Name: ______________________________ Problem 1: ______________ /60

Problem 1(60 points) Let V be the vector space $\mathbb{R}$, and let $\vec{v}_0$ be 0. Let W be the vector space $\mathbb{R}^2$, and let $\vec{w}_0$ be $\vec{0}$. Let $G : V \times W \to W$ be the following continuously differentiable function,

$$\left(x, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) \mapsto \begin{bmatrix} y_1 + y_2 + y_1^2 - x \\ y_1 - y_2 - y_2^2 + x \end{bmatrix}.$$

For every $\vec{v} \in V$, denote by $G_{\vec{v}} : W \to W$ the function $\vec{w} \mapsto G(\vec{v}, \vec{w})$. It is true that $G_{\vec{v}_0}(\vec{w}_0)$ equals 0 and $T := d(G_{\vec{v}_0})_{\vec{w}_0}$ is an invertble linear transformation.

An *associated contraction* is a triple $(\tilde{V}, \tilde{W}, K)$ of an open ball centered at $\vec{v}_0$, $\vec{v}_0 \in \tilde{V} \subset V$, a positive radius, bounded, closed ball centered at $\vec{w}_0$, $\vec{w}_0 \in \tilde{W} \subset W$, and a continuously differentiable function $K : \tilde{V} \times \tilde{W} \to \tilde{W}$ with $d(K_{\vec{v}_0})_{\vec{w}_0} = 0$ and such that for every $\vec{v} \in \tilde{V}$, $K_{\vec{v}} : \tilde{W} \to \tilde{W}$ is a contraction whose unique fixed point $\vec{w}$ is the unique element of $\tilde{W}$ such that $G_{\vec{v}}(\vec{w})$ equals 0. An *implicit function* is a continuously differentiable function $f : \tilde{V} \to \tilde{W}$ such that for every $\vec{v} \in \tilde{V}$ and $\vec{w} \in \tilde{W}$, we have $G(\vec{v}, \vec{w})$ equals 0 if and only if $\vec{w} = f(\vec{v})$.

Do all of the following.

(i) (10 points) Compute the linear transformation $T = d(G_{\vec{v}_0})_{\vec{w}_0}$.

(ii) (10 points) Compute the inverse linear transformation T^{-1}.

(iii) (20 points) Write down a continuously differentiable function $K : V \times W \to W$ such that for some choice of $\tilde{V}$ and $\tilde{W}$ (which you **do not** need to find) the triple $(\tilde{V}, \tilde{W}, G)$ is an associated contraction. Write out each component of your answer; do not leave matrix multiplications unevaluated. Also simplify as much as possible (gather like terms and do all appropriate cancellations).

(iv) (20 points) Starting with the constant function $f_0 : \tilde{V} \to \tilde{W}$ given by $f_0(x) = \vec{w}_0$, compute explicitly the iterates f_1 and f_2 in the approximating sequence determined by K which uniformly converges to the implicit function f.

(i) For a differentiable function $g : \mathbb{R}^n \to \mathbb{R}^m$ by $\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \mapsto \begin{bmatrix} g_1(y_1, y_2, \dots, y_n) \\ \vdots \\ g_m(y_1, y_2, \dots, y_n) \end{bmatrix}$, the total derivative at $\vec{w}_0 = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ is given by (left) multiplication by the Jacobian matrix

$$\left[\frac{\partial g_i}{\partial y_j}\right]_{\vec{w}_0} = \begin{bmatrix} \frac{\partial g_1}{\partial y_1}(a_1, \dots, a_n) & \cdots & \frac{\partial g_1}{\partial y_n}(a_1, \dots, a_n) \\ \vdots & & \\ \frac{\partial g_m}{\partial y_1}(a_1, \dots, a_n) & \cdots & \frac{\partial g_m}{\partial y_n}(a_1, \dots, a_n) \end{bmatrix}.$$

In our case, $g = G_{\vec{v}_0} : \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \mapsto \begin{bmatrix} y_1 + y_2 + y_1^2 \\ y_1 - y_2 - y_2^2 \end{bmatrix}$, so $\left[\frac{\partial g_i}{\partial y_j}\right] = \begin{bmatrix} 1 + 2y_1 & 1 \\ 1 & -1 - 2y_2 \end{bmatrix}$, and $\boxed{d(G_{\vec{v}_0})_{\vec{w}_0} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}$

(ii) Either by Cramer's Rule, row reduction of $[T \,\vdots\, Id_{2\times 2}]$ to $[Id_{2\times 2} \,\vdots\, T^{-1}]$, or guess-and-check, $\boxed{T^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}$.

(iii) The formula for the contraction is $K_{\vec{v}}(\vec{w}) = \vec{w} - T^{-1} \circ G_{\vec{v}}(\vec{w})$.

$$T^{-1} \circ G_{\vec{v}}(\vec{w}) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} y_1 + y_2 + y_1^2 - x \\ y_1 - y_2 - y_2^2 - x \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2y_1 + y_1^2 - y_2^2 \\ 2y_2 + y_1^2 + y_2^2 - 2x \end{bmatrix} = \begin{bmatrix} y_1 + \frac{1}{2}(y_1^2 - y_2^2) \\ y_2 + \frac{1}{2}(y_1^2 + y_2^2) - x \end{bmatrix}.$$

So $$K_{\vec{v}}(\vec{w}) = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \begin{bmatrix} y_1 + \frac{1}{2}(y_1^2 - y_2^2) \\ y_2 + \frac{1}{2}(y_1^2 + y_2^2) - x \end{bmatrix} = \boxed{\begin{bmatrix} \frac{1}{2}(y_2^2 - y_1^2) \\ -\frac{1}{2}(y_2^2 + y_1^2) + x \end{bmatrix}}$$

(iv) The approximating sequence is given by $f_{n+1}(x) = K_x(f_n(x)) = \ldots = (K_x)^n(f_0(x))$.

Explicitly, if $f_n(x)$ equals $\begin{bmatrix} f_{n,1}(x) \\ f_{n,2}(x) \end{bmatrix}$, then (iii) gives

$$\begin{bmatrix} f_{n+1,1}(x) \\ f_{n+1,2}(x) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left((f_{n,2}(x))^2 - (f_{n,1}(x))^2\right) \\ \frac{-1}{2}\left((f_{n,2}(x))^2 + (f_{n,1}(x))^2\right) + x \end{bmatrix}.$$ Starting from $f_0(x) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, this gives

$$\begin{bmatrix} f_{1,1}(x) \\ f_{1,2}(x) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(0^2 - 0^2) \\ -\frac{1}{2}(0^2 + 0^2) + x \end{bmatrix} = \boxed{\begin{bmatrix} 0 \\ x \end{bmatrix}}, \quad \begin{bmatrix} f_{2,1}(x) \\ f_{2,2}(x) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(x^2 - 0^2) \\ -\frac{1}{2}(x^2 + 0^2) + x \end{bmatrix} = \boxed{\begin{bmatrix} \frac{1}{2}x^2 \\ x - \frac{1}{2}x^2 \end{bmatrix}}.$$

Name: ______________________ **Problem 2:** __________ /80

Problem 2(80 points) Consider the following 2nd order, constant coefficient, initial value problem

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = 0, \quad x(t_0) = b_0, \quad \frac{dx}{dt}(t_0) = b_1.$$

(a)(10 points) Find a 2×2-matrix A with real entries such that for every $(b_0, b_1)^\dagger \in \mathbb{R}^2$, the solution of the 1st order IVP

$$\frac{d\vec{x}}{dt} = A\vec{x}, \quad \vec{x}(t_0) = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}, \quad \vec{x}(t) = \begin{bmatrix} x_0(t) \\ x_1(t) \end{bmatrix}$$

gives a solution of the original 2nd order IVP by $x(t) = x_0(t)$.

(b)(15 points) Find the characteristic polynomial of A, find the factorization into a product of linear factors (each of which will be real), and find all eigenvalues of A.

(c)(20 points) Find an invertible 2×2 matrix U, a diagonal 2×2 matrix $\tilde{S}$, and a 2×2 matrix $\tilde{N}$ which is upper triangular (or lower triangular if you prefer) such that $\tilde{S}\tilde{N} = \tilde{N}\tilde{S}$ and such that $AU = U(\tilde{S} + \tilde{N})$.

(d)(20 points) Compute $\exp(\tilde{S}(t - t_0))$, $\exp(\tilde{N}(t - t_0))$ and $\exp(A(t - t_0))$. In your answer, write out each entry of the matrix; do not leave matrix multiplications unevaluated. All entries of your matrices should involve only polynomials in t and exponentials in t, no unevaluated power series.

(e)(15 points) Find the general solution $\vec{x}(t)$ of the 1st order IVP above. Write out each component of $\vec{x}(t)$; do not leave matrix multiplications unevaluated. Both components should involve only polynomials in t and exponentials in t.

(a). Set $x_0(t) = x(t)$ and $x_1(t) = \frac{dx}{dt}(t)$. Then $\frac{d}{dt}x_0(t)$ equals $x_1(t)$. And $\frac{d}{dt}x_1(t)$ equals $\frac{d^2x}{dt^2}$, which equals $2\frac{dx}{dt} - x = -1\,x_0(t) + 2\,x_1(t)$, by the 2^{nd} order ODE.

Thus $\begin{cases} \frac{d}{dt}x_0(t) = 0\,x_0(t) + 1\,x_1(t) \\ \frac{d}{dt}x_1(t) = -1\,x_0(t) + 2\,x_1(t) \end{cases}$, or $\frac{d}{dt}\begin{bmatrix} x_0(t) \\ x_1(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_0(t) \\ x_1(t) \end{bmatrix}$, $\boxed{A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}}$

(b). Trace$(A) = 0 + 2 = 2$ and Det$(A) = 0\cdot 2 - 1(-1) = 1$. Thus the characteristic polynomial is $C_A(\lambda) = \mathrm{Det}(\lambda \, Id_{2\times 2} - A) = \lambda^2 - \mathrm{Trace}(A)\lambda + \mathrm{Det}(A) = \lambda^2 - 2\lambda + 1 = \boxed{(\lambda - 1)^2}$.

There is one repeated eigenvalue, $\boxed{\lambda_1 = 1.}$

(c). The matrix $A - \lambda_1 Id_{2\times2}$ is $\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$. The kernel is span$\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$.

One vector in $\mathbb{R}^2 \setminus Ker(A-\lambda_1 Id_{2\times2})$ is $\mathbb{b}_2 := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

And $(A-\lambda_1 Id_{2\times2})\mathbb{b}_2$ is $\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Denote this by $\mathbb{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Then $U := [\mathbb{b}_1 \,\vdots\, \mathbb{b}_2] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ satisfies $(A-\lambda_1 Id_{2\times2})U = \left[(A-\lambda_1 Id)\mathbb{b}_1 \,\vdots\, (A-\lambda_1 Id)\mathbb{b}_2\right] =$

$[\vec{0} \,\vdots\, \mathbb{b}_1] = [0\mathbb{b}_1 + 0\mathbb{b}_2 \,\vdots\, 1\mathbb{b}_1 + 0\mathbb{b}_2] = [\mathbb{b}_1 \,\vdots\, \mathbb{b}_2]\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = U\tilde{N}$ where $\tilde{N} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Thus, for $\boxed{U = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \ \tilde{S} = \lambda_1 Id_{2\times2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \tilde{N} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}$, AU equals $U(\tilde{S}+\tilde{N})$.

$U^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

(d). $\tilde{S}(t-t_0)$ equals $\begin{bmatrix} t-t_0 & 0 \\ 0 & t-t_0 \end{bmatrix}$, so that $\exp(\tilde{S}(t-t_0))$ equals $\boxed{\begin{bmatrix} e^{t-t_0} & 0 \\ 0 & e^{t-t_0} \end{bmatrix}}$.

$\tilde{N}(t-t_0)$ equals $\begin{bmatrix} 0 & t-t_0 \\ 0 & 0 \end{bmatrix}$, which has square equal to $O_{2\times2}$. Therefore

$\exp(\tilde{N}(t-t_0)) = Id_{2\times2} + \frac{1}{1!}\tilde{N}(t-t_0) + \frac{1}{2!}(\tilde{N}(t-t_0))^2 + \cdots$ equals $Id_{2\times2} + \tilde{N}(t-t_0) = \boxed{\begin{bmatrix} 1 & t-t_0 \\ 0 & 1 \end{bmatrix}}$.

Since $\tilde{S}(t-t_0)$ and $\tilde{N}(t-t_0)$ commute, $\exp(\tilde{S}(t-t_0) + \tilde{N}(t-t_0))$ equals $\exp \tilde{S}(t-t_0) \cdot \exp \tilde{N}(t-t_0)$

$= \begin{bmatrix} e^{t-t_0} & 0 \\ 0 & e^{t-t_0} \end{bmatrix}\begin{bmatrix} 1 & (t-t_0) \\ 0 & 1 \end{bmatrix} = e^{t-t_0}\begin{bmatrix} 1 & (t-t_0) \\ 0 & 1 \end{bmatrix}$. Finally, since $A(t-t_0)$ equals $U(\tilde{S}(t-t_0) + \tilde{N}(t-t_0))U^{-1}$,

$\exp(A(t-t_0))$ equals $U\exp(\tilde{S}(t-t_0)+\tilde{N}(t-t_0))U^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} e^{t-t_0} \begin{bmatrix} 1 & t-t_0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} =$

$e^{t-t_0}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1-(t-t_0) & t-t_0 \\ -1 & 1 \end{bmatrix} = \boxed{e^{t-t_0}\begin{bmatrix} 1-(t-t_0) & +(t-t_0) \\ -(t-t_0) & 1+(t-t_0) \end{bmatrix}}$

(e) The solution is $\vec{x}(t) = \exp(A(t-t_0))\vec{x}(t_0) = e^{t-t_0}\begin{bmatrix} 1-(t-t_0) & +(t-t_0) \\ -(t-t_0) & 1+(t-t_0) \end{bmatrix}\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} =$

$\boxed{\begin{bmatrix} (b_0 + (b_1 - b_0)(t-t_0))e^{t-t_0} \\ (b_1 + (b_1 - b_0)(t-t_0))e^{t-t_0} \end{bmatrix}}$.

Name: ______________________ Problem 3: __________ /60

Problem 3(60 points) Let X be a set. Let $(X_n)_{n=1}^{\infty}$ be a collection of subset $X_n \subset X$ such that $X = \cup_{n=1}^{\infty} X_n$ and such that $X_n \subset X_{n+1}$ for every n. Let $(X_n, \mathcal{M}_n, \mu_n)_{n=1}^{\infty}$ be a collection of (nonnegative) measure spaces such that $\mathcal{M}_{n+1} \cap \mathcal{P}(X_n)$ equals $\mathcal{M}_n$ and μ_{n+1} restricted to $\mathcal{M}_n$ equals μ_n for every n.

Define $\mathcal{M} \subset \mathcal{P}(X)$ to be the collection of sets $E \subset X$ such that for every n, $E \cap X_n$ is in $\mathcal{M}_n$. And for every such E, define

$$\mu(E) := \sup_{n \in \mathbb{N}} \mu_n(E \cap X_n).$$

(a)(15 points) Prove that $\mathcal{M}$ is a σ-algebra on X.

(b)(20 points) Prove that $\mu(\emptyset) = 0$, that μ is finitely additive, and thus that μ is monotone, i.e., $\mu(E) \leq \mu(F)$ if $E \subset F$.

(c)(25 points) Prove that μ is countably additive, thus a measure function.
Hint. Part (b) gives one inequality. Because of this, it suffices to consider a sequence $(E_k)_{k=1}^{\infty}$ of pairwise disjoint sets in $\mathcal{M}$ such that $\sum_k \mu(E_k)$ is finite. Thus for every real $\epsilon > 0$, there exists an integer $K > 0$ such that for every integer $L > 0$,

$$\sum_{k=1}^{L} \mu(E_k) \leq \sum_{k=1}^{\infty} \mu(E_k) \leq \left(\sum_{k=1}^{K} \mu(E_k) \right) + \epsilon.$$

And for every integer $k > 0$, and for every real $\epsilon_k > 0$, there exists an integer n_k such that for all $n \in \mathbb{N}$, we have

$$\mu_n(E_k \cap X_n) \leq \mu(E_k) \leq \mu_{n_k}(E_k \cap X_{n_k}) + \epsilon_k.$$

(a) Axiom 1. $\emptyset \in \mathcal{M}$. For every $n = 1, 2, \ldots$, $\emptyset$ is in $\mathcal{M}_n$. So for every n, $\emptyset \cap X_n = \emptyset$ is in $\mathcal{M}_n$. Hence $\emptyset$ is in $\mathcal{M}$.

Axiom 2. $E \in \mathcal{M}$ implies $X \setminus E$ is in $\mathcal{M}$. Let E be in $\mathcal{M}$. Then for every n, $E \cap X_n$ is in $\mathcal{M}_n$. Since $\mathcal{M}_n$ is an algebra on X_n, the complement $X_n \setminus (E \cap X_n)$ is in $\mathcal{M}_n$. Since $(X \setminus E) \cap X_n$ equals $X_n \setminus (E \cap X_n)$, $(X \setminus E) \cap X_n$ is in $\mathcal{M}_n$ for every n. Hence $X \setminus E$ is in $\mathcal{M}$.

Axiom 3. $\mathcal{M}$ stable under countable union. Let $(E_i)_{i=1}^{\infty}$ be a sequence of elements $E_i \in \mathcal{M}$. Then for every n & for every i, $E_i \cap X_n$ is in $\mathcal{M}_n$. Since $\mathcal{M}_n$ is a σ-algebra, $\bigcup_{i=1}^{\infty} (E_i \cap X_n)$ is in $\mathcal{M}_n$. Since $(\bigcup_{i=1}^{\infty} E_i) \cap X_n$ equals $\bigcup_{i=1}^{\infty} (E_i \cap X_n)$, $(\bigcup_{i=1}^{\infty} E_i) \cap X_n$ is in $\mathcal{M}_n$ for every n. Hence $\bigcup_{i=1}^{\infty} E_i$ is in $\mathcal{M}$.

Since $\mathcal{M}$ satisfies Axioms 1, 2 and 3, $\mathcal{M}$ is a σ-algebra.

Let E be an element of $\mathcal{M}_n$. For every $m \leq n$, since $X_m \in \mathcal{P}(X_m) \cap \mathcal{M}_n = \mathcal{M}_m$, also $E \cap X_m$ is in $\mathcal{P}(X_m) \cap \mathcal{M}_n = \mathcal{M}_m$ (since $\mathcal{M}_n$ is an algebra). And for every $m \geq n$, $E \cap X_m = E$ is in $\mathcal{M}_n \subseteq \mathcal{M}_m$. Thus E is in $\mathcal{M}$. Also for $m \leq n$, $\mu_m(E \cap X_m) = \mu_n(E \cap X_n)$ (by $\mu_{m+1}|_{\mathcal{M}_m} = \mu_m$ + induction) $\leq \mu_n(E)$ by monotonicity of μ_n. And for $m \geq n$ $\mu_m(E \cap X_n) = \mu_m(E) = \mu_n(E)$.
Therefore $\mu(E) = \sup \mu_m(E \cap X_m)$ equals $\mu_n(E)$. So $\mathcal{M}_n \subseteq \mathcal{M}$ and $\mu|_{\mathcal{M}_n}$ equals μ_n.

(b). $\mu(\emptyset) = 0$. For every n, since μ_n is a measure function, $\mu_n(\emptyset \cap X_n) = \mu_n(\emptyset)$ equals 0. Hence $\mu(\emptyset) = \sup_n \mu_n(\emptyset \cap X_n) = \sup_n 0$, which equals 0.

μ is finitely additive. Let E, F be sets in $\mathcal{M}_n$ with $E \cap F = \emptyset$. Then for every n, $E \cap X_n$ and $F \cap X_n$ are disjoint in $\mathcal{M}_n$. Since μ_n is finitely additive, $\mu_n((E \cap X_n) \cup (F \cap X_n))$ equals $\mu_n(E \cap X_n) + \mu_n(F \cap X_n)$. Therefore $\mu(E \cup F) = \sup_n \mu_n((E \cup F) \cap X_n) = \sup_n (\mu_n(E \cap X_n) + \mu_n(F \cap X_n))$, which is $\leq \sup_n \mu_n(E \cap X_n) + \sup_n \mu_n(F \cap X_n) = \mu(E) + \mu(F)$.

If $\mu(E \cup F)$ equals $+\infty$, then also $\mu(E) + \mu(F)$ equals $+\infty$, so $\mu(E \cup F)$ equals $\mu(E) + \mu(F)$. Thus assume $\mu(E \cup F)$ is finite, i.e. $\sup_n (\mu_n(E \cap X_n) + \mu_n(F \cap X_n))$ is finite. Since each $\mu_n(F \cap X_n)$ is ≥ 0, $\mu(E) = \sup_n \mu_n(E \cap X_n) \leq \sup_n (\mu_n(E \cap X_n) + \mu_n(F \cap X_n)) < +\infty$. Similarly $\mu(F) < \infty$. Since the sequences $(\mu_n(E \cap X_n))_{n=1}^{\infty}$ and $(\mu_n(F \cap X_n))_{n=1}^{\infty}$ are nondecreasing and bounded, they are convergent with limit equal to the supremum. Thus also $(\mu_n(E \cap X_n) + \mu_n(F \cap X_n))_{n=1}^{\infty}$ is convergent with limit equal to the supremum (in fact this was our hypothesis), and the limit laws give

$$\mu(E \cup F) = \sup_n (\mu_n(E \cap X_n) + \mu_n(F \cap X_n)) = \lim_{n \to \infty} (\mu_n(E \cap X_n) + \mu_n(F \cap X_n)) = \text{(by the limit laws)}$$

$$\lim_{n \to \infty} \mu_n(E \cap X_n) + \lim_{n \to \infty} \mu_n(F \cap X_n) = \sup_n \mu_n(E \cap X_n) + \sup_n \mu_n(F \cap X_n) = \mu(E) + \mu(F).$$

Hence μ is finitely additive. Thus μ is also monotone.

(c). Leaving the hint aside, by an exercise from Problem Set 7, a finitely additive μ is countably additive if and only if it is continuous from below.
Let $E_1 \subset E_2 \subset \cdots$ be a sequence of sets in $\mathcal{M}$. By monotonicity, $\sup_i \mu(E_i) \leq \mu(\bigcup_{i=1}^{\infty} E_i)$ (since $\mathcal{M}$ is a σ-algebra, $\bigcup_{i=1}^{\infty} E_i$ is in $\mathcal{M}$). By definition, $\mu(\bigcup_{i=1}^{\infty} E_i) = \sup_n \mu_n(\bigcup_i E_i \cap X_n)$

Of course $E_1 \cap X_n \subset E_2 \cap X_n \subset \cdots$ is an increasing sequence in $\mathcal{M}_n$. Since μ_n is countably additive, it is continuous from below, so $\mu_n(\bigcup_i (E_i \cap X_n))$ equals $\sup_i \mu_n(E_i \cap X_n)$.

So, $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sup_{n \in \mathbb{Z}_{>0}} \sup_{i \in \mathbb{Z}_{>0}} \mu_n(E_i \cap X_n) = \sup_{(n,i) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}} \mu_n(E_i \cap X_n)$

$= \sup_{i \in \mathbb{Z}_{>0}} \sup_{n \in \mathbb{Z}_{>0}} \mu_n(E_i \cap X_n) = \sup_{i \in \mathbb{Z}_{>0}} \mu(E_i)$. Thus μ is continuous from below.