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MAT 544 Problem Set 4

Problems.

Problem 1 Look over the problems and solutions from Midterm 1, now posted on the exams page of the course webpage. Understand the correct solutions to each of the problems. Each student will get full credit for this problem; there is nothing to write up or turn in (but I do expect each student to actually understand the solutions).

Problem 2 For an $\mathbb{R}$-vector space U, recall that two norms $\|\bullet\|_U$ and $\|\bullet\|'_U$ are *equivalent* if there exist real numbers $0 < m, M$ such that for every $\vec{u} \in U$,

$$m\|\vec{u}\|_U \leq \|\vec{u}\|'_U \leq M\|\vec{u}\|_U.$$

For normed vector spaces $(V, \|\bullet\|_V)$ and $(W, \|\bullet\|_W)$, an open subset $O \subset V$, and a continuous function $F : O \to W$, recall that the function F is (Frechet) differentiable at $\vec{v}_0 \in O$ if there exists a bounded linear operator

$$dF_{\vec{v}_0} : V \to W$$

such that for every real number $\epsilon > 0$, there exists a real number $\delta > 0$ with

$$\|F(\vec{v}_0 + \vec{v}) - F(\vec{v}_0) - dF_{\vec{v}_0}(\vec{v})\|_W \leq \epsilon\|\vec{v}\|_V$$

whenever $\|\vec{v}\|_V < \delta$. In this case the bounded linear operator $dF_{\vec{v}_0}$ is called the (Frechet) derivative of F at $\vec{v}_0$.
Prove that if $\|\bullet\|'_V$ is a norm on V which is equivalent to $\|\bullet\|_V$, then F is differentiable at $\vec{v}_9$ with respect to $\|\bullet\|_V$ if and only if it is differentiable at $\vec{v}_0$ with respect to $\|\bullet\|'_V$, and in this case the derivatives are equal. Similarly, if $\|\bullet\|'_W$ is a norm on W which is equivalent to $\|\bullet\|_W$, prove that F is differentiable at $\vec{v}_0$ with respect to $\|\bullet\|_W$ if and only if it is so with respect to $\|\bullet\|'_W$, and in this case the derivatives are equal. In particular, if V and W are finite dimensional, then all norms on V, respectively on W, are equivalent, hence differentiability is an intrinsic property.

Problem 3 Let V be a finite dimensional real vector space. Denote by $L(V, V)$ the vector space of linear transformations $T : V \to V$ (all of which are automatically bounded with respect to all norms on V). Denote by $\mathrm{Det}(T)$ and $\mathrm{Tr}(T)$ the determinant and trace of T (with respect to one, and hence any, basis for V). Thus

$$\mathrm{Det} : L(V, V) \to \mathbb{R}$$

is a continuous function and

$$\mathrm{Tr} : L(V, V) \to \mathbb{R}$$

is a (bounded) linear transformation. Prove that Det is differentiable at Id_V and $d(Det)_{\mathrm{Id}_V}$ equals Tr.

Problem 4 Let (X, d_X) be a metric space. For this problem, an *almost contraction* is a continous map $T : (X, d_X) \to (X, d_X)$ such that for all $x, x' \in X$ with $x \neq x'$, we have

$$d_X(T(x), T(x')) < d_X(x, x'),$$

where this is strict inequality.

(a) Prove that an almost contraction has at most one fixed point x, and this is the same as a point of X at which the following function attains a minimum

$$f : X \to \mathbb{R}_{\geq 0}, \quad f(x) = d_X(x, T(x)).$$

(b) Prove that every almost contraction of a compact metric space has a fixed point.

Problem 5 Let $d_{\mathbb{R}}$ be the usual distance function on $\mathbb{R}$, namely $d_{\mathbb{R}}(s, t) = |t - s|$. Find an example of an almost contraction $f : \mathbb{R} \to \mathbb{R}$ which has no fixed point. Conclude that in Problem 4 it does not suffice to replace "compact" by "complete" (although, of course, "complete" is sufficient in the Banach contraction mapping fixed point theorem).