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MAT 544 Midterm 1 Review

The policies regarding exams are posted on the exams part of the course webpage. The exam is closed book, closed notes, no electronic devices are allowed, and you need only bring a writing implement.

Review Topics. Please be familiar with all of the following concepts.

1. Metric spaces. Open balls in a metric space. Open and closed sets in a metric space.

2. Sequences. Subsequences. Convergent sequence. Limits of sequences and closed sets. Cauchy sequences.

3. Continuity. Uniform continuity. Lipschitz continuity. Behavior of convergent sequences with respect to a continuous function. Behavior of open sets, respectively closed sets, with respect to a continuous function. Behavior of Cauchy sequences with respect to continuous and uniformly continuous functions.

4. Completeness. Completion of a metric space. Behavior of completion with respect to uniformly continuous functions. Properties of the image of a complete metric space under an isometric embedding.

5. Compactness, both sequential and topological. Properties of the image of a compact metric space under a continuous map. Total boundedness. Relation of compactness to both completeness and total boundedness.

6. Uniformly convergent sequences. The uniform norm on the real vector space of bounded, continuous real-valued functions on a metric space. Equicontinuity. The Arzela-Ascoli theorem.

7. Banach spaces. Absolute convergence. Criterion for completeness of a normed vector space in terms of absolute convergence. Bounded linear maps between normed vector spaces. The operator norm on the vector space of bounded linear maps. Completeness of the space of bounded linear maps between normed vector spaces.

8. Bounded linear functional. Closed linear subspaces. The Hahn-Banach theorem. The double dual of a normed vector space and the completion of a normed vector space. Reflexive Banach spaces.

9. Real inner product spaces and Hermitian (complex) inner product spaces. Hilbert spaces. The Cauchy-Scwartz inequality. Orthogonality. Orthogonal projection. The Gram-Schmidt algorithm.

Some Practice Problems.

Problem 1 State the definition of continuity in terms of $\epsilon - \delta$. Give an equivalent criterion in terms of inverse images of open sets, respectively closed sets. Give an equivalent criterion in terms of convergent sequences.

Problem 2 Let (X, d_X) and (Y, d_Y) be metric spaces. Let $F : X \to Y$ be a function. Assume that for every continuous function $g : Y \to \mathbb{R}$ (where $\mathbb{R}$ has its usual metric), the composition $g \circ F : X \to \mathbb{R}$ is continuous. Prove that F is continuous.

Problem 3 Find an example of metric spaces (X, d_X) and (Y, d_Y) together with a function $F : X \to Y$ such that for every continuous function $h : \mathbb{R} \to X$ the composition $F \circ h : \mathbb{R} \to Y$ is continuous, yet F is not continuous. Can you find a metric space (Z, d_Z) such that for every (X, d_X), (Y, d_X) and $F : X \to Y$, the map F is continuous if and only if $F \circ h$ is continuous for every continuous $h : Z \to X$?

Problem 4 State the completeness property and the least upper bound property of the real numbers. Are the real numbers compact? Are the real numbers totally bounded? Which subsets of the real numbers are complete, respectively compact, totally bounded?

Problem 5 Prove the Intermediate Value Theorem: for a continuous function $f : [0, 1] \to \mathbb{R}$ such that $f(0) < 0$ and $f(1) > 0$, there exists $t \in (0, 1)$ such that $f(t)$ equals 0. **Hint.** Consider the subset of t such that f is nonpositive on $[0, t)$. What does the least upper bound property imploy about this set? What does continuity imply?

Problem 6 Find an example of a compact metric space (X, d_X) and a continuous function $f : X \to \mathbb{R}$ such that f takes on negative values, f takes on positive values, but f never equals 0.

Problem 7 Explain the notion of "uniform convergence" of a sequence of functions. Prove that a uniform limit of bounded continuous functions is bounded and continuous. Give an example of a sequence of bounded continuous functions which converge pointwise to a discontinuous function.

Problem 8 Prove that a product of two complete metric spaces is complete. Prove that a product of two sequentially compact metric spaces is sequentially compact.

Problem 9 Let X be a sequentially compact metric space. Prove that every open covering of X has a finite subcovering. (This is the easier direction in the equivalence between sequential compactness and topological compactness.)

Problem 10 Prove that the image of a compact metric space under a continuous map is a closed subset of the target. Give an example showing this is false if "compact" is replaced by "complete". What if "compact" is replaced by "complete" and "continuous map" is replaced by "isometric embedding"?

Problem 11 Prove that a subset of a metric space is closed if and only if it contains all of its limit points. Let Y be a non-closed subset of a metric space (X, d_X). Prove that there exists a continuous function $f : Y \to \mathbb{R}$ which does not extend to a continuous function on the closure of Y.

Problem 12 Prove that every Cauchy sequence is totally bounded. Prove that for a totally bounded metric space, every sequence has a Cauchy subsequence.

Problem 13 Prove that a metric spae (X, d_X) is complete if and only if for every continuous function $f : X \to \mathbb{R}$, the restriction of f to every totally bounded subset is bounded. (**Hint**. What does Problem 11 say when you consider X to be a subset of its completion?)

Problem 14 What does it mean to say that a series in a normed vector space is absolutely convergent? Is every subseries of an absolutely convergent series also absolutely convergent? If one sequence is pointwise dominated by a second sequence (in norm), and the second gives an absolutely convergent series, what can you say about the first?

Problem 15 Give an example of a sequence of elements in a normed vector space such that the corresponding sequence of partial sums is convergent, yet the series is not absolutely convergent.

Problem 16 Let $(V, \| \bullet \|)$ be a normed vector space. Let W be a linear subspace. Prove that the closure of W equals the intersection of all linear subspace $\mathrm{Ker}(\phi)$, asϕ ranges over all bounded linear functionals $\phi : V \to \mathbb{R}$ such that $W \subset \mathrm{Ker}(\phi)$.

Problem 17 Let $(V, \| \bullet \|_V)$ and $(W, \| \bullet \|_W)$ be normed linear spaces. Assume that the closed unit ball in V is compact. Prove directly (without using finite dimensionality) that for every bounded linear operator $T : V \to W$, there exists $\vec{v} \in V$ with $\|\vec{v}\|_V = 1$ and with $\|T(\vec{v})\|_W = \|T\|_{\mathrm{op}}$.

Problem 18 Let $BC([0,1), \mathbb{R})$ be the vector space of bounded, continous functions on $\mathbb{R}$ with the uniform metric. Show that the following linear transformation is bounded

$$T : BC([0,1), \mathbb{R}) \to BC([0,1), \mathbb{R}), T(f(x)) = xf(x).$$

Show that $\|T\|_{\mathrm{op}}$ equals 1, yet there is no bounded continuous function $f(x)$ in $BC([0,1), \mathbb{R})$ such that $\|f\|_{\mathrm{uni.}} = 1$ and $\|T(f)\|_{\mathrm{uni.}}$ equals 1. Thus in Problem 17 it is not sufficient to replace “compact” by “complete”.

Problem 19 Let $(H, \langle \bullet, \bullet \rangle)$ be an inner product space (either real or Hermitian). Assume that the natural map $i : H \to H^*$ is an isometric bijection, where $i(\vec{v})$ is the bounded linear functional $\vec{w} \mapsto \langle \vec{w}, \vec{w} \rangle$. Prove that H is a Hilbert space.

Problem 20 Let X_1, X_2, Y_1 and Y_2 be metric spaces. Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be continuous functions. Prove that the following map between the product metric spaces is continuous.

$$f : X_1 \times X_2 \to Y_1 \times Y_2, \quad (x_1, x_2) \mapsto (f_1(x_1), f_2(x_2)).$$

Problem 21 Let $f : X \to Y$ be a continuous map which is bijective. If X is compact, prove that the inverse bijection $f^{-1} : Y \to X$ is continuous. Give a counterexample when X is not compact.

Problem 22 A set of functions is *uniformly bounded* if there is a common bound for all of the functions. Give an example of a metric space and an equicontinous sequence of uniformly bounded functions on that metric space which has no uniformly convergent subsequence. What if the metric space is compact?

Problem 23 Let $(V, \| \bullet \|_V)$ and $(W, \| \bullet \|_W)$ be normed vector spaces. Let $T : V \to W$ be a bounded linear transformation. Prove that there exists a bounded linear transformation $T^\dagger : W^* \to V^*$ such that for every bounded linear functional $\phi : W \to \mathbb{R}$ and every $\vec{v} \in V$, $(T^\dagger \phi)(\vec{v}) = \phi(T\vec{v})$. What is the relationship between $\|T\|_{\text{op}}$ and $\|T^\dagger\|_{\text{op}}$?

Problem 24 Let $(V, \| \bullet \|)$ be a normed vector space. Let $W \subset V$ be a linear subspace. If W is closed in V, prove that also $(W^*)^*$ is closed in $(V^*)^*$. Does the converse hold?

Problem 25 Let $(H, \langle \bullet, \bullet \rangle)$ be an inner product space. Let $\vec{v}_1, \dots, \vec{v}_n$ be elements in H. If they are pairwise orthogonal, prove that for every sequence of scalars $(t_1, \dots, t_n)$, we have

$$\left\| \sum_{i=1}^{n} t_i \vec{v}_i \right\| = \sum_{i=1}^{n} |t_i| \, \|\vec{v}_i\| .$$

Does the converse hold?

Problem 26 Let $\langle \bullet, \bullet \rangle$ be the standard inner product on $\mathbb{R}^2$. Let W be the subspace spanned by a nonzero vector $[a, b]^\dagger$. Compute the matrix M of the orthogonal projection of $\mathbb{R}^2$ onto W. Compute $M \cdot M$ and directly verify it equals M.

Problem 27 Same problem as above, but in $\mathbb{R}^3$.

Problem 28 For the following ordered sequence of vectors $(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ in $\mathbb{R}^3$, perform the Gram-Schmidt algorithm to find an orthonormal basis $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ such that $\text{span}(\vec{v}_1, \dots, \vec{v}_i) = \text{span}(\vec{u}_1, \dots, \vec{u}_i)$ and $\langle \vec{v}_i, \vec{u}_i \rangle$ is a positive real number for $i = 1, 2, 3$.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \vec{v}_1 = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} \vec{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$$

What if you reverse the order of the vectors?

Problem 29 Same problem as above for the following vectors.

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix} \vec{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \vec{v}_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$