x}$. Thus, after replacing r by ur, there exists a regular system of parameters $r, r_2, \dots, r_n$ for $\widehat{\mathcal{O}}_{U,x}$ with respect to which the pullback homomorphism π^* is the unique local homomorphism with

$$\pi^* t = r^m \text{ and } \pi^*(dt) = m r^{m-1} dr.$$

In particular, the pullback homomorphism π^* on relative differentials locally factors through $\pi^*\Omega_B((e-1)D) = \pi^*\Omega_B(D_{\pi,\log})$. Moreover the stalk of the cokernel of the induced homomorphism is the free module generated by $dr_2, \dots, dr_n$. $\square$

The locally free quotient $\Omega_{\pi,\log}$ of Ω_π is called the sheaf of *log relative differentials*. Of course it equals the torsion-free quotient of Ω_π. But its true importance comes from the following lemma: given a base change $V \to B$ for which the normalized fiber product $\widetilde{U \times_B V}$ is smooth over V, the sheaf $\Omega_{\widetilde{U \times_B V}/V}$ of relative differentials of $\widetilde{U \times_B V}$ over V equals the pullback of $\Omega_{\pi,\log}$. Thus the relative deformation theory of $\widetilde{U \times_B V}$ over V is already captured by the sheaf $\Omega_{\pi,\log}$ on U. Before stating the lemma precisely, there is some setup.

Let

$$\pi : U \to B, \quad \varpi : V \to B$$

be two good morphisms with respective log divisors $D_{\pi,\log}$ and $E_{\varpi,\log}$. Let b be a closed point of B. Let D be a prime divisor of U in $\text{Supp}(D_{\pi,\log}) \cap \pi^{-1}(b)$, and let E be a prime divisor of V in $\text{Supp}(E_{\varpi,\log}) \cap \varpi^{-1}(b)$. Denote by $m_D - 1$, resp. $m_E - 1$, the coefficient of D in $D_{\pi,\log}$, resp. the coefficient of E in $E_{\varpi,\log}$. The *normalized fiber product* of U and V along D and E is the normalization $\widetilde{U \times_B V}$ of $U \times_B V$ along $D \times_{\{b\}} E$. Denote by

$$\text{pr}_U : U \times_B V \to U, \quad \text{pr}_V : U \times_B V \to V$$

the two projections, and denote by

$$\widetilde{\text{pr}}_U : \widetilde{U \times_B V} \to U, \quad \widetilde{\text{pr}}_V : \widetilde{U \times_B V} \to V$$

the compositions with the normalization morphism. Denote by Exc the exceptional locus of the morphism, i.e.,

$$\text{Exc} := (\widetilde{\text{pr}}_U^{-1}(D) \cap \widetilde{\text{pr}}_V^{-1}(E))_{\text{reduced}}.$$

From this point forward we explicitly assume that $\text{char}(k)$ equals 0.

Hypothesis 4.4. The algebraically closed ground field k has characteristic 0. In particular, this implies Hypothesis 4.1.

The sheaves Ω_π and $\Omega_{\pi,\log}$ agree on a dense open subset of U, namely $U - \mathrm{Supp}(D_{\pi,\log})$. Because $\widetilde{\mathrm{pr}}_V$ and pr_V are isomorphic over a dense open subset of V (namely $V - E$) also $\Omega_{\widetilde{\mathrm{pr}}_V}$ agrees with $\widetilde{\mathrm{pr}}_V^*\Omega_\pi$ on a dense open subset of $\widetilde{U \times_B V}$. Therefore also Ω_{ϖ} agrees with $\widetilde{\mathrm{pr}}_V^*\Omega_{\pi,\log}$ on a dense open subset of $\widetilde{U \times_B V}$.

Lemma 4.5. *The morphism*

$$\widetilde{pr}_V : \widetilde{U \times_B V} \to V$$

is smooth at every point of Exc if and only if m_D divides m_E. In this case the reduced normalization equals the blowing up of $U \times_B V$ along the closed subscheme $pr_U^{-1}(D) \times pr_V^{-1}((m_E/m_D)E)$ and Exc is contained in the maximal open neighborhood of $\widetilde{U \times_B V}$ on which $\Omega_{\widetilde{\pi}}$ agrees with $(\widetilde{pr_V})^*\Omega_{\pi,log}$.

Proof. This is proved in much the same way as the second part of Lemma 4.3. For every closed point x of U and y of V with common image point $b = \pi(x) = \varpi(y)$, there exist a regular system of parameters $(r, r_2, \dots, r_n)$ for $\widehat{\mathcal{O}}_{U,x}$, resp. $(s, s_2, \dots, s_p)$ for $\widehat{\mathcal{O}}_{V,y}$, and a regular parameter t for $\widehat{\mathcal{O}}_{B,b}$ such that

$$\pi^* t = r^{m_D} \text{ and } \varpi^* t = s^{m_E},$$

and thus,

$$\widehat{\mathcal{O}}_{U \times_B V,(x,y)} = k\left[\!\left[r, r_2, \dots, r_n, s, s_2, \dots, s_p\right]\!\right] / \langle r^{m_D} - s^{m_E}\rangle.$$

Denoting by m the greatest common factor of m_D and m_E, the stalk of the normalization equals

$$k\left[\!\left[u, r, r_2, \dots, r_n, s, s_2, \dots, s_p\right]\!\right] / \langle r - u^{m_E/m}, s - u^{m_D/m}\rangle.$$

Thus it is formally smooth as a $k\left[\!\left[s, s_2, \dots, s_p\right]\!\right]$-algebra if and only if m_D/m equals 1, i.e., if and only if m_D divides m_E. In this case it is easy to see that the normalization is the blowing up at the ideal $\langle s, t^{m_E/m_D}\rangle$ and it is easy to see that the module of relative differentials is the free module generated by $dr_2, \dots, dr_n$, i.e., it is the pullback of $\Omega_{\pi,\log}$. $\square$

Having introduced the ideas need to deal with the ramification issues, we now resume the proof of Theorem 1.9. So from this point on we assume the following.

Hypothesis 4.6. The following hypotheses of Theorem 1.9 hold.

(i) The algebraically closed ground field k has characteristic 0, i.e., Hypothesis 4.4 holds.
(ii) The smooth k-curve B is projective and connected.
(iii) The reduced, finite type k-scheme X is normal and projective.
(iv) And the geometric generic fiber of the flat morphism $\pi : X \to B$ is a normal, integral scheme whose smooth locus contains a very free curve.

Definition 4.7. A *log preflexible curve* is a connected, smooth, proper curve $C \subset U$ such that

(i) the generic fiber of C over B is contained in the very free locus of the generic fiber of U over B,
(ii) $\pi_U(C)$ equals B,

FIGURE 4. Where a log preflexible curve intersects a reduced, possibly singular, fiber the map from the curve to B is unramified.

(iii) and every intersection point of C with $\mathrm{supp}(D_{\pi,\log})$ is transverse, i.e., the tangent direction of C at the intersection point is not contained in the tangent space of $\mathrm{supp}(D_{\pi,\log})$.

A *linked log preflexible curve* is a B-morphism from a linked curve $j : C_{\mathrm{link}} \to U$ such that the handle C is log preflexible and for every link L_i the image in B of L_i is disjoint from the image in B of $D_{\pi,\log}$.

A log preflexible curve C is a *log flexible curve* if

$$h^1(C, T_{\pi,\log}|_C) \text{ equals } 0, \text{ where } T_{\pi,\log} := \mathit{Hom}_{\mathcal{O}_U}(\Omega_{\pi,\log}, \mathcal{O}_U).$$

A linked log preflexible curve is a *linked log flexible curve* if

$$h^1(C_{\mathrm{link}}, j^*T_{\pi,\log}) \text{ equals } 0.$$

Figure 4 shows a log preflexible curve intersecting a singular, but reduced fiber. Because the curve is transverse to the fiber, the morphism to B is unramified. On the other hand, Figure 5 shows a log preflexible curve intersecting a nonreduced fiber – the middle component has multiplicity 2. Necessarily the map from the curve to B is ramified.

LEMMA 4.8. *There exists a log preflexible curve C. In fact, every intersection of X with $\dim(X) - 1$ general hyperplanes is a log preflexible curve.*

PROOF. Because $X - U$ has codimension 2 in X, a general complete intersection curve in X is disjoint from $X - U$, i.e., it is contained in U. By hypothesis, $U_{\pi,\mathrm{v.f.}}$ is

FIGURE 5. Where a log preflexible curve intersects a nonreduced fiber – e.g., the middle component has multiplicity 2 – the map from the curve to B is necessarily ramified.

a dense open subset of U and thus a general complete intersection curve intersects this open. Finally, by Bertini's theorem a general complete intersection curve in U is smooth and intersects $\mathrm{supp}(D_{\pi,\log})$ transversally. $\square$

An important consequence of the smoothing combs technique is the following result.

PROPOSITION 4.9. *There exists a log flexible curve in X. In fact, for every comb in X with log preflexible handle C and with sufficiently many very free teeth in fibers of π_U attached at general points of C and with general tangent directions, there exists a one-parameter deformation of the comb whose general member is a log flexible curve.*

PROOF. By hypothesis, C intersects the very free locus $U_{\pi,\text{v.f.}}$ of the morphism π_U. By the same argument as in the proof of Proposition 3.6, $U_{\pi,\text{v.f.}}$ is open. Therefore all but finitely many points of C are contained in $U_{\pi,\text{v.f.}}$. By Proposition 3.6 applied to 1-jets, i.e., to tangent directions, for each such point c there exists a very free rational curve in $U_{\pi_U(c)}$ containing c and whose tangent direction at c is a general tangent direction in $U_{\pi_U(c)}$.

Let C_{comb} be a comb obtained by attaching to C a number of teeth $L_1, \ldots, L_N$ as in the previous paragraph at general points of C (in particular, points where $C \to B$ is unramified) and with general tangent directions in $U_{\pi_U(c)}$. These tangent directions are the same as normal directions to C in U. By the same argument as in the proof of Proposition 3.7, if N is sufficiently large there is a one-parameter deformation

$$\mathcal{C} \subset \Pi \times_k U$$

of C_{comb} such that $\mathcal{C}_t$ is smooth for general t in Π. The properties (i), (ii) and (iii) of Definition 4.7 are all open properties and hold for $\mathcal{C}_0 = C_{\text{comb}}$, thus also hold for $\mathcal{C}_t$ so long as t is general.

For each tooth L_i in a fiber U_{b_i}, $T_{\pi,\log}|_{L_i}$ equals $T_{U_{b_i}}|_{L_i}$. Since L_i is very free, this is an ample locally free sheaf. Thus, by Lemma 3.9 with the pullback of $T_{\pi,\log}$ in the place of $\mathcal{E}$, we have that $h^1(\mathcal{C}_t, T_{\pi,\log}|_{\mathcal{C}_t})$ equals 0 for t a general point of Π. Therefore, for t a general point of Π, $\mathcal{C}_t$ is a log flexible curve. $\square$

Because the fibers of π are rationally connected, every log preflexible curve, resp. log flexible curve, extends to a linked log preflexible curve, resp. linked log flexible curve.

LEMMA 4.10. *For every linked curve C_{link} such that each point $b_i = \pi_{C,link}(L_i)$ is disjoint from $\pi_U(D_{\pi,log})$, and for every B-morphism $j_0 : C \to X$ mapping C isomorphically to a log preflexible curve, resp. log flexible curve, and mapping each fiber C_{b_i} into the very free locus $U_{\pi,v.f.}$ of π_U, there exists a B-morphism $j : C_{link} \to X$ which is linked log preflexible, resp. linked log flexible, and restricting to j_0 on C.*

PROOF. Let L_i be a link of C_{link}. Let L_i intersect C in m points $t_1, \ldots, t_m$ contained in the fiber over a general point b_i of B. Let $x_1, \ldots, x_m$ be the images $j(t_1), \ldots, j(t_m)$ in $U_{b_i,\text{v.f.}}$. By Corollary 3.8, there exists a morphism

$$j_i : (L_i, t_1, \ldots, t_m) \to ((U_{b_i,\text{v.f.}}, x_1, \ldots, x_m)$$

such that

$$j_i^* T_{U_{b_i}} \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n) \text{ for integers } a_1, \ldots, a_n \geq m-1.$$

Because of this,

$$h^1(L_i, j_i^* T_{U_{b_i}}(-(t_1 + \cdots + t_m))) \text{ equals } 0.$$

Define $j : C_{\text{link}} \to U$ to be the unique morphism restricting to j_0 on C and restricting to j_i on each link L_i. Because $j_i(t_k) = j_0(t_k)$ for every link L_i and for every node t_k contained in L_i, this morphism is defined. It is clearly log preflexible.

Next assume that j_0 is log flexible. The claim is that j is also log flexible. To see this, consider the short exact sequence

$$0 \longrightarrow \oplus_i j_i^* T_{U_{b_i}}(-C_{b_i}) \longrightarrow j^* T_{\pi,\log} \longrightarrow j_0^* T_{\pi,\log} \longrightarrow 0.$$

By the hypothesis that j_0 is log flexible, the third term has vanishing h^1. And by the construction of j_i, $j_i^* T_{U_{b_i}}(-C_{b_i})$, i.e., $j_i^* T_{U_{b_i}}(-(t_1 + \cdots + t_m))$, has vanishing h^1. Thus, by the long exact sequence of cohomology, also $h^1(C_{\text{link}}, j^* T_{\pi,\log})$ equals 0. Therefore $j : C_{\text{link}} \to U$ is a linked log flexible curve. $\square$

5. Existence of log deformations

There is a definition of one-parameter deformation that takes the divisor $D_{\pi,\log}$ into account. Unfortunately, not every curve over B admits a log deformation specializing to a section curve, e.g., étale covers of B are rigid. However, after attaching a sufficient number of links, the linked curve does admit a log deformation specializing to a section curve.

DEFINITION 5.1. Let $(C_{\text{link}}, \pi_{C,\text{link}})$ be a linked curve with handle C. Let $D_C \subset C$ be an effective, reduced, Cartier divisor contained in the smooth locus of C_{link}. A *one-parameter log deformation* of $(C_{\text{link}}, \pi_{C,\text{link}}, D_C)$ is a one-parameter deformation of $(C_{\text{link}}, \pi_{C,\text{link}})$,

$$(\rho, \pi_{\mathcal{C}}) : \mathcal{C} \to \Pi \times_k B$$

together with an effective Cartier divisor $D_{\mathcal{C}} \subset \mathcal{C}$ such that

(i) the pullback of $D_{\mathcal{C}}$ to $\mathcal{C}_0 = C_{\text{link}}$ equals D_C

(ii) and $\pi_{\mathcal{C}}(D_{\mathcal{C}})$ equals $\pi_C(D_C)$, i.e., $D_{\mathcal{C}}$ is vertical over B.

LEMMA 5.2. *For every finite morphism of smooth, projective curves $\pi_C : C \to B$ and for every effective, reduced, Cartier divisor D_C of C, after attaching sufficiently many links to C over general points of B, there exists a one-parameter log deformation specializing to a section curve.*

PROOF. For all sufficiently positive integers e, for a general morphism $g : C \to \mathbb{P}^1$ of degree e, the induced morphism $(\pi_C, g) : C \to B \times_k \mathbb{P}^1$ is unramified and is injective except for finitely many double points, none of which intersects the image of D_C. Denote by $\Sigma \to B \times_k \mathbb{P}^1$ the blowing up along the finitely many double points of $(\pi_C, g)(C)$. Then there is a B-morphism $h : C \to \Sigma$ which is an embedding.

For each point p of D_C, denote by m_p the multiplicity of p in the Cartier divisor $\pi_C^*(\pi_C(p))$. Denote by $\nu_p : \Sigma'_p \to \Sigma$ the m_p-fold iterated blowup of Σ first at p, then at the image of p in the strict transform of $h(C)$, etc. Denote by E_p the final exceptional divisor of this sequence of blowups. The point of this construction is that the strict transform of $h(C)$ intersects E_p at p, and E_p occurs with multiplicity m_p in the Cartier divisor $\Sigma'_p \times_B \{\pi_C(p)\}$. Denote by $\nu : \Sigma' \to \Sigma$ the fiber product over all points p in D_C of $\nu_p : \Sigma'_p \to \Sigma$. Denote by E the Cartier divisor in Σ' being the sum over all p of the pullback of E_p from Σ'_p. Denote by $\pi_{\Sigma'} : \Sigma' \to B$ the composition of $\Sigma' \to \Sigma \to B \times_k \mathbb{P}^1$ with pr_B. Denote by $h' : C \to \Sigma'$ the strict transform of $h(C)$. The point of this construction is that E is a Cartier divisor in Σ' which is vertical over B and such that h^*E equals D_C.

Denote by d the degree of π_C and let $t_1, \dots, t_d$ be closed points of $\mathbb{P}^1$ such that the Cartier divisor $B \times_k \{t_1, \dots, t_d\}$ of $B \times_k \mathbb{P}^1$ is disjoint from all double points of $(\pi_C, g)(C)$ and disjoint from $(\pi_C, g)(D)$. Denote by T the strict transform of $B \times_k \{t_1, \dots, t_d\}$ in Σ'. Form the invertible sheaf $\mathcal{O}_{\Sigma'}(h'(C) - T)$ and the pushforward $\mathcal{E} := \pi_{\Sigma',*}\mathcal{O}_{\Sigma'}(h'(C) - T)$ on B. Because $\pi_{\Sigma'}$ is flat and because $\mathcal{O}_{\Sigma'}(h'(C) - T)$ is locally free, $\mathcal{E}$ is torsion-free. For every point b in $B - \pi_C(D_C)$, Σ'_b is isomorphic to $\mathbb{P}^1$ (via the projection $\Sigma' \to B \times_k \mathbb{P}^1 \to \mathbb{P}^1$). And $\Sigma'_b \cap h'(C)$ and $\Sigma'_b \cap T$ are divisors of the same degree d. Thus $\mathcal{O}_{\Sigma'}(h'(C) - T)|_{\Sigma'_b}$ is isomorphic to $\mathcal{O}_{\Sigma'_b} \cong \mathcal{O}_{\mathbb{P}^1}$. Therefore $\mathcal{E}|_b$ is isomorphic to $H^0(\Sigma'_b, \mathcal{O}_{\Sigma'_b})$, which is one-dimensional. Therefore $\mathcal{E}$ is an invertible sheaf.

By Riemann-Roch and Serre duality, for every sufficiently large degree, for a general effective divisor Δ on B of that degree, $\mathcal{E} \otimes_{\mathcal{O}_B} \mathcal{O}_B(\Delta)$ is globally generated. Choose Δ to be disjoint from $\pi_C(D_C)$ and from the image in B of the finitely many intersection points of $h'(C)$ and T. Since $\mathcal{E} \otimes_{\mathcal{O}_B} \mathcal{O}_B(\Delta)$ is globally generated, there exists a section which is nonzero at every point of Δ. Of course a nonzero section of this sheaf (up to scaling) is precisely the same thing as a divisor V on Σ' such that

$$h'(C) + \pi_{\Sigma'}^*\Delta \sim T + V.$$

For b in $B - \pi_C(D_C)$, if the section is nonzero at b then V does not intersect Σ'_b. The same does not necessarily hold for points b of $\pi_C(D_C)$ since b may lie in the support of $R^1\pi_{\Sigma',*}\mathcal{O}_{\Sigma'}(h'(C)-T)$. Therefore V is a sum of finitely many irreducible components of fibers of $\pi_{\Sigma'}$ (possibly with multiplicity) lying over points not in Δ.

The linked curve $(C_{\text{link}}, \pi_{C,\text{link}})$ is $h'(C) + \pi_{\Sigma'}^*\Delta$ together with the restriction of $\pi_{\Sigma'}$. Denote by Π the pencil of divisors in Σ' spanned by the divisors $h'(C) + \pi_{\Sigma'}^*\Delta$ and $T + V$, with these two divisors marked as 0 and ∞ respectively. Denote by $\mathcal{C} \subset \Pi \times_k \Sigma'$ the corresponding family of divisors. By Bertini's theorem, the general member $\mathcal{C}_t$ is smooth away from the base locus. Now the only singular points of $h'(C) + \pi_{\Sigma'}^*\Delta$ are the points $h'(\pi_C^{-1}(\Delta))$. Since V does not intersect $\pi_{\Sigma'}^*\Delta$, these singular points are not in the base locus. Since $\mathcal{C}_0$ is nonsingular at every basepoint, the same is true for $\mathcal{C}_t$ for t general. Thus a general member $\mathcal{C}_t$ is smooth everywhere.

Define $D_{\mathcal{C}}$ to be the pullback to $\mathcal{C}$ of the Cartier divisor E in Σ'. Because E is vertical over B and because h^*E equals D_C, the deformation $\mathcal{C}$ together with the effective Cartier divisor $D_{\mathcal{C}}$ is a one-parameter log deformation of $(C_{\text{link}}, \pi_{C,\text{link}}, D_C)$. And it specializes at $t = \infty$ to a union of section curves and vertical curves. $\square$

6. Completion of the proof

We are finally prepared for the proof of Theorem 1.9.

Proof of Theorem 1.9. By Proposition 4.9, there exists a log flexible curve $j_0 : C \to U$. Denote by D_C the reduced scheme of the intersection $C \cap D_{\pi,\log}$. By Lemma 5.2, after attaching finitely many links to C over the points of a general divisor Δ of B, the linked curve C_{link} together with D_C admits a one-parameter log deformation

$$(\rho, \pi_{\mathcal{C}}) : \mathcal{C} \to \Pi \times_k B, \quad D_{\mathcal{C}} \subset \mathcal{C}$$

of (C_{link}, D_C) specializing to a section curve (in fact $\mathcal{C}_\infty$ is a union of section curves and vertical curves).

By Proposition 3.6, the relative very free locus $U_{\pi,\text{v.f.}}$ is open in U. Thus $C \cap U_{\pi,\text{v.f.}}$ is open in C. So its complement is finitely many points in C. Thus a general divisor Δ is disjoint from the finite set $\pi_U(D_{\pi,\log})$ and from the finite set $\pi_C(C - C \cap U_{\pi,\text{v.f.}})$. Then, by Lemma 4.10, there exists an extension of j_0 to a linked log flexible curve

$$j : C_{\text{link}} \to U.$$

Form the fiber product

$$U_{\mathcal{C}} := \mathcal{C} \times_{\pi_{\mathcal{C}}, B, \pi_U} U.$$

Since π_U is flat, also the projection

$$\text{pr}_{\mathcal{C}} : U_{\mathcal{C}} \to \mathcal{C}$$

is flat. Since $\pi_{\mathcal{C}}$ is surjective, the geometric generic fiber of $\text{pr}_{\mathcal{C}}$ equals the geometric generic fiber of π_U, which is integral. Since $\text{pr}_{\mathcal{C}}$ is flat with integral geometric generic fiber, $U_{\mathcal{C}}$ is integral. Define

$$\nu : \widetilde{U}_{\mathcal{C}} \to U_{\mathcal{C}}$$

to be the blowing up of $U_{\mathcal{C}}$ along the closed subscheme $D_{\mathcal{C}} \times_B D_{\pi,\log}$. Since $U_{\mathcal{C}}$ is integral, also $\widetilde{U}_{\mathcal{C}}$ is integral. And the composition

$$\widetilde{U}_{\mathcal{C}} \to U_{\mathcal{C}} \to \mathcal{C} \to \Pi$$

is surjective. Since Π is a smooth curve, the morphism

$$\widetilde{\rho} : \widetilde{U}_{\mathcal{C}} \to \Pi$$

is flat.

Consider the graph,

$$\Gamma_j : C_{\text{link}} = \mathcal{C}_0 \to \mathcal{C}_0 \times_B U = U_{\mathcal{C},0}.$$

Because the links of C_{link} do not intersect $D_{\pi,\log}$, the image of Γ_j is smooth at every point of intersection with $D_C \times_B D_{\pi,\log}$. Since ν is birational, Γ_j gives a rational transformation from C_{link} to $\widetilde{U}_{\mathcal{C},0}$. Since ν is proper, and since C_{link} is smooth at every point of intersection with $D_C \times_B D_{\pi,\log}$, the valuative criterion of properness implies this rational transformation is actually a regular morphism

$$\widetilde{\Gamma}_j : \mathcal{C}_0 \to \widetilde{U}_{\mathcal{C},0}.$$

Clearly this is a section of the projection morphism

$$\text{pr}_{\mathcal{C}_0} : \widetilde{U}_{\mathcal{C},0} \to \mathcal{C}_0.$$

For every point t in $C_{\text{link}} - D_C$, the morphism $\pi_U : U \to B$ is smooth at $j(t)$. Therefore also $U_{\mathcal{C},0} \to \mathcal{C}_0$ is smooth at $\Gamma_j(t)$. And since ν is an isomorphism over $\Gamma_j(t)$, also $\text{pr}_{\mathcal{C}_0} : \widetilde{U}_{\mathcal{C},0} \to \mathcal{C}_0$ is smooth at $\widetilde{\Gamma}_j(t)$. Also the vertical tangent bundle equals the pullback of the vertical tangent bundle of $\pi_U : U \to B$, which also equals $T_{\pi,\log}$ (since $j(t)$ is not in $D_{\pi,\log}$).

Let t be a point of D_C and let D_t be the unique irreducible component of $D_{\pi,\log}$ containing $j(t)$. Give D_t the reduced structure. Because $j_0(C)$ is transverse to D_t at $j_0(t)$, the ramification index $m_C - 1$ of $\pi_C : C \to B$ at t equals the ramification index $m_D - 1$ of π_U along D_t. Therefore, by Lemma 4.5, the projection

$$\text{pr}_{\mathcal{C}_0} : \widetilde{U}_{\mathcal{C},0} \to \mathcal{C}_0$$

is smooth over the preimage of $\{t\} \times D_t$ for every t and the vertical tangent bundle equals the pullback of $T_{\pi,\log}$. Since $\Gamma_j(t)$ is in $\{t\} \times D_t$, this implies that $\text{pr}_{\mathcal{C}_0}$ is smooth at every point of the image of $\widetilde{\Gamma}_j$ and the vertical tangent bundle of $\text{pr}_{\mathcal{C}_0}$ equals the pullback of $T_{\pi,\log}$.

Since $\widetilde{\Gamma}_j$ is a section with image in the smooth locus of $\text{pr}_{\mathcal{C}_0}$, the normal sheaf $\mathcal{N}$ equals the restriction of the vertical tangent bundle. Therefore $\widetilde{\Gamma}_j^*\mathcal{N}$ equals $j^*T_{\pi,\log}$. Since $j : C_{\text{link}} \to U$ is log flexible, $h^1(C_{\text{link}}, j^*T_{\pi,\log})$ equals 0. Therefore, by Proposition 3.1, the relative Hilbert scheme $\text{Hilb}(\widetilde{U}_{\mathcal{C}}/\Pi)$ is smooth over Π at the point $0' := [\text{Image}(\widetilde{\Gamma}_j)]$. Thus for a general complete intersection curve Π' containing $0'$, the morphism $\Pi' \to \Pi$ is smooth at $0'$.

Replace Π' by the unique irreducible component containing $0'$, and then replace this by its normalization. The result is that Π' is a smooth, projective, connected curve together with a morphism $\Pi' \to \text{Hilb}(\widetilde{U}_{\mathcal{C}}/\Pi)$ so that the induced morphism $\Pi' \to \Pi$ is smooth at $0'$. In particular it is flat, so surjective. Let ∞' denote a closed point of Π' mapping to ∞. Then $(\Pi', 0', \infty') \to (\Pi, 0, \infty)$ is a flat morphism of 2-pointed smooth curves. Thus, by Lemma 2.2, the base change $\Pi' \times_\Pi \mathcal{C}$ is a one-parameter deformation of C_{link} over $(\Pi', 0', \infty')$ specializing to a section curve.

Denote by

$$Z \subset \Pi' \times_\Pi \widetilde{U}_{\mathcal{C}}$$

the pullback of the universal closed subscheme $\mathrm{Univ}(\widetilde{U}_{\mathcal{C}}/\Pi)$ by the morphism $\Pi' \to \mathrm{Hilb}(\widetilde{U}_{\mathcal{C}}/\Pi)$. The composition with $\mathrm{pr}_{\mathcal{C}}$ is a projective morphism

$$Z \subset \Pi' \times_{\Pi} \widetilde{U}_{\mathcal{C}} \to \Pi' \times_{\Pi} \mathcal{C}$$

of flat Π'-schemes. Moreover, the fiber over $0' \in \Pi'$ is an isomorphism since the projection $\widetilde{\Gamma}_j(C_{\mathrm{link}}) \to C_{\mathrm{link}}$ is an isomorphism. Therefore the morphism is an isomorphism over $N \times_{\Pi} \mathcal{C}$ for some open neighborhood N of $0'$ in Π'. (This is well-known; a complete proof is given in [**dJS03**, Lemma 4.7].) Invert this isomorphism and compose it with the morphism

$$\Pi' \times_{\Pi} \widetilde{U}_{\mathcal{C}} \to \widetilde{U}_{\mathcal{C}} \to U_{\mathcal{C}} \to U.$$

The result is precisely an extension

$$j_N : N \times_{\Pi} \mathcal{C} \to X$$

of j for the one-parameter deformation $\Pi' \times_{\Pi} \mathcal{C}$. Therefore, by Lemma 2.3, there exists a section $s : B \to X$ of π. □

7. Corollaries

There are a number of consequences of Theorem 1.9 and its generalization to positive characteristic in [**dJS03**]. Many of these consequences were recognized before Conjecture 1.8 was proved.

Corollary 7.1. [**Kol96**, Conjecture IV.5.6] *Conjecture 1.7 is true. Moreover, for every smooth, projective, irreducible variety X over an algebraically closed field of characteristic 0, there exists a dense open $X^0 \subset X$ and a projective, smooth morphism $q_0 : X^0 \to Q^0$ such that every fiber of q_0 is rationally connected, and every projective closure of Q^0 is nonuniruled.*

Corollary 7.2. [**GHS03**, Corollary 1.7] *The uniruledness conjecture implies Mumford's conjecture. To be precise, assume that for every smooth, projective, irreducible variety X over an algebraically closed field k of characteristic 0, if X is nonuniruled then $h^0(X, \omega_X^{\otimes n})$ is nonzero for some $n > 0$. Then for every smooth, projective, irreducible variety X over k, if X is not rationally connected then $h^0(X, \Omega_X^{\otimes n})$ is nonzero for some $n > 0$.*

The next corollary is a fixed point theorem. In characteristic 0 it can be proved using the Atiyah-Bott fixed point theorem. But in positive characteristic it is a new result. There are examples due to Shioda proving one cannot replace "separably rationally connected" by "rationally connected", cf. [**Shi74**].

Corollary 7.3. [**Kol03**] *Let Y be a smooth, projective, separably rationally connected variety over a field k and let $f : Y \to Y$ be a k-automorphism. If $\mathrm{char}(k)$ is positive, say p, assume in addition that f has finite order n not divisible by p^2. Then the fixed locus of f is nonempty.*

Proof. Of course it suffices to prove the case when k is algebraically closed, since the fixed locus of the base change equals the base change of the fixed locus. First assume f has finite order n. If n is prime to $\mathrm{char}(k)$, let B' denote $\mathbb{P}^1$ and let $\mathbb{Z}/n\mathbb{Z}$ act on $\mathbb{P}^1$ by multiplication by a primitive n^{th} root of unity. Note this action fixes ∞ and has trivial generic stabilizer. If $\mathrm{char}(k) = p$ is positive and if

$n = pm$ where m is prime to p, let B' be the normal, projective completion of the affine curve

$$\mathbb{V}(y^m - (x^p - x)) \subset \mathbb{A}^2_k.$$

Let ζ be a primitive m^{th} root of unity, and let a generator of $\mathbb{Z}/m\mathbb{Z}$ act by $(x, y) \mapsto (x, \zeta y)$. Similarly, let a generator of $\mathbb{Z}/p\mathbb{Z}$ act by $(x, y) \mapsto (x + 1, y)$. Clearly these actions commute, thus define an action of $\mathbb{Z}/n\mathbb{Z}$ on B'. Note this action fixes the unique point ∞ not in the affine chart above, and the action has trivial generic stabilizer.

Let $\mathbb{Z}/n\mathbb{Z}$ act diagonally on $Y \times_k B'$, and let X be the quotient. Also let B be the quotient of the $\mathbb{Z}/n\mathbb{Z}$-action on B'. The projection $\pi : X \to B$ satisfies the hypotheses of Theorem 1.9 (or its generalization in [**dJS03**]). Therefore there exists a section. This is the same as $\mathbb{Z}/n\mathbb{Z}$-equivariant k-morphism $f : B' \to Y$. In particular, since ∞ is a fixed point in B', $f(\infty)$ is a fixed point in Y.

Next assume k has characteristic 0. By general limit arguments there exists an integral, finitely generated $\mathbb{Z}$-algebra R, a ring homomorphism $R \hookrightarrow k$, a smooth, projective morphism $Y_R \to \text{Spec } R$ whose relative very free locus is all of Y_R, and an R-automorphism $f_R : Y_R \to Y_R$ such that the base change $Y_R \otimes_R k$ equals Y and the base change of f_R equals f. The intersection $(Y_R)^{f_R}$ of the graph of f_R and the diagonal of $Y_R \times_R Y_R$ is the fixed subscheme of f_R (actually its image under the diagonal morphism). Since $(Y_R)^{f_R}$ is a proper scheme over Spec R, the image in Spec R is a closed subscheme of Spec R. To prove this closed subscheme equals all of Spec R, and thus contains the image of Spec k, it suffices to prove it contains a Zariski dense set of closed points.

Choose an f-invariant very ample sheaf, choose a basis for the space of global sections, and let A be the $N \times N$ matrix with entries in R giving the action of f on global sections with respect to this basis. The set of maximal ideal in Spec R with residue field of characteristic $p > N$ are Zariski dense in Spec R. Every invertible matrix over a characteristic p field with order divisible by p^2 has a Jordan block with eigenvalue 1 and size divisible by p. Thus, since $p > N$, the finite order of f_R modulo the prime is not divisible by p^2. Therefore, by the previous case, the reduction of f_R modulo the prime has nonempty fixed locus. Therefore the original automorphism f has nonempty fixed locus. □

This fixed point theorem implies that separably rationally connected varieties are simply connected. When the field k is $\mathbb{C}$, this was first proved by Campana using analytic methods, cf. the excellent reference by Debarre, [**Deb01**, Corollary 4.18].

Corollary 7.4 (Campana, Kollár). [**Cam91**], [**Deb03**, 3.6] *Let X be a smooth, projective, separably rationally connected variety over an algebraically closed field k. The algebraic fundamental group of X is trivial. If $k = \mathbb{C}$, then the topological fundamental group of X is also trivial.*

Kollár has generalized this considerably to prove a result for open subschemes of rationally connected varieties, cf. [**Kol03**].

Proof. The full proof is included in the beautiful survey by Debarre, [**Deb03**, 3.6]. Here is a brief sketch. First of all, for every quasi-projective, (not necessarily separably) rationally chain connected variety, Campana proved that the algebraic fundamental group is finite and also the topological fundamental group is finite when k equals $\mathbb{C}$ (so that the topological fundamental group is defined). Thus the universal cover $\widetilde{X} \to X$ is finite. Since X is smooth, projective and separably rationally connected, also $\widetilde{X}$ is smooth, projective and separably rationally connected. If the fundamental group of X is nonzero, then it contains a cyclic subgroup $\mathbb{Z}/n\mathbb{Z}$ such that p^2 does not divide n. Of course the action of this group on $\widetilde{X}$ is fixed-point-free. But Corollary 7.3 implies there exists a fixed point. Thus X is simply connected. $\square$

Theorem 1.9 also plays an important role in the proof of a "converse" to that same theorem.

THEOREM 7.5. [**GHMS05**] *Let $\pi : X \to B$ be a surjective morphism of normal, projective, irreducible varieties over an algebraically closed field k of characteristic zero. Assume that for some sufficiently large, algebraically closed field extension K/k, for every k-morphism $C \to B$ from a smooth, projective, K-curve to B, the pullback $\pi_C : C \times_B X \to C$ has a section. Then there exists a closed subvariety $Y \subset X$ such that the geometric generic fiber of $\pi|_Y : Y \to B$ is nonempty, irreducible and rationally connected.*

One corollary of this theorem, in fact the motivation for proving it, was to answer a question first asked by Serre and left unresolved by Theorem 1.9: could it be that a smooth, projective variety X over the function field of a curve has a rational point if it is $\mathcal{O}$*-acyclic*, i.e., if $h^i(X, \mathcal{O}_X)$ equals 0 for all $i > 0$? One reason to ask this is that the corresponding question has a positive answer if "function field" is replaced by "finite field" thanks to N. Katz's positive characteristic analogue of the Atiyah-Bott fixed point theorem, [**DK73**, Exposé XXII, Corollaire 3.2], recently generalized by Esnault, [**Esn03**]. Nonetheless, the answer is negative over function fields.

COROLLARY 7.6. [**GHMS05**] *There exists a surjective morphism $\pi : X \to B$ of smooth, projective varieties over $\mathbb{C}$ such that B is a curve and the geometric generic fiber of π is an Enriques surface, but π has no section. Thus, to guarantee a fibration over a curve has a section, it is not sufficient to assume the geometric generic fiber is $\mathcal{O}$-acyclic.*

In fact G. Lafon found an *explicit* morphism π as in Corollary 7.6 where B is $\mathbb{P}^1_{\mathbb{C}}$, or in fact $\mathbb{P}^1_k$ for any field k with $\mathrm{char}(k) \neq 2$, and there does not even exist a power series section near $0 \in \mathbb{P}^1_k$, cf. [**Laf04**].

CHAPTER 4

The Period-Index theorem of de Jong

1. Statement of the theorem

This chapter and the next are extracted from an earlier set of lecture notes, [**Sta08**]. The results in this chapter are joint work with A. J. de Jong. In particular, a full proof of Theorem 4.2 is presented in [**Sta08**]. For this reason we will not reproduce the proof here, but we will reproduce enough of the proof to prove de Jong's Period-Index theorem.

Recall Problem 4.15: for a given field K, what is the precise relationship between the index of division algebras and the period, or exponent, of division algebras? Of course if K is algebraically closed, or more generally if K has cohomological dimension ≤ 1, Theorem 3.3(i) implies that the only division algebra over K is K itself. Thus both the period and the index equal 1. And in this sense the power $r(K)$ of K equals 0. In particular, by Chevalley's theorem and Tsen's theorem discussed in Chapter 2, this holds if K is a finite field or if K is the fraction field of a curve over an algebraically closed field. The following question was explicitly asked by Colliot-Thélène, cf. [**CT01**].

Question 1.1. Let K be the function field of a variety of dimension $d \geq 2$ over $\mathbb{C}$. For every division algebra D over K, does $\text{index}(D)$ divide $\text{period}(D)^{d-1}$? In other words, is $r(K)$ equal to $d-1$?

There are examples showing $r(K)$ is $\geq d-1$, so this question is sharp. One can also ask this question with $\mathbb{C}$ replaced by any algebraically closed field k, perhaps with the restriction that $\text{char}(k)$ does not divide $\text{period}(D)$. Even more generally, one could ask this question with K replaced by any C_d field, e.g., a function field of a variety of dimension $d-1$ over a finite field.

The first major advance in this direction (from which subsequent advances have flowed) is the following theorem of A. J. de Jong.

Theorem 1.2 (de Jong's Period-Index theorem). [**dJ04**] *Let k be an algebraically closed field. Let K/k be the function field of a surface over k. Then the power of K is* 1*, i.e., for every central simple algebra A over K, the period of A equals the index of A.*

Initially de Jong proved this under the additional hypothesis that $\text{char}(k)$ does not divide the period of A. Using the technique of discriminant avoidance, described below, de Jong and I removed this hypothesis, cf. [**dJS05**, Theorem 1.0.2]. The original proof of de Jong used an analysis of the moduli space of *Azumaya algebras*, i.e., sheaves of central simple algebras, on a smooth, projective surface. The second proof of de Jong and myself will be presented below. Lieblich used moduli spaces of α-twisted sheaves, as introduced by Căldăraru and Yoshioka, to give a third proof. Moreover, Lieblich's proof proves more.

Theorem 1.3. [**Lie08**] *For K the function field of a surface over a finite field k, for every division algebra D over K, if char(k) does not divide period(D), then index(D) divides period$(D)^3$. If, moreover, D is the generic fiber of an Azumaya algebra on a smooth, proper model of K, then index(D) equals period(D).*

Before explaining the second proof due to de Jong and myself, there are 3 more reformulations of Theorem 1.2. The first reformulation has to do with the map Δ_n from Theorem 3.15.

Lemma 1.4. *For a field K, the power $r(K)$ equals 1, i.e., the period equals the index for every division algebra over K, if and only if for every integer $n \geq 1$ the map*

$$\Delta_n : H^1(Gal(K^s/K), \mathbf{PGL}_{n,K}(K^s)) \to Br'(K)[n]$$

is surjective, and thus bijective.

Proof. Assume first that $r(K)$ equals 1. By Theorem 4.7 and Theorem 4.12, every element of $\mathrm{Br}'(K)$ is of the form $\Delta[D]$ for some division algebra D. Since $r(K)$ equals 1, the order of $\Delta[D]$ equals the index of D. Thus, if the order r of $\Delta[D]$ divides n, say $n = rl$, then $A = D \otimes_K \mathrm{Mat}_{l\times l,K}$ gives an element of $H^1(\mathrm{Gal}(K^s/K), \mathbf{PGL}_{n,K}(K^s))$ mapping to $\Delta[D]$ in $\mathrm{Br}'(K)[n]$. Thus Δ_n is surjective.

Next assume that Δ_n is surjective for every n. Let D be a division algebra over K, and denote by n the order of $\Delta(D)$. Since Δ_n is surjective, there exists an Azumaya algebra A with $\dim_K(A) = n^2$ such that $\Delta_n(A)$ equals $\Delta(D)$. By Theorem 4.7, A is Morita equivalent to D. By Theorem 4.12, $A = D \otimes_K \mathrm{Mat}_{l\times l,K}$ for some integer $l \geq 1$. Comparing dimensions of these K-vector spaces, the period n is at least as large as the index of D. Since the period divides the index, also the index is at least as large as the period, and thus both are equal. $\square$

The second reformulation requires the auxiliary notion of *generalized Brauer-Severi varieties*. Let A be a central simple algebra over K with $\dim_K(A) = n^2$ and let $r \geq 1$ be an integer. The functor

$$K\text{-algebras} \to \text{Sets}, \quad R \mapsto \{I \subset A\otimes_K R \mid I \text{ a free } R\text{-module of rank } rn \text{ and a right ideal }\}$$

is representable.

Definition 1.5. [**Bla91**], [**SVdB92**] The *generalized Brauer-Severi variety*, $\mathrm{Grass}_{r,A}$, is the scheme representing the functor above. Stated differently, if E denotes the $\mathbf{PGL}_{n,K}$-torsor associated to A, then $\mathrm{Grass}_{r,A}$ equals the twist ${}_E\mathrm{Grass}_K(r, K^{\oplus n})$ of the usual Grassmannian $\mathrm{Grass}_K(R, K^{\oplus n})$ of locally free, rank r, locally direct summands of $K^{\oplus n}$.

Lemma 1.6. *For a field K, the power $r(K)$ equals 1 if and only if for every central simple algebra A, for $r =$ period(A), Grass$_{r,A}$ has a K-point.*

Proof. Assume first that $r(K)$ equals 1. Let A be a central simple algebra with $\dim_K(A) = n^2$. By Theorem 4.12, A is isomorphic to $D \otimes_K \mathrm{Mat}_{l\times l,K}$ for some division algebra D over K and some integer $l \geq 1$, i.e.,

$$A = \mathrm{End}_{D-\mathrm{mod}}(D^{\oplus l}).$$

Since $r(K)$ equals 1, $\mathrm{period}(A) = \mathrm{period}(D)$ equals $\mathrm{index}(D)$. Denote by $D^{\oplus 1} \subset D^{\oplus l}$ the left D-submodule of $D^{\oplus l}$ generated by the first basis element. Then

$$I = \mathrm{Hom}_{D-\mathrm{mod}}(D^{\oplus l}, D^{\oplus 1})$$

is a right ideal in A with $\dim_K(I) = rn$. Thus I gives a K-point of $\mathrm{Grass}_{r,A}$.

Next assume that $\mathrm{Grass}_{r,A}$ always has a K-point. Let D be a division algebra of order r. Since $\mathrm{Grass}_{r,D}$ has a K-point, there is a right ideal I in D with

$$\dim_K(I) = \mathrm{period}(D) \cdot \mathrm{index}(D).$$

Since the only nonzero right ideal in D is D itself, I equals D. Since $\dim_K(D) = \mathrm{index}(D)^2$, dividing gives

$$\mathrm{period}(D) = \mathrm{index}(D).$$

□

Among all the generalized Brauer-Severi varieties $\mathrm{Grass}_{r,A}$, the one with $r = \mathrm{period}(A)$ has a special property.

LEMMA 1.7. *Let K be a field and let A be a central simple algebra over K with $\dim_K(A) = n^2$. Let r be an integer, $1 \leq r \leq n$. There exists an invertible sheaf $\mathcal{L}$ on $\mathrm{Grass}_{r,A}$ such that $\mathrm{Pic}(\mathrm{Grass}_{r,A} \otimes_K K^s) = \mathbb{Z} \cdot (\mathcal{L} \otimes_K K^s)$, i.e., the base change of $\mathcal{L}$ to $\mathrm{Grass}_{r,A} \otimes_K K^s = \mathrm{Grass}_{K^s}(r, (K^s)^{\oplus n})$ generates the Picard group, if and only if period(A) divides r.*

PROOF. Consider the commutative diagram of central extensions of smooth group schemes (similar to the one from the proof of Theorem 3.15).

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{G}_{m,K} \arrow[r] \arrow[d, "(-)^r"'] & \mathbf{GL}_{n,K} \arrow[r] \arrow[d, "q"] & \mathbf{PGL}_{n,K} \arrow[r] \arrow[d, "="] & 1 \\ 1 \arrow[r] & \mathbb{G}_{m,K} \arrow[r, "s"'] & \mathbf{GL}_{n,K}/\boldsymbol{\mu}_{r,K} \cdot \mathrm{Id}_{n \times n} \arrow[r] & \mathbf{PGL}_{n,K} \arrow[r] & 1 \end{tikzcd}.$$

Using Proposition 2.16 and Remark 2.17, $\Delta_n(A)$ has order dividing r if and only if the associated $\mathbf{PGL}_{n,K}$-torsor of A lifts to a torsor for $\mathbf{GL}_{n,K}/\boldsymbol{\mu}_{r,K}$. On the other hand, the image of the natural homomorphism

$$\bigwedge^r : \mathbf{GL}_{n,K} \to \mathbf{GL}_{\binom{n}{r},K}$$

is precisely the quotient $\mathbf{GL}_{n,K}/\boldsymbol{\mu}_{r,K}$. Thus, from the commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{G}_{m,K} \arrow[r, "s"'] \arrow[d, "="'] & \mathbf{GL}_{n,K}/\boldsymbol{\mu}_{r,K} \arrow[r] \arrow[d, "\wedge^r"] & \mathbf{PGL}_{n,K} \arrow[r] \arrow[d, "\wedge^r"] & 1 \\ 1 \arrow[r] & \mathbb{G}_{m,K} \arrow[r] & \mathbf{GL}_{\binom{n}{r},K} \arrow[r] & \mathbf{PGL}_{\binom{n}{r},K} \arrow[r] & 1 \end{tikzcd}$$

a $\mathbf{PGL}_{n,K}$-torsor lifts to a torsor for $\mathbf{GL}_{n,K}/\boldsymbol{\mu}_{r,K}$ if and only if the associated $\mathbf{PGL}_{\binom{n}{r},K}$-torsor lifts to a torsor for $\mathbf{GL}_{\binom{n}{r},K}$, i.e., if it lifts from the group of projective equivalences on $\mathbb{P}(K^{\oplus\binom{n}{r}})$ the group of equivalences of $(\mathbb{P}(K^{\oplus\binom{n}{r}}), \mathcal{O}(1))$. Since the restriction of $\mathcal{O}(1)$ to $\mathrm{Grass}_K(r, K^{\oplus n})$ is the generator of the Picard group, it follows that $\Delta_n(A)$ has order dividing r if and only if there exists an invertible sheaf $\mathcal{L}$ on $\mathrm{Grass}_{r,A}$ whose base change to K^s is a generator of the Picard group. □

This is relevant to Lemma 1.6 because of the *elementary obstruction* to the existence of K-points.

Definition 1.8. [**CTS87**] Let K be a field and let X be a geometrically integral, quasi-projective, smooth K-scheme. The *elementary obstruction* to the existence of a K-point of X is the short exact sequence of $\mathrm{Gal}(K^s/K)$-modules,

$$1 \longrightarrow (K^s)^* \longrightarrow K^s(X)^* \longrightarrow K^s(X)^*/(K^s)^* \longrightarrow 1$$

where $K^s(X)$ is the fraction field of $X \otimes_K K^s$. Stated differently, the elementary obstruction is zero if and only if there exists a Galois-invariant splitting of $(K^s)^* \to K^s(X)^*$.

Remark 1.9. If X has a K-point, or even a 0-cycle of degree 1 over K, then the elementary obstruction is zero, cf. [**CTS87**, Prop. 2.2.2], [**Sko01**, Theorem 2.3.4]. And if X is projective with $\mathrm{Pic}(X \otimes_K K^s)$ isomorphic to $\mathbb{Z}$, then the elementary obstruction vanishes if and only if there exists an invertible sheaf $\mathcal{L}$ on X such that $\mathcal{L} \otimes_K K^s$ is a generator for the Picard group, cf. [**BCTS08**, Lemma 2.2(v)].

This leads to the final reinterpretation of Theorem 1.2

Lemma 1.10. *For a field K, the power $r(K)$ equals 1 if and only if for every central simple algebra A, denoting by n the integer such that $dim_K(A) = n^2$, for every integer r with $1 \leq r \leq n$, vanishing of the elementary obstruction for $Grass_{r,A}$ is a sufficient condition for existence of a K-point.*

This last reformulation suggests a problem, the refined version of Problem 1.1 for function fields of surfaces.

Problem 1.11. Let K be the function field of a surface over an algebraically closed field k. Let X be a smooth, projective K-scheme. Find sufficient conditions on $X \otimes_K K^s$ so that X has a K-point when the elementary obstruction vanishes, e.g., when there exists an invertible sheaf $\mathcal{L}$ on X such that $\mathcal{L} \otimes_K K^s$ is a generator for $\mathrm{Pic}(X \otimes_K K^s)$.

2. Abel maps over curves and sections over surfaces

Let κ be a characteristic 0 field, not necessarily algebraically closed. Let C be a smooth, projective, geometrically integral curve over κ. Let $\pi : X \to B$ be a projective, flat morphism whose geometric generic fiber is irreducible and smooth. And let $\mathcal{L}$ be a π-ample invertible sheaf on X.

Definition 2.1. For a κ-scheme S, a *family of sections of π parameterized by S* is a morphism of C-schemes

$$\tau : S \times_\kappa C \to X.$$

For an integer e, the family of sections has *degree e* if the invertible sheaf $\tau^*\mathcal{L}$ on $S \times_\kappa C$ has relative degree e over S, i.e., for every geometric point s of S, the basechange of the invertible sheaf to C_s has degree e. A pair (S, τ) as above is *universal* if for every κ-scheme S' and for every family of degree e sections of π parameterized by S,

$$\tau' : S' \times_\kappa C \to X,$$

there exists a unique κ-morphism $f : S' \to S$ such that τ' equals $\tau \circ (f, \mathrm{Id}_C)$.

Theorem 2.2 (Grothendieck). [**Gro62**, Part IV.4.c, p. 221-19] *For every integer e there exists a pair $(Sections^e(X/C/\kappa), \sigma)$ of a κ-scheme and a family of degree e sections of π parameterized by $Sections^e(X/C/\kappa)$,*

$$\sigma : Sections^e(X/C/\kappa) \times_\kappa C \to X,$$

such that $\sigma^*\mathcal{L}$ which is universal. *Moreover Sections$^e(X/C/\kappa)$ is a quasi-projective κ-scheme.*

Invertible sheaves on C of degree e are parameterized by the Picard scheme $\mathrm{Pic}^e_{C/\kappa}$. Thus, associated to the invertible sheaf $\sigma^*\mathcal{L}$ there is a morphism of κ-schemes

$$\alpha : \mathrm{Sections}^e(X/C/\kappa) \to \mathrm{Pic}^e_{C/\kappa}.$$

This morphism is the *Abel map* associated to $\mathcal{L}$.

The quasi-projective κ-scheme $\mathrm{Sections}^e(X/C/\kappa)$ is rarely projective. There is a projective scheme $\Sigma^e(X/C/\kappa)$ containing $\mathrm{Sections}^e(X/C/\kappa)$ as an open subscheme: the coarse moduli space of the stack of *stable sections*, i.e., stable maps to X whose stabilization under π is an isomorphism to C. Stable maps are an important tool in this area, but a discussion of the basics of stable maps would carry us rather far from the main story. Thus, suffice it to say that $\Sigma^e(X/C/\kappa)$ is a projective scheme, and there exists a κ-morphism

$$\alpha : \Sigma^e(X/C/\kappa) \to \mathrm{Pic}^e_{C/\kappa}.$$

whose restriction to $\mathrm{Sections}^e(X/C/\kappa)$ is the Abel map from the last paragraph. Although the following problem may at first appear to have little to do with Problem 1.11, in fact the two are intimately connected.

Problem 2.3. Let κ, $\pi : X \to C$ and $\mathcal{L}$ be as above. Assume that the base change of $\mathcal{L}$ to the geometric generic fiber of π generates the Picard group. Find conditions conditions under which there exists a sequence of closed subschemes $(Z_e \subset \Sigma^e(X/C/\kappa)_{e\gg 0}$ such that Z_e intersects $\mathrm{Sections}^e(X/C/\kappa)$ and such that the restriction of the Abel map

$$\alpha|_{Z_e} : Z_e \to \mathrm{Pic}^e_{C/\kappa}$$

is surjective with rationally connected geometric generic fiber.

In fact there do exist such conditions which capture, in some sense, the notion of “rational simple connectedness”.

3. Rational simple connectedness hypotheses

Hypothesis 1. The κ-scheme X is smooth and projective.

Hypothesis 2. The invertible sheaf $\mathcal{L}$ is π-ample and π-relatively globally generated.

All of the hypotheses are insensitive to replacing $\mathcal{L}$ by $\mathcal{L} \otimes_{\mathcal{O}_X} \pi^*\mathcal{M}$ for an invertible sheaf $\mathcal{M}$ on B. Thus, in Hypothesis 2, we may even assume $\mathcal{L}$ is ample and globally generated.

The remaining hypotheses are stated in terms of Y.

Hypothesis 3. The basechange $\mathcal{L}_Y$ of $\mathcal{L}$ to Y is very ample.

From now on we consider Y to be embedded in a projective space $\mathbb{P}^N_K$ by the complete linear system of $\mathcal{L}_Y$. In particular, $\overline{\mathcal{M}}_{0,1}(Y/K,1)$ is the projective K-scheme parameterizing pointed lines (p, L) in Y, i.e., L is a line contained in Y and p is a point of L. There is an “evaluation” K-morphism

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,1}(Y/K,1) \to Y, \quad (L,p) \mapsto p.$$

Hypothesis 4. The morphism ev above is surjective, the geometric generic fiber of ev is irreducible and rationally connected, and the fiber of ev over every codimension 1 point η of Y is "Morse", i.e., $\overline{\mathcal{M}}_{0,1}(Y/K,1)$ is smooth at every point of $\text{ev}^{-1}(\eta)$ and the only singularities on the geometric fiber over η are at worst finitely many ordinary double point singularities.

Of course there is a dense open subset of Y, Y_{free}, such that every line in Y intersecting Y_{free} is free. Thus

$$\text{ev} : \text{ev}^{-1}(Y_{\text{free}}) \to Y_{\text{free}}$$

is smooth and projective. In particular, the fiber of ev over the geometric generic point of Y is automatically smooth.

If one thinks of rational curves in Y as analogous to continuous paths in a CW complex, this hypothesis is analogous to asking that the space of continuous paths with one boundary point fixed is itself path-connected. Since continuous paths with one fixed boundary point can be retracted to a constant path to the basepoint, the condition always holds. The Morse condition is technical, but often easy to verify. Notice also that this hypothesis implies that $\mathcal{L}_Y$ is not itself a tensor power $\mathcal{K}^{\oplus d}$ for an integer $d > 1$.

The next hypothesis uses the space of chains of free lines. For an integer $n \geq 1$, a *chain of n free lines* is an n-tuple

$$(C, p_1, q_n) = ((L_1, p_1, q_1), (L_2, p_2, q_2), \ldots, (L_n, p_n, q_n))$$

where each L_i is a free line in Y, p_1 and q_i are K-points of L_i, and $q_i = p_{i+1}$ for $i = 1, \ldots, n-1$. The points p_1 and q_n are considered "boundary points" of the chain analogous to the boundary points 0 and 1 of the closed unit interval $[0, 1]$.

There is a quasi-projective K-scheme $\text{FreeChain}_2(Y/K, n)$ parameterizing chains of n free lines. And there is an "evaluation" K-morphism

$$\text{ev} : \text{FreeChain}_2(Y/K, n) \to Y \times_K Y, \quad (C, p_1, q_n) \mapsto (p_1, q_n).$$

Hypothesis 5. For some positive integer $n = n_0$, the morphism ev above is dominant and the geometric generic fiber is "birationally rationally connected", i.e., it is isomorphic as a scheme over $\overline{K(Y \times_K Y)}$ to an open subscheme of a projective, rationally connected scheme.

Under Hypotheses 3 and 4, ev is dominant if and only if Y is irreducible and $\mathcal{L}_Y$ is a generator of $\text{Pic}(Y)$. If the Picard number equals 1, the argument proving ev is dominant appears, for instance, in [**Kol96**, Corollary IV.4.14]. And the opposite direction follows from [**Kol96**, Proposition IV.3.13.3].

Together with Hypotheses 1 and 2, this implies a condition on the singular fibers of π: The specializations of the geometric generic fiber occurring as singular fibers of π are sufficiently mild that the ample generator of the Picard group of the geometric generic fiber extends to an ample invertible sheaf on the special fiber. In particular this implies all singular geometric fibers are irreducible and more.

The hypothesis that the fibers of ev are rationally connected is very strong. The fiber of ev is analogous to the space of continuous paths in a path connected CW complex connecting two points. The CW complex is simply connected if and only if this path space is simply connected. In this way we consider Hypothesis 5 to be a version of "rational simple connectedness".

The final hypothesis may seem technical, but it is essential. There are examples where Hypotheses 1 through 5 hold and yet the fibers of the Abel maps are not rationally connected.

Hypothesis 6. There exists a *very twisting scroll* in Y, i.e., a pair (R, D) consisting of a closed subscheme R of $\mathbb{P}^1_K \times_K Y$ and a Cartier divisor class D on R such that

(i) R is a *scroll*, i.e., $\mathrm{pr}_{\mathbb{P}^1} : R \to \mathbb{P}^1$ is a smooth morphism whose geometric fibers are lines in Y,
(ii) $\mathcal{O}_R(D)$ is globally generated and D has relative degree 1 over $\mathbb{P}^1$,
(iii) the normal sheaf $N_{R/\mathbb{P}^1\times Y}$ is globally generated, and
(iv) $h^1(R, N_{R/\mathbb{P}^1\times Y}(-D) \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1}(-2))$ equals 0.

In fact this complicated condition has a simple geometric meaning. By Bertini's theorem, a general member of the linear system $|D|$ is a smooth curve in R. By (ii), this curve is of the form $\tau(\mathbb{P}^1)$ for a section $\tau : \mathbb{P}^1 \to R$ of $\mathrm{pr}_{\mathbb{P}^1}$. The scroll R together with the section τ defines a family of pointed lines in Y parameterized by $\mathbb{P}^1_K$. Equivalently, it defines a K-morphism

$$\zeta : \mathbb{P}^1_K \to \overline{\mathcal{M}}_{0,1}(Y, 1).$$

The conditions (ii)–(iv) say that $\zeta(\mathbb{P}^1)$ is in the smooth locus of both morphisms

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,1}(Y, 1) \to Y$$

and

$$\rho : \overline{\mathcal{M}}_{0,1}(Y, 1) \to \overline{\mathcal{M}}_{0,0}(Y, 1),$$

that the pullback $\zeta^*T_\rho := \zeta^*(\Omega_\rho)^\vee$ is globally generated, and that the pullback $\zeta^*T_{\mathrm{ev}} := \zeta^*(\Omega^\vee_{\mathrm{ev}}$ is ample. In fact if there exists a morphism ζ with these properties, then there also exists a morphism such that both ζ^*T_ρ and ζ^*T_{ev} are ample. Then ζ is very free relative to both ev and ρ.

4. Rational connectedness of the Abel map

These are all the hypotheses. However, before stating the theorem there is one more definition.

Definition 4.1. Let $d \geq 0$ be an integer. For an algebraically closed extension field k of κ, a *d-free section* of π defined over k is a section $\sigma : C \otimes_\kappa k \to X$ defined over k such that for one (and hence every sufficiently general) effective Cartier divisor D of $C \otimes_\kappa k$ of degree d,

$$h^1(C \otimes_\kappa k, \sigma^* N_{\sigma(C\otimes_\kappa k)/X\otimes_\kappa k}(-D)) \text{ equals } 0.$$

In particular, when $d \geq \max 2g, 1$ the first cohomology vanishes if and only if the map on global sections

$$H^0(C \otimes_\kappa k, \sigma^* N_{\sigma(C\otimes_\kappa k)/X\otimes_\kappa k}) \to H^0(C \otimes_\kappa k, \sigma^* N_{\sigma(C\otimes_\kappa k)/X\otimes_\kappa k} \otimes_{\mathcal{O}_C} \mathcal{O}_D)$$

is surjective. We call a section *(g)-free* if it is d-free for some $d \geq \max 2g, 1$, i.e., for $g \geq 1$ it is $2g$-free and for $g = 0$ it is 1-free.

Theorem 4.2 (de Jong, Starr). *If $(X, \mathcal{L}, \pi)$ satisfies Hypotheses 1 through 6, then there exists a sequence $(Z_e)_{e\gg 0}$ of irreducible components Z_e of $\Sigma^e(X/C/\kappa)$ such that*

(i) *for each $e \gg 0$, a general point of Z_e parameterizes a free section of π,*
(ii) *for each $e \gg 0$, the Abel map*

$$\alpha|_{Z_e} : Z_e \to Pic^e_{C/\kappa}$$

is surjective with rationally connected geometric generic fiber, and

(iii) *for every algebraically closed field extension k of κ, for every (g)-free section $\sigma_0 : C \otimes_\kappa k \to X$ of π defined over k, and for all $\delta \gg 0$, every stable section obtained by attaching δ free lines in fibers of π to $\sigma_0(C)$ gives a k-point of $\Sigma^{e_0+\delta}(X/C/\kappa)$ lying in $Z_{e_0+\delta}$.*

Moreover every geometric fiber X_t of π is integral. And every point of the smooth locus $(X_t)_{smooth}$ is contained in a very free rational curve in $(X_t)_{smooth}$.

Note that (iii) implies that the sequence $(Z_e)_{e\gg 0}$ is *canonical*. In particular, it is Galois invariant. Thus, in order to verify the theorem over every characteristic 0 field κ, it suffices to verify the theorem over algebraically closed, characteristic 0 fields. Moreover, as the statements are geometric, it suffices to pass to "sufficiently large" algebraically closed fields, e.g., uncountable fields. This will be technically convenient later (although, of course, the interested reader can easily formulate all of our arguments over any characteristic 0 field). From this point on, we will assume that κ is an uncountable, algebraically closed field of characteristic 0.

The proof of this theorem goes beyond what we can present here. However we will present a proof in the special case necessary for de Jong's Period-Index theorem.

5. Rational simply connected fibrations over a surface

Theorem 4.2 is important because of its application to the existence of rational sections of fibrations over surfaces.

Corollary 5.1 (de Jong, Starr). *Let k be an algebraically closed field of characteristic 0. Let S be a smooth, integral, projective surface over k. Let $f : X \to S$ be a proper morphism. Assume there exists a Zariski open subset U of S and an invertible sheaf $\mathcal{L}$ on $f^{-1}(U)$ such that*

(i) *$S - U$ is a finite collection of k-points of S,*
(ii) *$f^{-1}(U)$ is smooth,*
(iii) *the restriction*
$$f|_{f^{-1}(U)} : f^{-1}(U) \to U$$
is flat and surjective
(iv) *$\mathcal{L}$ is f-ample and f-relatively globally generated, and*
(v) *the geometric generic fiber Y of f together with the base change $\mathcal{L}_Y$ of $\mathcal{L}$ satisfy Hypotheses 3–6 of Section 3.*

Then there exists a rational section of f.

Proof. There exists a Lefschetz pencil of ample divisors on S whose basepoints are all contained in U. So after replacing S by the blowing up of the base locus, replacing f and $\mathcal{L}$ by the pullbacks over the blowing up, and replacing U by its inverse image in the blowing up, we may assume in addition that there exists a surjective, projective morphism
$$r : S \to \mathbb{P}^1_k.$$
Theorem 2.2 generalizes to give a pair $(\text{Sections}^e(X/S/\mathbb{P}^1_k), \sigma)$ of a $\mathbb{P}^1_k$-scheme $\text{Sections}^e(X/S/\mathbb{P}^1_k)$ and an S-morphism
$$\sigma : \text{Sections}^e(X/S/\mathbb{P}^1_k) \times_{\mathbb{P}^1} S \to X$$

which is universal among such pairs. And there is a generalization of the definition of $\Sigma^e(X/C/\kappa)$ giving a $\mathbb{P}^1$-scheme $\Sigma^e(X/S/\mathbb{P}^1_k)$ and a family of stable sections

$$(\rho : \mathcal{C} \to \Sigma_e(X/S/\mathbb{P}^1_k), h : \mathcal{C} \to X)$$

of π having $\mathcal{L}$-degree e. And again using the methods of [**KM76**], there is an Abel map of $\mathbb{P}^1_k$-schemes

$$\alpha : \Sigma_e(X/S/\mathbb{P}^1_k) \to \mathrm{Pic}^e_{S/\mathbb{P}^1_k},$$

at least when we restrict over the dense open subset $(\mathbb{P}^1_k)_{\omega,\text{good}}$ – the maximal open subset over which ω is smooth (over the finitely many k-points of $\mathbb{P}^1_k - (\mathbb{P}^1_k)_{\omega,\text{good}}$ there are some technical difficulties in representing the relative Picard functor by a projective scheme).

Denote the function field $k(\mathbb{P}^1_k)$ by κ (of course it is not algebraically closed). So Spec $\kappa \to \mathbb{P}^1_k$ is the generic point of $\mathbb{P}^1_k$. The fibers of these $\mathbb{P}^1_k$ schemes and morphisms of $\mathbb{P}^1_k$-schemes over Spec κ are

$$\Sigma^e(X/S/\mathbb{P}^1_k) \times_{\mathbb{P}^1_k} \text{Spec } \kappa = \Sigma^e(X_\kappa/C/\kappa),$$

$$\mathrm{Pic}^e(X/S/\mathbb{P}^1_k) \times_{\mathbb{P}^1_k} \text{Spec } \kappa = \mathrm{Pic}^e_{C/\kappa}$$

where X_κ denotes $X \times_{\mathbb{P}^1_k}$ Spec κ and C denotes $S \times_{\mathbb{P}^1_k}$ Spec κ. Moreover the basechange of the $\mathbb{P}^1_k$-morphism α above is the Abel map

$$\alpha : \Sigma^e(X_\kappa/C/\kappa) \to \mathrm{Pic}^e_{C/\kappa}$$

defined in Section 2.

The hypotheses on $X, \mathcal{L}, U$ and Y imply Hypotheses 1–6 for $X_\kappa, \mathcal{L}_\kappa$ and Y (which hasn't actually changed). Thus, by Theorem 4.2, there exists a sequence $(Z_e)_{e\geq e_0}$ of irreducible components Z_e of $\Sigma^e(X_\kappa/C/\kappa)$ satisfying Conditions (i)–(iii) of Theorem 4.2. For every $e \geq e_0$, the closure of Z_e in $\Sigma^e(X/S/\mathbb{P}^1_k)$ is an irreducible component $Z_{\mathbb{P}^1_k,e}$ of $\Sigma^e(X/S/\mathbb{P}^1_k)$. Because of Conditions (i) for $(Z_e)_{e\geq e_0}$, a general point of $Z_{\mathbb{P}^1_k,e}$ intersects $\text{Sections}^e(X/S/\mathbb{P}^1_k)$ is a dense open subset of $Z_{\mathbb{P}^1_k,e}$. And because of Condition (ii), for $e \gg 0$ the Abel map

$$\alpha : Z_{\mathbb{P}^1_k,e} \to \mathrm{Pic}^e_{S/\mathbb{P}^1_k}$$

is surjective (at least over $(\mathbb{P}^1_k)_{\omega,\text{good}}$), and the geometric generic fiber is rationally connected.

For every integer e there exist sections

$$\sigma : (\mathbb{P}^1_k)_{\omega,\text{good}} \to \mathrm{Pic}^e_{S/\mathbb{P}^1_k}.$$

Indeed, the exceptional divisor over the blowing up of every basepoint of the Lefschetz pencil of divisors on S is a Cartier divisor on S which defines a section to $\mathrm{Pic}^1_{S/\mathbb{P}^1_k}$. So e times this section gives a section σ.

For $e \gg 0$, the geometric generic fiber of

$$\alpha : Z_{\mathbb{P}^1_k,e} \to \mathrm{Pic}^e_{S/\mathbb{P}^1_k}$$

is rationally connected. It may happen, nonetheless, that the fiber of

$$\alpha : Z_{\mathbb{P}^1_k,e} \to \mathrm{Pic}^e_{S/\mathbb{P}^1_k}$$

over the geometric generic point of $\sigma(\mathbb{P}^1_k)$ is not rationally connected. There is a standard way to deal with this issue. We can always realize $\sigma(\mathbb{P}^1_k)$ as an irreducible component of multiplicity 1 in a complete intersection curve C_0 in $\mathrm{Pic}^e_{S/\mathbb{P}^1_k}$, i.e., C_0 is a curve which is a complete intersection of $g = g(C)$ very ample divisors in $\mathrm{Pic}^e_{S/\mathbb{P}^1_k}$. And we can deform these divisors to obtain a family of complete intersection curves (C_t) in $\mathrm{Pic}^e_{S/\mathbb{P}^1_k}$ parameterized by a DVR $R = k[\![t]\!]$. By deforming the divisors appropriately, we can arrange that the fiber of C_t over the geometric generic point $\overline{k(\!(t)\!)}$ is smooth, and the fiber of α over the geometric generic point of C_t is rationally connected.

Thus after performing a base change from k to $L = \overline{k(\!(t)\!)}$, the restriction of α over the geometric generic fiber

$$C_L := C \otimes_{k[\![t]\!]} \overline{k(\!(t)\!)}$$

is a rationally connected fibration over the smooth, projective curve C_L over the algebraically closed field L. Thus, by Theorem 1.9, there exists a section. But by the usual limit theorems, every section over L can, in fact, be defined over the fraction field $k(R') = k\left(\!\left(t^{1/N}\right)\!\right)$ of some finite extension $R' = k\left[\!\left[t^{1/N}\right]\!\right]$ of R.

Since $\sigma(\mathbb{P}^1_k)$ is an irreducible component of multiplicity 1 in C_0, $\sigma(\mathbb{P}^1_k)$ is still an irreducible component of multiplicity one in the base change $C_t \otimes_R R'$. Thus $C_t \otimes_R R'$ is normal at the generic point of $\sigma(\mathbb{P}^1_k)$, which is a codimension 1 point of this 2-dimensional scheme. Finally, by the valuative criterion of properness applied to the proper $\mathbb{P}^1$-scheme $Z_{\mathbb{P}^1_t,e}$, the section defined over $C_t \otimes_R k(R')$ extends to the generic point of $\sigma(\mathbb{P}^1_k)$. Thus there exists a rational section

$$\tau : \sigma(\mathbb{P}^1_k) \dashrightarrow Z_{\mathbb{P}^1_k,e}$$

of α. Again by the valuative criterion of properness, this extends to all of $\sigma(\mathbb{P}^1_k)$.

It may happen that $\tau(\mathrm{Spec}\ \kappa)$ parameterizes a stable section rather than a section. But every stable section has a unique handle, i.e., the unique component which is the image of a section. Because the handle is canonical, it satisfies the cocycle condition for Galois descent, and thus is defined over the same field of definition as the stable section – in this case κ. Thus finally, quite possibly for some integer $e' < e$, there exists a κ-point of $\mathrm{Sections}^{e'}(X/S/\mathbb{P}^1_k)$. By the universal property of $\mathrm{Sections}^{e'}(X/S/\mathbb{P}^1_k)$, this is equivalent to a morphism

$$\mathrm{Spec}\ \kappa \times_{\mathbb{P}^1_k} S \to X.$$

The closure of the image of this morphism is an integral closed subscheme Z of X such that the projection $Z \to S$ is a proper morphism which is an isomorphism over C. Thus it is birational, i.e., Z is the closure of the image of a rational section of $X \to S$. Therefore $f : X \to S$ has a rational section. $\square$

6. Discriminant avoidance

Even though the theorems above require that $\mathrm{char}(\kappa)$ equals 0, there is a technique for "lifting" from positive characteristic. It is recalled very briefly below, but the proof will not be discussed.

Let $\mathcal{G}$ be a reductive group scheme over some base scheme T. Let $\mathcal{X}$ be a smooth, projective T-scheme on which $\mathcal{G}$ acts. For every T-scheme S and every $\mathcal{G}$-torsor $\mathcal{T}$ over S, there is an associated S-scheme

$$\mathcal{X}_{\mathcal{T}} := \mathcal{X} \times_T \mathcal{T}/\mathcal{G},$$

the quotient by the free action of $\mathcal{G}$. Let U be a dense open subscheme of T. Let c be a nonnegative integer. Consider the following two properties of the datum $(T, \mathcal{G}, \mathcal{X})$ and the integer c.

PROPERTY 6.1. For every algebraically closed field over U, Spec $k \to U$, for every projective, integral k-scheme S of dimension c, and for every $\mathcal{G}$-torsor $\mathcal{T}$ over S, the projection

$$\mathcal{X}_{\mathcal{T}} \to S$$

admits a rational section.

PROPERTY 6.2. For every algebraically closed field over T, Spec $k \to U$, for every quasi-projective, integral k-scheme S of dimension c, and for every $\mathcal{G}$-torsor $\mathcal{T}$ over S, the projection

$$\mathcal{X}_{\mathcal{T}} \to S$$

admits a rational section.

The basic technique of "discriminant avoidance" proves the following.

PROPOSITION 6.3. [**dJS05**] *If Property 1 holds, then Property 2 holds.*

COROLLARY 6.4. *To prove Theorem 1.2, it suffices to prove Theorem 4.2 in the special case that the morphism*

$$X \otimes_\kappa \overline{\kappa} \to C \otimes_\kappa \overline{\kappa}$$

is of the form $\mathrm{Grass}_{C \otimes_\kappa \overline{\kappa}}(r, \mathcal{E})$ *for a locally free sheaf* $\mathcal{E}$ *of rank* n *divisible by* r *on* $C \otimes_\kappa \overline{\kappa}$.

PROOF. By Lemma 1.6, to prove Theorem 1.2 it suffices to prove that for every pair of positive integers r, n with r dividing n, for every field K as in Theorem 1.2, for every central simple algebra A with $\dim_K(A) = n^2$ and period(A) dividing r, the generalized Brauer-Severi variety $\mathrm{Grass}_{r,A}$ has a K-point.

Define $T = \mathrm{Spec}\ \mathbb{Z}_p$ and define $U = \mathrm{Spec}\ \mathbb{Q}_p$. Let $\mathcal{G}$ be the group scheme $\mathbf{GL}_{n,T}/\boldsymbol{\mu}_{r,T}$ appearing in the proof of Lemma 1.7. And let $(\mathcal{X}, L)$ be $(\mathrm{Grass}_T(r, \mathcal{O}_T^{\oplus n}), \mathcal{O}(1))$. By the proof of Lemma 1.7, for every algebraically closed field k, for every fraction field K of a surface over k, and for every central simple algebra A over K with $\dim_K(A) = n^2$ and period(A) dividing r, there exists a quasi-projective surface S over k with fraction field K, and there exists a $\mathcal{G}$-torsor $\mathcal{T}$ over S such that $\mathrm{Grass}_{r,A}$ is the generic fiber of the S-scheme $\mathcal{X}_{\mathcal{T}}$. Thus by Proposition 6.3, to prove $\mathrm{Grass}_{r,A}$ always has a K-point, it suffices to prove that for every *characteristic* 0 algebraically closed field k, for every *projective* surface S over k, and for every $\mathcal{G}$-torsor $\mathcal{T}$ over S, $\mathcal{X}_{\mathcal{T}}$ has a rational section over S. Define $\mathcal{L}$ on $\mathcal{X}_{\mathcal{T}}$ to be the twist of L on $\mathcal{X}$.

By resolution of singularities on surfaces in characteristic 0 (which is reasonably easy), we may as well assume S is smooth. Then the proof of Corollary 5.1 reduces existence of a rational section to Theorem 4.2 for the restriction of $\mathcal{X}_{\mathcal{T}}$ and $\mathcal{L}$ to the generic fiber C of a Lefschetz pencil of divisors on S. By Corollary 2.12 and by Proposition 3.2(i), the base change of $\mathcal{T}$ to $C \otimes_\kappa \overline{\kappa}$ lifts to a torsor for $\mathbf{GL}_{n,\overline{\kappa}}$. This is equivalent to a locally free sheaf $\mathcal{E}$ of rank n on $C \otimes_\kappa \overline{\kappa}$. Thus the base change of $\mathcal{X}_{\mathcal{T}}$ to $C \otimes_\kappa \overline{\kappa}$ equals $\mathrm{Grass}_{C \otimes_\kappa \overline{\kappa}}(r, \mathcal{E})$. $\square$

7. Proof of the main theorem for Grassmann bundles

We will not prove Theorem 4.2, but we will prove it for Grassmann bundles coming from Azumaya algebras over C. The following proof is an *ad hoc* argument using the well-known fact that the moduli space of stable sheaves of fixed rank and determinant on a curve is unirational.

PROPOSITION 7.1. *Let κ be a characteristic 0 field, let C be a smooth, projective, geometrically integral κ-scheme, let*

$$\pi : X \to C$$

be a flat, projective morphism, and let $\mathcal{L}$ be a π-ample invertible sheaf on X whose base change to the geometric generic fiber of π generates the Picard group. Assume further that the morphism

$$\pi \otimes Id : X \otimes_\kappa \overline{\kappa} \to C \otimes_\kappa \overline{\kappa}$$

is a Grassmann bundle $Grass_{C\otimes_\kappa\overline{\kappa}}(r, \mathcal{E})$ for a locally free sheaf $\mathcal{E}$ of rank n divisible by r on $C \otimes_\kappa \overline{\kappa}$. Then the conclusion of Theorem 4.2 holds. Therefore, by Corollary 6.4, de Jong's Period-Index theorem, Theorem 1.2, also holds.

PROOF. Thus $\text{Sections}^e(X/C/\kappa) \otimes_\kappa \overline{\kappa}$ is the scheme parametrizing locally free, rank r quotients $\mathcal{Q}$ of $\mathcal{E}^\vee$ of degree e. And the Abel map associates to $\mathcal{Q}$ the determinant $\det(\mathcal{Q}) = \bigwedge^r \mathcal{Q}$ (up to a constant translation which makes no difference in the following).

Denote by V_e the subscheme V_e of $\text{Sections}^e(X/C/\kappa) \otimes_\kappa \overline{\kappa}$ parametrizing quotients with are stable sheaves. Since stability is an open condition in families, V_e is an open subscheme. Moreover, since $\mathcal{Q}$ is stable if and only if every twist of $\mathcal{Q}$ by an invertible sheaf is stable, V_e is independent of the choice of lift of the torsor $(\mathcal{T} \times_S C) \otimes_\kappa \overline{\kappa}$ to a $\mathbf{GL}_{n,L}$-torsor over $C \otimes_\kappa L$, where $\mathcal{T}$ is as in the proof of Corollary 6.4. Therefore V_e is invariant under the action of $\text{Gal}(\overline{\kappa}/\kappa)$, i.e., V_e equals $U_e \otimes_\kappa \overline{\kappa}$ for some open subscheme U_e of $\text{Sections}^e(X/C/\kappa)$. Finally, for $e \gg 0$, there exist quotients $\mathcal{Q}$ which are stable so that U_e is nonempty. Define Z_e to be the closure of U_e.

By standard arguments it is not hard to show that for $e \gg 0$ *every* stable sheaf of rank r and degree e on $C \otimes_\kappa L$ appears as a quotient $\mathcal{Q}$, and also that

$$h^1(C, Hom_{\mathcal{O}_{C\otimes_\kappa L}}(\mathcal{E}, \mathcal{Q}) \text{ equals } 0.$$

Therefore the forgetful morphism

$$\Phi : U_e \to \text{Bun}_{r,e,\text{stable}}(C \otimes_\kappa L)$$

to the moduli space of stable rank r, degree e, locally free sheaves on $C \otimes_\kappa L$ is surjective and the fiber over every point $\mathcal{Q}$ is an open subset of the constant rank affine space $\text{Hom}_{\mathcal{O}_{C\otimes_\kappa L}}(\mathcal{E}, \mathcal{Q})$, i.e., Φ is an open subscheme of an affine bundle over the moduli space of stable sheaves. And the Abel map factors through the "usual" Abel map

$$\det : \text{Bun}_{r,e,\text{stable}}(C \otimes_\kappa L) \to \text{Pic}^e_{C\otimes_\kappa L/L}.$$

As is well-known, the morphism det above has rationally connected (and even unirational) fibers of dimension $(r^2 - 1)(g(C) - 1)$, i.e., the moduli space of stable sheaves of rank r and fixed determinant is unirational. Thus the geometric generic fiber of

$$\alpha|_{U_e} : U_e \to \text{Pic}^e_{C/\kappa}$$

is isomorphic to an open subscheme of an affine bundle over a rationally connected variety, which is itself an open subscheme of a rationally connected projective variety. But it is also isomorphic to a dense open subset of the projective variety which is the geometric generic fiber of

$$\alpha|_{Z_e} : Z_e \to \mathrm{Pic}^e_{C/\kappa}.$$

Since rational connectedness is a birational invariant among projective varieties, the geometric generic fiber of $\alpha|_{Z_e}$ is rationally connected. $\square$

CHAPTER 5

Rational simple connectedness and Serre's "Conjecture II"

1. Generalized Grassmannians are rationally simply connected

This chapter is extracted from an earlier set of lecture notes, [**Sta08**]. The omitted proofs can be found there. The results of this chapter are joint work with A. J. de Jong and Xuhua He, heavily relying on a strategy proposed by Philippe Gille.

The proof of Proposition 7.1 used an *ad hoc* argument. But in fact one can directly verify the hypotheses of Theorem 4.2 for Grassmannians. The most difficult hypothesis to verify is the existence of a very twisting scroll. This is not too bad for Grassmannians. But for *generalized Grassmannians*, i.e., minimal (positive dimensional) homogeneous spaces, for other groups, e.g., groups of type E_8, it becomes unreasonable to verify this "by hand". Thus it is extremely fortunate that Xuhua He has given a very simple, elegant proof that every generalized Grassmannian satisfies the hypotheses. Moreover, the proof uses only general results about root systems – it does not use the classification of simple algebraic groups. The conclusion is the following.

Corollary 1.1 (de Jong, He, Starr). *Let k be an algebraically closed field of characteristic 0. Let X be a smooth, connected, projective k-scheme of positive dimension which is a homogeneous space for a linear algebraic group scheme G over k. Assume there exists an invertible sheaf $\mathcal{L}$ on X which is an ample generator for Pic(X). Then X and $\mathcal{L}$ satisfy Hypotheses 3–6 of Section 3.*

Corollary 1.2 (de Jong, He, Starr). *Let k be an algebraically closed field of characteristic 0. Let S be a smooth, integral, projective surface over k. Let $f : X \to S$ be a proper morphism. Assume there exists a Zariski open subset U of S and an invertible sheaf $\mathcal{L}$ on $f^{-1}(U)$ such that*

(i) *$S - U$ is a finite collection of k-points of S,*
(ii) *$f^{-1}(U)$ is smooth,*
(iii) *the restriction*
$$f|_{f^{-1}(U)} : f^{-1}(U) \to U$$
is flat and surjective
(iv) *$\mathcal{L}$ is f-relatively globally generated, and*
(v) *the geometric generic fiber Y of f is a homogeneous space for a linear algebraic group over $K = \overline{k(S)}$ and $\mathcal{L}_Y$ is an ample generator for Pic(Y).*

Then there exists a rational section of f.

2. Statement of the theorem

Serre's Conjecture II, Conjecture 5.5, has been proved for some important classes of fields, but in general it remains open. One class of fields satisfying the hypotheses of the conjecture is the class of function fields K of surfaces over an algebraically closed field k. Notice that if $\mathrm{char}(k)$ is positive, then K is not perfect but does satisfy the conditions of Remark 5.6.

For these fields, there has been tremendous progress. In particular, work of Merkurjev–Suslin, Ojanguren–Parimala, Colliot-Thélène–Gille–Parimala and Gille has reduced Serre's Conjecture II for all function fields K of surfaces over an algebraically closed field k to the special case for such fields when G is an simple algebraic group of type E_8, cf. [**CTGP04**, Theorem 1.2(v)]. The main application of Theorem 4.2 and Corollary 5.1 settles the "split case" of Serre's Conjecture II for function fields K.

Theorem 2.1 (de Jong, He, Starr). *Let k be an algebraically closed field and let K/k be the function field of a surface. Let G be a connected, simply connected, semisimple algebraic group over k. Every G-torsor over K is trivial.*

In particular, if G_K is a simple algebraic group over K of type E_8, then G_K is itself split and thus every G_K-torsor over K is trivial. So Serre's Conjecture II holds over function fields for groups of type E_8.

Remark 2.2. As we explain further below, in addition to Corollary 1.2, the main input to Theorem 2.1 is a strategy due to Philippe Gille.

3. Reductions of structure group

There is a statement that implies both Theorem 2.1 as well as again proving Theorem 1.2 (it is essentially the same proof as above). The statement below is a generalization of a consequence of Theorem 4.2 proved by Philippe Gille. Gille's strategy is absolutely fundamental in the proof of Theorem 2.1. Let k be an algebraically closed field. Let G be a (smooth) connected, simply connected, semisimple algebraic group over k, and let P be a (reduced) parabolic subgroup of G. The center Z_G of G is a finite group scheme which is contained in P. There is a maximal quotient $P \twoheadrightarrow T_P$ which is an algebraic torus. Denote by $Z_{G,P}$ the kernel of the induced homomorphism $Z_G \to T_P$. The natural action of G on G/P lifts canonically to a linear action on every invertible sheaf over G/P. The finite subgroup scheme $Z_{G,P}$ is the maximal subgroup acting trivially on G/P and on every invertible sheaf over G/P. Thus $G/Z_{G,P}$ is the maximal quotient of G acting on G/P whose action lifts to a linear action on every invertible sheaf over G/P.

Theorem 3.1. *Let K/k be the function field of a surface over k. For every torsor $\mathcal{T}$ for $G/Z_{G,P}$ over K, the associated K-variety $X = \mathcal{T}/P$ has a K-point. Equivalently, the torsor $\mathcal{T}$ has a reduction of structure group to $P/Z_{G,P}$.*

Proof. Let T denote Spec of the Witt ring of k, e.g., $T = \mathrm{Spec}\ k$ if k has characteristic 0 and $T = \mathrm{Spec}\ \mathbb{Z}_p$ if k equals $\mathbb{Z}/p\mathbb{Z}$. And denote by U the open subset of T which consists of the generic point only.

Associated to the root datum for G, we can construct a smooth, linear algebraic group scheme G_T over T. And associated to the parabolic P, we can construct a closed subgroup scheme P_T over T. The definition of $Z_{G,P}$ extends to give a finite, flat group scheme Z_{G_T,P_T} over T. The quotient group scheme $\mathcal{G} = G_T/Z_{G_T,P_T}$ is a reductive group scheme since it is T-flat and the closed fiber is reductive. And the T-scheme $\mathcal{X} = G_T/P_T$ is smooth and quasi-projective. Since the closed fiber is proper over k, $\mathcal{X}$ is projective over T. Thus $\mathcal{G}$ and $\mathcal{X}$ satisfy the hypotheses in Section 6. The goal is to prove Property 6.2 for $c = 2$. Because of Proposition 6.3, it suffices to prove Property 6.1 for $c = 2$. In particular, to prove the theorem it suffices to assume that k has characteristic 0.

Now we use an induction argument proposed by Philippe Gille. The induction is on the corank, $\mathrm{rank}(G) - \mathrm{rank}(P)$. The base case is when P is a maximal parabolic. Again applying Proposition 6.3, to prove the result for all $G/Z_{G,P}$-torsors over fraction fields K of surfaces over k, it suffices to prove the result for each torsor which is the generic fiber of a $G/Z_{G,P}$-torsor over a smooth, projective, connected surface S over k. In this case the S-scheme $\mathcal{X}_S = [\mathcal{T}_S \times (G/P)]/(G/Z_{G,P})$ satisfies the hypotheses of Corollary 1.2. Thus Corollary 1.2 implies the result in this case.

By way of induction, assume the corank is > 1 and the result is known for all smaller values of the corank. Since the corank is > 1, P is not a maximal parabolic. Let Q be a maximal parabolic containing P. Then $Z_{G,P}$ is contained in $Z_{G,Q}$. For every $G/Z_{G,P}$-torsor over K, by the base case, the associated $G/Z_{G,Q}$-torsor has a reduction of structure group to $Q/Z_{G,Q}$. Thus the original $G/Z_{G,P}$-torsor has a reduction of structure group to $Q/Z_{G,P}$ (observe $(G/Z_{G,P})/(Q/Z_{G,P})$ is the same as $(G/Z_{G,Q})/(Q/Z_{G,Q})$ since both are just G/Q).

Now Q has a filtration by normal subgroup schemes,

$$Q = Q_0 \supset Q_1 \supset Q_2,$$

where Q_2 is the unipotent radical of Q and where Q_0/Q_1 is the maximal quotient of Q which is of multiplicative type, i.e., isomorphic to $\mathbf{G}_{m,k}$. By Hilbert's Theorem 90, every Q_0/Q_1-torsor over K is trivial, thus there is a reduction of structure group to $Q_1/Z_{G,P}$ (by construction $Z_{G,P}$ is contained in $P \cap Q_1$). And over a characteristic 0 field, every torsor for a unipotent group is trivial. Thus this $Q_1/Z_{G,P}$-torsor has a reduction of structure group to $(P \cap Q_1)/Z_{G,P}$ if and only if the associated $Q_1/Z_{G,P}Q_2$ torsor has a reduction of structure group to $(P \cap Q_1)/Z_{G,P}(P \cap Q_2)$. But Q_1/Q_2 is again a semisimple, simply connected algebraic group, G' $(P \cap Q_1)/(P \cap Q_2)$ is a parabolic subgroup P', and $Z_{G',P'}$ equals the image of $Z_{G,P}$. Since the corank of P' in G' is 1 less than the corank of P in G, by the induction hypothesis every $G'/Z_{G',P'}$-torsor over the fraction field of a surface K has a reduction of structure group to $P'/Z_{G',P'}$. Thus every $G/Z_{G,P}$-torsor over K has a reduction of structure group to $P/Z_{G,P}$. Therefore the result is proved by induction on the corank. $\square$

Proof of Theorem 1.2. This is roughly the same as the proof already presented. We include this proof to illustrate how Theorem 3.1 connects the Period-Index problem, Problem 4.15, and Serre's Conjecture II, Conjecture 5.5. Let G be $\mathbf{SL}_{n,k}$. Let m be an integer $1 < m < n$ and which divides n. Let P be the maximal parabolic subgroup of $\mathbf{SL}_{n,k}$ consisting of upper block matrices with the upper right block of size m and another diagonal block of size $n - m$. The center Z_G of $\mathbf{SL}_{n,k}$ is the group scheme μ_n of n^{th} roots of unity. And $Z_{G,P}$ is the subgroup scheme μ_m.

One can prove that a torsor for $G/Z_G = \mathbf{PGL}_{n,k}$ has a reduction of structure group to $G/Z_{G,P}$ if and only if the order of the corresponding element in the Brauer group $H^2(\mathrm{Gal}(K), \mu_n)$ divides m. And then, by Theorem 3.1, there is a reduction of structure group to $P/Z_{G,P}$.

Here is a reformultation in terms of central simple algebras. Let C be a central simple algebra with center K and with $\dim_K(C) = n^2$. Let $\mathcal{T}_K$ be the K-scheme whose set of A-points for each commutative K-algebra A equals the set of A-algebra isomorphisms of $C \otimes_K A$ with $\mathrm{Mat}_{n\times n,A}$. Since the automorphism group of $\mathrm{Mat}_{n\times n}$ is $\mathbf{PGL}_n$, $\mathcal{T}_K$ is a $\mathbf{PGL}_{n,k}$-torsor over K. By the previous paragraph, if the order of $[C]$ in the Brauer group of K divides m, then there is a reduction of structure group to $P/Z_{G,P}$. But this is the same thing as an isomorphism of C with $\mathrm{Mat}_{m\times m,K} \otimes_K B$ for some central simple algebra B. In particular, if D is a division algebra over K with $\dim_K(D) = n^2$, then D is not isomorphic to $\mathrm{Mat}_{m\times m,K} \otimes_K B$. Thus the order of $[D]$ does not divide m. Since this holds for every proper divisor m of n, the conclusion is that the order of $[D]$ equals n. $\square$

Proof of Theorem 2.1. By the same argument as in the proof of Theorem 3.1, it suffices to prove the case when k has characteristic 0. Denote by B a Borel subgroup of G. Then $Z_{G,B}$ is the trivial group scheme. So by Theorem 3.1, every G-torsor over K has a reduction of structure group to a B-torsor over K. Denote by $R_u(B)$ the unipotent radical of B. Since B is connected and solvable, $B/R_u(B)$ is of multiplicative type, i.e., isomorphic to $\mathbf{G}_{m,k}^r$ where r is the rank of G. By Hilbert's Theorem 90, every torsor for $B/R_u(B)$ is trivial. Thus there is a reduction of structure group to $R_u(B)$. But every torsor for a unipotent group over a characteristic 0 field is trivial. Thus the B-torsor is trivial, and hence the original G-torsor was also trivial. $\square$

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