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MAT 544 Problem Set 2 Solutions

Problems.

Problem 1 A metric space is *separable* if it contains a dense subset which is finite or countably infinite. Prove that every totally bounded metric space X is separable. Also give an example of a bounded, separable metric space which is not totally bounded.

Problem 2 Recall from lecture that for a sequence $(x_n)_{n\in\mathbb{Z}_{>0}}$ with no convergent subsequence, the infinite set $S = \{x_n | n \in \mathbb{Z}_{>0}\}$ has a covering by balls $(B_{r(x)}(x))_{x\in S}$ such that every $B_{r(x)} \cap S = \{x\}$.

(a) Prove that the covering $(B_{r(x)/2}(x))_{x\in S}$ has the same property as above, and further for every pair of distinct points x, y of S, the balls $B_{r(x)/x}(x)$ and $B_{r(y)/2}(y)$ are disjoint in X.

(b) Define a continuous function $f : X \to \mathbb{R}$ which equals 0 on the complement of all of the balls $B_{r(x)/2}(x)$ and which is unbounded.

(c) Conclude that a metric space X is sequentially compact if and only if every continuous function $f : X \to \mathbb{R}$ is bounded.

Problem 3 Theorem 4.5.1 on p. 211 of the textbook proves that a continuous function $f : X \to Y$ is uniformly continuous if X is compact. Give a second proof by applying Lemma 4.5.3 to the inverse images of ϵ-balls of Y.

Problem 4 For a metric space X, recall the definition of the associated metric space,

$$C^\infty(X, \mathbb{R}) := \{f : X \to \mathbb{R} | f \text{ continuous and bounded }\},$$

$$\|f\| := \text{ least upper bound } (|f(x)|)_{x\in X}, \quad d(f, g) := \|f - g\|.$$

Prove that for bounded, continuous functions $(f_n : X \to \mathbb{R})_{n\in\mathbb{Z}_{>0}}$ and $f : X \to \mathbb{R}$, the sequence (f_n) converges to f in this metric space if and only if the sequence of functions converges uniformly to f. This justifies the names “uniform norm” and “uniform metric”.

Problem 5 Let X be a metric space and let $(f_n : X \to \mathbb{R})_{n\in\mathbb{Z}_{>0}}$ be an **equicontinuous** sequence of bounded, continuous functions which converge pointwise to the zero function. Recall this means that for every $\epsilon > 0$ and for every $x \in X$, there exists $\delta = \delta(x, \epsilon) > 0$ such that $d_X(y, x) < \delta$ implies for all $n \in \mathbb{Z}_{>0}$ that $|f_n(y) - f_n(x)| < \epsilon$.

(a) If X is compact, prove that the sequence converges uniformly.

Hint. Show that for every $\epsilon > 0$ and for every $x \in X$, there exists a $\delta = \delta(x, \epsilon) > 0$ and a positive integer $N = N(x, \epsilon)$ such that $d_X(y, x) < \delta$ implies for all $n \geq N$ that $|f_n(y)| < \epsilon$. Then apply Theorem 4.5.3 to this covering of X by δ-balls.

(b) Let $(x_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence in X with no convergent subsequence. With notation as above, find an equicontinuous sequence of bounded, continuous functions $(f_x : X \to \mathbb{R})_{x \in S}$ such that f_x is zero on the complement of $B_{r(x)/2}(x)$, and thus the sequence $(f_x)_{x \in S}$ automatically converges pointwise to the zero function, and yet the functions do not converge uniformly to the zero function.

Solutions to Problems.

Solution to (1) Let $\epsilon > 0$ be a real number. A subset S of a metric space (X, d_X) is ϵ-*dense* for every $x \in X$, there exists $y \in S$ with $d(x, y) < \epsilon$. Note that if S is ϵ-dense, then for every real $\delta > \epsilon$, also S is δ-dense. Also if S contains a subset Y, and if Y is ϵ-dense, then also S is ϵ-dense – the element $y \in Y$ in the definition above is also an element of S. A subset S is *dense* if for every real number $\epsilon > 0$ it is ϵ-dense. Because every real number $\epsilon > 0$ is larger than some fraction $1/k$ with k a positive integer, it is equivalent to say that S is $(1/k)$-dense for every positive integer k.

A metric space (X, d_X) is *totally bounded* if for every real number $\epsilon > 0$, there exists a finite subset $X_\epsilon \subset X$ which is ϵ-dense, i.e., for every $x \in X$ there exists $y \in X_\epsilon$ such that $d_X(x, y) < \epsilon$. As above, it is equivalent to say that for every positive integer k, there exists a finite subset $X_{1/k} \subset X$ which is $(1/k)$-dense. Assume that (X, d_X) is totally bounded. Using the countable axiom of choice, choose one such finite subset $X_{1/k}$ for each positive integer k. Define S to be the union of these countably many sets,

$$S = \cup_{k=1}^{\infty} X_{1/k}.$$

Since every set $X_{1/k}$ is finite, the union S is either finite or countably infinite. And for every positive integer k, S is $(1/k)$-dense since already the finite subset $X_{1/k}$ of S is $(1/k)$-dense. Therefore S is dense.

The converse does not hold, essentially because a countably infinite set S which is $(1/k)$-dense may contain no finite subset Y which is also $(1/k)$-dense, e.g., the set of integers inside $\mathbb{R}$ is 1-dense but contains no finite subset which is 1-dense. There do exist separable, bounded metric spaces which are not totally bounded. The closed unit ball in any infinite dimensional, separable Banach space gives an example (for more on this, see Problem Set 3). Here is an even easier example that uses the same trick from the solution to Problem 3 from the online solutions to Problem Set 1. Recall that given a metric space (X, d_X) and a positive real number ϵ, we define a new function

$$d'_X : X \times X \to \mathbb{R}, \quad d'_X(x, y) := \min(d_X(x, y), \epsilon).$$

It is straightforward to check that this is again a metric. Also the topology induced by d_X equals the topology induced by d'_X. In particular, a subset $S \subset X$ is dense for d_X if and only if it is dense for d'_X, and thus X is separable for d_X if and only if it is separable for d'_X. Moreover, total boundedness is equivalent to the existence of finite subsets of X which are δ-dense for all real numbers $\delta > 0$ which are *sufficiently small* (in fact, just for a sequence of such numbers δ which converges to 0). And for $\delta \leq \epsilon$, a subset of X is δ-dense for d_X if and only if it is δ-dense for d'_X.

Thus X is totally bounded for d_X if and only if it is totally bounded for d'_X. (In fact, more generally separableness and total boundedness are preserved upon replacing the metric by any "equivalent" metric, i.e., another metric such that the identity map is bi-Lipschitz).

However one property of the metric space is not the same for both d_X and d'_X: whether or not (X, d_X) is bounded, the metric space (X, d'_X) is always bounded, in fact equal to the 2ϵ-ball about any element. So for a metric space (X, d_X) that is separable yet not totally bounded, the metric space (X, d'_X) is separable and bounded yet not totally bounded. Now $\mathbb{R}$ with its standard metric, $d_{\mathbb{R}}(s,t) = |t-s|$, is a metric space which is separable since the countably infinite subset $\mathbb{Q}$ is dense in $\mathbb{R}$. Yet $\mathbb{R}$ is not totally bounded since it is not even bounded. Therefore $(\mathbb{R}, d'_{\mathbb{R}})$ is a metric space which is separable and which is not totally bounded, yet which is bounded.

Solution to (2) First of all, for an element $x \in S$, let us recall the proof that there exists $r(x) > 0$ with $B_{r(x)}(x) \cap S = \{x\}$. By way of contradiction, suppose for every integer $k > 0$, the ball $B_{1/k}(x)$ contains an element x_n of the sequence different from x, i.e., $0 < d_X(x, x_n) < 1/k$. We will construct a subsequence $(x_{m_r})_{r=1,2,\dots}$ which converges to x. Define m_1 to be an integer n as above so that $0 < d(x, x_{m_1}) < 1/1$. And by way of induction, assume we have constructed integers $m_1 < m_2 < \dots < m_{r-1}$ such that for $k = 1, \dots, r-1$ we have $0 < d_X(x, x_{m_k}) < 1/k$. Let ϵ equal the minimum of $1/(r-1)$ and the finitely many positive real numbers $d_X(x, x_n)$ with $n = 1, \dots, m_{r-1}$ and with $x_n \neq x$. By the Archimedean property, there exists an integer k such that $1/k < \epsilon$. And by hypothesis, there exists an integer m_r such that $0 < d_X(x, x_{m_r}) < 1/k$. Note, in particular, that $1/k < 1/(r-1)$ so that $k \geq r$. And since $0 < d_X(x, x_{m_r}) < d_X(x, x_n)$ for all $n = 1, \dots, m_{r-1}$ with $d_X(x, x_n) > 0$, it follows that m_r equals no integer n with $n = 1, \dots, m_{r-1}$, i.e., $m_{r-1} < m_r$. So by induction, and by the countable axiom of choice, there exists a subsequence $(x_{m_r})_{r=1,2,\dots}$ converging to x. This contradicts that the sequence contains no convergent subsequence. Therefore for every $s \in S$ there exists a real number $r(x) > 0$ such that $B_{r(x)}(x) \cap S$ equals $\{x\}$.

Solution to (a). This follows from the triangle inequality. By way of contradiction, assume that there exists $z \in B_{r(x)/2}(x) \cap B_{r(y)/2}(y)$. Then by the triangle inequality we have

$$d_X(x,y) \leq d_X(x,z) + d_X(z,y) < \frac{r(x)}{2} + \frac{r(y)}{2} = \frac{r(x)+r(y)}{2}.$$

By trichotomy for real numbers, either $r(y) \leq r(x)$ or $r(x) \leq r(y)$, i.e., either $(r(x)+r(y))/2$ is $\leq r(x)$ or $\leq r(y)$. Thus either $d_X(x,y) < r(x)$ or $d_X(x,y) < r(y)$. The first conclusion contradicts that $B_{r(x)}(x) \cap S$ equals $\{x\}$ and the second conclusion contradicts that $B_{r(y)}(y) \cap S$ equals $\{y\}$. Either way there is a contradiction. Therefore $B_{\leq r(x)/2}(x)$ and $B_{\leq r(y)/2}(y)$ are disjoint. Form the subset

$$U := \cup_{x \in S} B_{r(x)/2}(x).$$

Since it is a union of open balls, U is an open set. And by the argument above, it is a disjoint union

$$U = \sqcup_{x \in S} B_{r(x)/2}(x).$$

Solution to (b) For every element x in S, let $n(x)$ denote the smallest integer n such that x_n equals x. And define $\delta(x) = \min(r(x)/3, 1/n(x))$. Since $\delta(x) < r(x)/2$, the closed ball $B_{\leq \delta(x)}(x)$ is contained in the ball $B_{r(x)/2}(x)$. So by the solution to (a), for $x, y \in S$ with $x \neq y$, $B_{\leq \delta(x)}(x)$ and $B_{\leq\delta(y)}(y)$ are disjoint, i.e., the following union is a disjoint union

$$C = \cup_{x\in S} B_{\leq\delta(x)}(x) = \sqcup_{x\in S} B_{\leq\delta(x)}(x).$$

Claim 0.1. *The subset C of X is closed, i.e., for every sequence $(z_m)_{m=1,2,\dots}$ in C which converges to a limit $z_\infty \in X$, the limit z_∞ is in C.*

Claim 0.1 will follow from a second claim.

Claim 0.2. *For every sequence $(z_m)_{m=1,2,\dots}$ in X (not necessarily entirely in C), the subset $\{z_m | m = 1, 2, \dots\}$ intersects only finitely many of the closed balls $B_{\leq\delta(x)}(x)$.*

Claim 0.2 is proved by contradiction. If the sequence intersects infinitely many balls, then this gives a subsequence (z_{m_k}) each of which is contained in a ball $B_{\delta(x_{n_k})}(x_{n_k})$ for a distinct x_{n_k}. But since the radii $\delta(x)$ limit to 0, it follows that the sequences $(z_{m_k})_{k=1,2,\dots}$ and $(x_{n_k})_{k=1,2,\dots}$ are equivalent in the sense of equivalence of Cauchy sequences. Since $(z_{m_k})_{k=1,2,\dots}$ converges to z_∞, the same holds for $(x_{n_k})_{k=1,2,\dots}$. But this contradicts the hypothesis that no subsequence of $(x_n)_{n=1,2,\dots}$ converges. Therefore, by contradiction, the set $\{z_m | m = 1, 2, \dots\}$ intersects only finitely many of the balls $B_{\delta(x)}(x)$, i.e., Claim 0.2 is proved. Next suppose that $(z_m)_{m=1,2,\dots}$ is a sequence entirely contained in C, i.e., every z_m is contained in some ball $B_{\leq\delta(x)}(x)$. By Claim 0.2, the sequence intersects only finitely many balls. So there must be a single ball $B_{\leq\delta(x)}(x)$ which contains an infinite subsequence $(z_{m_k})_{k=1,2,\dots}$. The subsequence (z_{m_k}) still converges to z_∞. And by Problem 3 from Problem Set 1, the set $B_{\leq\delta(x)}(x)$ is closed. Therefore the limit z_∞ is contained in $B_{\leq\delta(x)}(x)$, and thus also contained in C. This proves Claim 0.1, C is closed.

Since C is closed, its complement $V := X \setminus C$ is open. Observe that the intersection of the open sets U and V is the disjoint union

$$U \cap V = \sqcup_{x\in S} \left(B_{r(x)/2}(x) \setminus B_{\leq\delta(x)}(x)\right).$$

This brings us to an elementary, but nonetheless fundamental, fact about continuous functions. Let (A, d_A) and (B, d_B) be metric spaces (or even just topological spaces). Let $(A_i)_{i\in I}$ be an open covering of A. For every $i \in I$, let $f_i : A_i \to B$ be a function. Assume that for every $i, j \in I$, the restrictions of f_i and f_j to $A_i \cap A_j$ are equal. Then there exists a unique function $f : A \to B$ whose restriction to every open subset A_i equals f_i.

Claim 0.3. *The function $f : A \to B$ is continuous if and only if for every $i \in I$ the restriction to the open subset A_i, $f_i : A_i \to B$, is continuous (where A_i has the topology induced from A).*

Of course the restriction of a continuous function to a subset is continuous. So if f is continuous, then so is every restriction f_i. Conversely suppose that every f_i is continuous. To prove that f is continuous, we must prove that for every open $W \subset B$, the preimage $f^{-1}(W)$ is open in A. Since A equals the union $\cup_{i\in I} A_i$, also $f^{-1}(W)$ equals $\cup_{i\in I}(f^{-1}(W) \cap A_i)$, i.e., it equals $\cup_{i\in I} f_i^{-1}(W)$. For every $i \in I$, since f_i is continuous, $f_i^{-1}(W)$ is open in A_i. And also A_i is open in A. Therefore $f_i^{-1}(W)$ is open in A. And every union of open subsets is an open subset. Therefore $\cup_{i\in I} f_i^{-1}(W)$ is open in A, which proves Claim 0.3.

Because of Claim 0.3, to construct a continuous function $f : X \to \mathbb{R}$, it is equivalent to construct continuous functions $f_U : U \to \mathbb{R}$ and $f_V : V \to \mathbb{R}$ which are equal on $U \cap V$. Similarly, since U is the disjoint union of the opens $B_{r(x)/2}(x)$, to construct a continuous function $f_U : U \to \mathbb{R}$, it is equivalent to construct for every $x \in S$, a continuous function $f_x : B_{r(x)/2}(x) \to \mathbb{R}$. First of all, define $f_V : V \to \mathbb{R}$ to be identically zero. This is trivially continuous. Next for every $x \in S$, define $f_x : B_{r(x)/2}(x) \to \mathbb{R}$ by

$$f_x(z) = \begin{cases} n(x)\left(1 - \frac{1}{\delta(x)} d_X(x,z)\right), & d_X(z,x) < \delta(x), \\ 0, & d_X(z,x) = \delta(x), \\ 0, & \delta < d_X(z,x) < r(x)/2 \end{cases}$$

Observe that on the closed ball $B_{\leq \delta(x)}$, f_x is expressed as $n(x)(1 - d_X(x,z)/\delta(x))$. Since the metric function d_X is continous, f_x is continuous on the closed ball $B_{\leq\delta(x)}(x)$. And on the (relatively) closed subset $B_{r(x)/2}(x) \setminus B_{\delta(x)}(x)$ of $B_{r(x)/2}$, f_x is the constant function 0, which is also continuous. And on the intersection of these sets, i.e., on the subset $\{z \in B_{r(x)/2}(x) | d_X(x,z) = \delta(x)\}$, the functions agree.

This brings us to the closed counterpart of Claim 0.3. As before, let (A, d_A) and (B, d_B) be metric spaces (or topological spaces). Let J be a *finite* index set, and let $(C_j)_{j\in J}$ be a collection of closed subsets of A such that $A = \cup C_j$. For every $j \in J$, let $f_j : C_j \to B$ be a function. Assume that for every $i, j \in I$, the restrictions of f_i and f_j to $C_i \cap C_j$ are equal. Then there exists a unique function $f : A \to B$ whose restriction to every subset C_j equals f_j.

Claim 0.4. *The function $f : A \to B$ is continuous if and only if for each of the finitely many elements $j \in J$ the restriction to the closed subset C_j, $f_j : C_j \to B$, is continuous (where C_j has the topology induced from A).*

As before, if f is continuous then its restriction to every subset is continuous, so each f_j is continuous. Conversely suppose that every f_j is continuous. To prove that f is continuous, we must prove that for every closed subset $D \subset B$, the preimage $f^{-1}(D)$ is closed in A. Since A equals the union $\cup_{j\in J} C_C$, also $f^{-1}(D)$ equals $\cup_{j\in J}(f^{-1}(D) \cap C_j)$, i.e., it equals $\cup_{j\in J} f_j^{-1}(D)$. For every $j \in J$, since f_j is continuous, $f_j^{-1}(D)$ is closed in C_j. And also C_j is closed in A. Therefore $f_j^{-1}(D)$ is closed in A. And every *finite* union of closed subsets is a closed subset. Therefore $\cup_{j\in J} f_j^{-1}(D)$ is closed in A, which proves Claim 0.3.

In particular, applying Claim 0.4 to the pair of (relatively) closed subsets $B_{\leq\delta(x)}(x)$ and $B_{r(x)/2}(x) \setminus B_{\delta(x)}(x)$ of $B_{r(x)/2}(x)$, it follows that f_x is continuous on $B_{r(x)/2}$. Since U is the disjoint union over $x \in S$ of the open sets $B_{r(x)/2}$, there is a unique function $f_U : U \to \mathbb{R}$ whose restriction to every subset $B_{r(x)/2}(x)$ equals f_x. And by Claim 0.3, the function f_U is continous on U. And the restriction of both f_U and f_V to

$$U \cap V = \sqcup_{x\in S} \left(B_{r(x)/2}(x) \setminus B_{\leq\delta(x)}(x)\right)$$

is the zero function. So the restrictions are equal on $U \cap V$. Therefore there is a unique function $f : X \to \mathbb{R}$ whose restrictions to U and V are f_U and f_V respectively. And by Claim 0.3 once more, f is a continuous function.

The final claim is that f is unbounded. Indeed, for every $x \in S$, $f(x) = f_x(x) = n(x)$. Since the set S is infinite, the set of distinct, positive integers $\{n(x) | x \in S\}$ is also infinite, hence unbounded. So $f : X \to \mathbb{R}$ is a continuous, unbounded function on X.

Solution to (c) As proved in lecture, the image of a compact metric space under a continuous map is compact. By the Heine-Borel theorem, the compact subsets of $\mathbb{R}$ are precisely the closed, bounded subsets of $\mathbb{R}$. Hence a continuous function from a compact metric space to $\mathbb{R}$ has image equal to a closed, bounded subset of $\mathbb{R}$, and thus the function is bounded. On the other hand, the example above proves that on every non-compact metric space there exists a continuous, unbounded function to $\mathbb{R}$. Therefore a metric space is compact if and only if every continuous function from the metric space to $\mathbb{R}$ is bounded.

Solution to Problem 3 Let (X, d_X) be a metric space. Let $f : X \to \mathbb{R}$ be a continuous function. Let $\epsilon' > 0$ be a real number. For every $x \in X$, the ball $B_{\epsilon'}(f(x))$ is an open subset of $\mathbb{R}$. Since f is continuous, the inverse image $U_{f,\epsilon',x} := f^{-1}(B_{\epsilon'}(f(x)))$ is an open subset of X containing x. Thus the collection $(U_{f,\epsilon',x})_{x\in X}$ is an open covering of X. By Lemma 4.5.3, if X is compact then there exists a real number $\delta > 0$ such that for every $y \in X$, there exists $x \in X$ with $B_\delta(y) \subset U_{f,\epsilon',x}$. For every $z \in B_\delta(y)$, since z is in $U_{f,\epsilon',x}$, we have $|f(z) - f(x)| \leq \epsilon'$. By the triangle inequality we have

$$|f(z) - f(y)| \leq |f(z) - f(x)| + |f(y) - f(x)| < \epsilon' + \epsilon' = 2\epsilon'$$

Therefore, choosing $\epsilon' = \epsilon/2$, it follows that for every pair of elements $y, z \in X$ with $d_X(y, z) < \delta$, $|f(z) - f(y)| < \epsilon$. Since this holds for every $\epsilon > 0$, f is uniformly continuous.

Solution to Problem 4 This is simply a matter of parsing the definitions carefully. No solution will be written up.

Solution to Problem 5

Solution to (a) Fix a real number $\epsilon > 0$ and an element $x \in X$. Since $(f_n(x))$ converges to 0, there exists an integer $N = N(x, \epsilon)$ such that $n \geq N$ implies that $|f_n(x)| < \epsilon/2$. Since the sequence $(f_n)_{n\geq N}$ is equicontinuous, there exists a real number $\delta = \delta(x, \epsilon) > 0$ such that for all $y \in B_{\delta(x,\epsilon)}(x)$, $|f_n(y) - f_n(x)| < \epsilon/2$. Then by the triangle inequality,

$$|f_n(y)| \leq |f_n(y) - f_n(x)| + |f_n(x)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

For fixed $\epsilon > 0$, the collection of balls $\{B_{\delta(x,\epsilon)}(x) | x \in X\}$ is an open covering of X. Since X is topologically compact, there is a finite subcover, i.e., there is a finite subset $X_\epsilon \subset X$ such that the finitely many sets $\{B_{\delta(x,\epsilon)}(x) | x \in X_\epsilon\}$ cover X. Denote by N be the maximum of the positive integers $N(x, \epsilon)$ as x varies in the finite set X_ϵ. For every $y \in X$, there exists $x \in X_\epsilon$ such that $y \in B_{\delta(x,\epsilon)}(x)$. Thus, by the inequality above, for all $n \geq N(x, \epsilon)$, $|f_n(y)| < \epsilon$. In particular, this holds for all $n \geq N$. So for every $n \geq N$, for every $y \in X$, $|f_n(y)| < \epsilon$. By definition of the uniform norm, $\|f_n\| \leq \epsilon$. Thus for every real number $\epsilon > 0$, there exists an integer $N > 0$ such that $n \geq N$ implies that $\|f_n\| \leq \epsilon$. This precisely says that (f_n) converges to 0 in the uniform metric.

Solution to (b) Notations here are as in the solution to **Problem 2**. For every $x \in S$, define $g_{n(x)}$ to be the function which agrees with f on the ball $B_{r(x)/2}(x)$ and which equals 0 outside of $B_{\leq\delta(x)}(x)$. And for integer $n > 0$ which are not of the form $n(x)$, define g_n to be identically zero. The function $g_{n(x)}$ is continuous by the same arguments as in the solution to **Problem 2**. Notice that $g_{n(x)}$ is nonzero only on $B_{\delta(x)}(x)$. By **Problem 2(a)**, for every $y \in X$, there exists at most one $x \in S$ such that $y \in B_{r(x)/2}(x)$. And if so, then for all $n > n(x)$, g_n is zero on y. Thus $(g_n(y))_{n=1,2,\dots}$ converges to 0, i.e., the sequence (g_n) converges pointwise to 0.

For essentially the same reason, (g_n) is also equicontinuous. Let $\epsilon > 0$ be a real number. Let y be an element of X. We need to prove that there exists $\delta = \delta(y, \epsilon)$ such that for every $n = 1, 2, \dots,$ for every $z \in B_{\delta(y,\epsilon)}(y)$, $|g_n(z) - g_n(y)| < \epsilon$. There are two cases depending on where y lies. If y is in U, then since U is open, there exists $\delta = \delta(y)$ such that $B_{\delta(y)}(y)$ is in U. Thus every g_n is identically zero on $B_{\delta(y)}(y)$. So for every $n = 1, 2, \dots,$ for every $z \in B_{\delta(y)}(y)$, $|g_n(z) - g_n(y)| = 0 < \epsilon$. Next, if y is in some subset $B_{r(x)/2}(x)$, then since $B_{r(x)/2}(x)$ is open, there exists $\delta' = \delta(y)$ such that $B_{\delta'}(y) \subset B_{r(x)/2}(x)$. Hence for $x' \in S$ with $x' \neq x$, by **Problem 2(a)**, $B_{\delta'(y)}(y)$ is disjoint from $B_{r(x')/2}(x')$. So for $n \neq n(x)$, g_n is identically zero on $B_{\delta'(y)}(y)$. Thus for $n \neq n(x)$ and for $z \in B_{\delta'(y)}(y)$, $|g_n(z) - g_n(y)| = 0 < \epsilon$. Finally, since $g_{n(x)}$ is continuous at y, there exists $\delta''(y, \epsilon) > 0$ such that $z \in B_{\delta''(y,\epsilon)}(y)$ implies that $|g_{n(x)}(z) - g_{n(x)}(y)| < \epsilon$. Taking $\delta(y, \epsilon) = \min(\delta'(y), \delta''(y, \epsilon)$, for every $n = 1, 2, \dots,$ and for every $z \in B_{\delta(y,\epsilon)}(y)$, $|g_n(y) - g_n(z)| < \epsilon$. Thus in both cases, (g_n) is equicontinuous at y. So the sequence is equicontinuous.

So (g_n) is an equicontinuous sequence of functions which converges to 0 pointwise. Yet the sequence does not converge to 0 uniformly. Indeed $g_{n(x)}(x)$ equals $n(x)$, so that $\|g_{n(x)}\| \geq n(x)$. Since the integers $n(x)$ are unbounded, the sequence $(\|g_n\|)$ is unbounded, hence does not converge to 0.