Name: ______________________________ **Problem 1:** ____________ /20

Problem 1(20 points) In each of the following statements, circle T if it is true and F if it is false. Each part is worth **only 2 points out of 100 total points**. Remember to use your time wisely. There is no need to show your work on this problem.

$\boxed{\text{T}}$ F (a) If both $f(x)$ and $g(x)$ are continuous at $x = a$, then also $f(x)g(x)$ is continuous at $x = a$.

T $\boxed{\text{F}}$ (b) If both $f(x)$ and $g(x)$ are discontinuous at $x = a$, then also $f(x)g(x)$ is discontinuous at $x = a$.

T $\boxed{\text{F}}$ (c) Each function $f(x)$ which is everywhere continuous is also everywhere differentiable.

$\boxed{\text{T}}$ F (d) An even function cannot have 2 different horizontal asymptotes.

$\boxed{\text{T}}$ F (e) If $f(x)$ is even and differentiable, then the derivative $f'(x)$ is odd.

$\boxed{\text{T}}$ F (f) The function $f(x) = x^3 + \cos(x)$ is zero for at least one real number x.

T $\boxed{\text{F}}$ (g) $\lim\limits_{x\to\infty} \tan(x) = \infty$.

T $\boxed{\text{F}}$ (h) If $\lim_{x\to\infty} f(x)$ does not exist, then $\lim_{x\to\infty} f(x)$ equals either ∞ or $-\infty$.

T $\boxed{\text{F}}$ (i) Let $f(x)$ be a continuous function defined on the interval $[a, b]$. Let L be a real number which is not between $f(a)$ and $f(b)$. Then by the Intermediate Value Theorem, there does not exist a number c between a and b such that $f(c) = L$.

$\boxed{\text{T}}$ F (j) If the function $f(x)$ is everywhere defined and is invertible, and if $g(y)$ is everywhere defined and is invertible, then also $g(f(x))$ is everywhere defined and is invertible.

Name: ________________________ Problem 2: __________ /20

Problem 2(20 points) The function $f(x)$ is defined by the following formula.

$$f(x) = \frac{3x-5}{2-x}.$$

(**a**)(5 points) Find the vertical and horizontal asymptotes for $y = f(x)$. Remember to show work justifying your answers.

Vertical Asymptote. $x=a$ where $\lim_{x\to a^\pm} f(x) = \pm\infty$

$2-x=0$, $x=2$. Check. $\lim_{x\to 2^-} \frac{3x-5}{2-x} = \frac{1}{0^+} = +\infty$ ✓ & $\lim_{x\to 2^+} \frac{3x-5}{2-x} = \frac{1}{0^-} = -\infty$ ✓

Horizontal Asymptote. $y=L$ where $\lim_{x\to\pm\infty} f(x) = L$

$+\infty$. $\frac{3x-5}{2-x} \to \frac{3x}{-x} = -3$, $\lim_{x\to+\infty} \frac{3x-5}{2-x} = -3$

$-\infty$. $\frac{3x-5}{2-x} \to \frac{3x}{-x} = -3$, $\lim_{x\to-\infty} \frac{3x-5}{2-x} = -3$

Vertical Asymptote. $\boxed{x=2}$ Horizontal Asymptote. $\boxed{y=-3}$

(**b**)(5 points) Find a formula for the inverse function $f^{-1}(x)$. Show your work.

$f(x)$. $y = \frac{3x-5}{2-x}$

$\downarrow$

$f^{-1}(x)$. $x = \frac{3y-5}{2-y}$

$\to x(2-y) = 3y-5 \to 2x - xy = 3y-5 \to 2x+5 = xy+3y$

$2x+5 = (x+3)y \to \frac{2x+5}{x+3} = y$.

$f^{-1}(x) = \boxed{\frac{2x+5}{x+3}}$

Double-check. $3y-5 = \frac{3(2x+5)}{x+3} - \frac{5(x+3)}{(x+3)} = \frac{6x+15-5x-15}{x+3} = \frac{x}{x+3}$,

$2-y = \frac{2(x+3)}{x+3} - \frac{(2x+5)}{x+3} = \frac{1}{x+3}$

$\frac{3y-5}{2-y} = \frac{x/x+3}{1/x+3} = x$

(**c**)(5 points) Find all vertical and horizontal asymptotes for $y = f^{-1}(x)$. Remember to show work justifying your answers.

Vertical Asymptote. $x=a$ where $\lim_{x\to a^\pm} f(x) = \pm\infty$

$x+3=0$, $x=-3$. Check. $\lim_{x\to -3^-} \frac{2x+5}{x+3} = \frac{-1}{0^-} = +\infty$ ✓ & $\lim_{x\to -3^+} \frac{2x+5}{x+3} = \frac{-1}{0^+} = -\infty$ ✓

Horizontal Asymptote. $y=L$ where $\lim_{x\to\pm\infty} f(x) = L$

$+\infty$. $\frac{2x+5}{x+3} \to \frac{2x}{x} = 2$, $\lim_{x\to+\infty} \frac{2x+5}{x+3} = 2$

$-\infty$. $\frac{2x+5}{x+3} \to \frac{2x}{x} = 2$, $\lim_{x\to-\infty} \frac{2x+5}{x+3} = 2$

Vertical Asymptote. $\boxed{x=-3}$ Horizontal Asymptote. $\boxed{y=2}$

DOUBLE-CHECK

$y=f(x)$ | $y=f^{-1}(x)$

H.A. $\leftrightarrow$ V.A., V.A. $\leftrightarrow$ H.A.

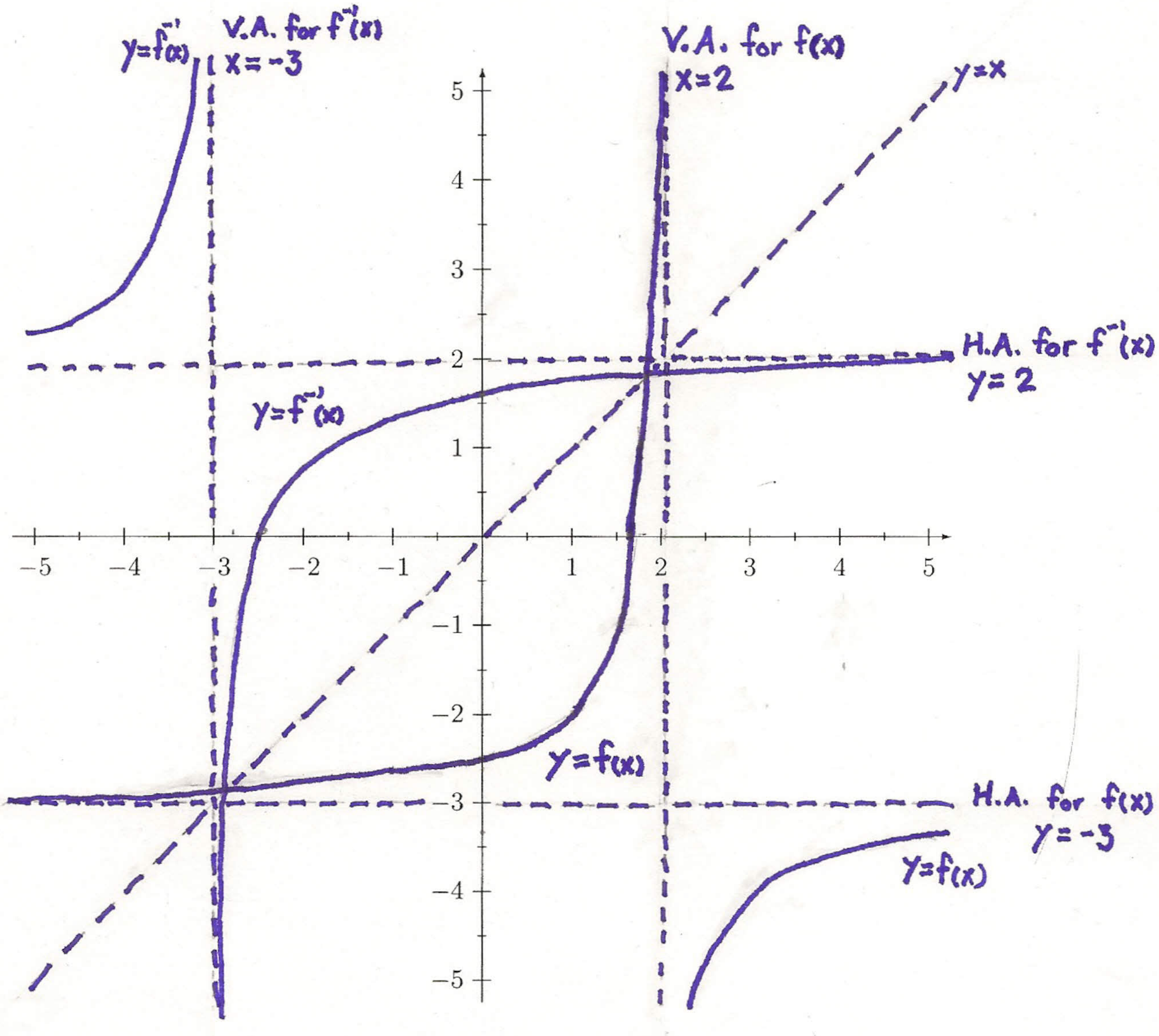
$y = \frac{3x-5}{2-x}$ | $y = \frac{2x+5}{x+3}$

H.A. $y=-3$ | H.A. $y=2$ ✓

V.A. $x=2$ | V.A. $x=-3$ ✓

Name: ______________________________ **Problem 2, continued**

(d)(5 points) On the grid below, sketch the graph of both $y = f(x)$ and $y = f^{-1}(x)$. Each graph has **no** local maximum, **no** local minimum, and **no** inflection point. Carefully label all vertical and horizontal asymptotes of each curve. Your graph should make clear how each curve approaches each asymptote from each relevant side.



Name: ______________________________ Problem 3: ____________ /25

Problem 3(25 points) Consider the function $f(x) = -2 + 3\sqrt{x}$.

Definition

(a)(20 points) Use the limit definition of the derivative to compute $f'(4)$. **Remember to show all your work.**

$$f'(4) = \lim_{h\to 0} \frac{f(4+h)-f(4)}{h}$$

$$f(4+h)-f(4) = \left(-2+3\sqrt{4+h}\right)-\left(-2+3\sqrt{4}\right) = 3\left(\sqrt{4+h}-\sqrt{4}\right),$$

$$\frac{f(4+h)-f(4)}{h} = \frac{3\left(\sqrt{4+h}-\sqrt{4}\right)}{h} \times \frac{\left(\sqrt{4+h}+\sqrt{4}\right)}{\left(\sqrt{4+h}+\sqrt{4}\right)} = \frac{3\left((4+h)-4\right)}{h\left(\sqrt{4+h}+\sqrt{4}\right)} = \frac{3h}{h\left(\sqrt{4+h}+\sqrt{4}\right)} = \frac{3}{\sqrt{4+h}+\sqrt{4}}$$

$$\lim_{h\to 0} \frac{f(4+h)-f(4)}{h} = \lim_{h\to 0} \frac{3}{\sqrt{4+h}+\sqrt{4}} = \frac{3}{\sqrt{4}+\sqrt{4}} = \frac{3}{2+2} = \frac{3}{4}$$

$$\boxed{f'(4) = \frac{3}{4}}$$

(b)(5 points) Find the equation of the tangent line at $(4, f(4))$. Write your answer in **slope-intercept form**, $y = mx + b$.

Equation of the Tangent Line: $y - f(a) = f'(a)(x-a)$.

$a = 4$

$f(a) = -2+3\sqrt{4} = -2+3\times 2 = -2+6 = 4$

$f'(a) = f'(4) = \frac{3}{4}$

$$y - 4 = \frac{3}{4}(x-4) = \frac{3}{4}x - 3 \rightarrow y = \frac{3}{4}x - 3 + 4 = \frac{3}{4}x + 1$$

$y =$ $\boxed{\frac{3}{4}x + 1}$

Name: ______________________ Problem 4: __________ /25

Problem 4(25 points) Consider the piecewise-defined function given by the following formula.

$$f(x) = \begin{cases} \sqrt{x^2-2x} - x, & x \geq 2 \\ \frac{|x|-2}{|x|-1} & x < 2 \text{ and } |x| \neq 1, \\ 3 & |x| = 1 \end{cases}$$

(a)(12 points) At each of the following points, circle **Cont.** if the function is continuous at that point, and circle **Discont.** if the function is discontinuous at that point. If the function is discontinuous, also circle the letter of the type of discontinuity: **R** for **R**emovable, **J** for **J**ump, or **I** for **I**nfinite. Show your work.

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(\sqrt{x^2-2x} - x) = \sqrt{2^2 - 2\cdot 2} - 2 = \underline{-2}$; $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{|x|-2}{|x|-1} = \frac{|2|-2}{|2|-1} = \frac{0}{1} = \underline{0}$.

Since $\lim_{x\to 2^+} f(x)$ & $\lim_{x\to 2^-} f(x)$ exist and are <u>unequal</u>, $f(x)$ has a <u>jump</u> discontinuity at $x=2$.

$x = +2$. Cont. or **Discont.** If discont., the type is: R **J** I

$x>0$. $\frac{|x|-2}{|x|-1} = \frac{x-2}{x-1}$, $\lim_{x\to 1^+} \frac{x-2}{x-1} = \frac{-1}{0^+} = \underline{-\infty}$, $\lim_{x\to 1^-} \frac{x-2}{x-1} = \frac{-1}{0^-} = \underline{+\infty}$.

Since $\lim_{x\to 1^+} f(x)$ equals $-\infty$ (& also $\lim_{x\to 1^-} f(x)$ equals $+\infty$), $f(x)$ has an <u>infinite</u> discontinuity at $x=1$.

$x = +1$. Cont. or **Discont.** If discont., the type is: R J **I**

<u>$x>0$</u>. $\frac{|x|-2}{|x|-1} = \frac{x-2}{x-1}$, <u>$x<0$</u>. $\frac{|x|-2}{|x|-1} = \frac{-x-2}{-x-1}$, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x-2}{x-1} = \frac{-2}{-1} = \underline{2}$. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{-x-2}{-x-1} = \frac{-2}{-1} = \underline{2}$.

Since $\lim_{x\to 0^-} f(x)$ & $\lim_{x\to 0^+} f(x)$ exist and are equal, $\lim_{x\to 0} f(x)$ equals 2. And $f(0)=2$. So $f(x)$ is <u>continuous</u> at $x=0$.

$x = 0$. **Cont.** or Discont. If discont., the type is: R J I

$x<0$. $\frac{|x|-2}{|x|-1} = \frac{-x-2}{-x-1} = \frac{x+2}{x+1}$, $\lim_{x\to -1^+} \frac{x+2}{x+1} = \frac{1}{0^+} = \underline{+\infty}$, $\lim_{x\to -1^-} \frac{x+2}{x+1} = \frac{1}{0^-} = \underline{-\infty}$.

Since $\lim_{x\to -1^+} f(x)$ equals $+\infty$ (& also $\lim_{x\to -1^-} f(x)$ equals $-\infty$), $f(x)$ has an <u>infinite</u> discontinuity at $x=-1$.

$x = -1$. Cont. or **Discont.** If discont., the type is: R J **I**

Name: ______________________________ Problem 4, continued

(c)(13 points) Compute both limits

$$\lim_{x\to+\infty} f(x) \text{ and } \lim_{x\to-\infty} f(x).$$

Then state the equations of all horizontal asymptotes. Show your work.

$\underline{x \geq 2}$. $f(x) = \sqrt{x^2-2x} - x$. $\lim_{x\to+\infty} f(x) = ?$ $\underline{\text{Plug-in}}$. $\sqrt{\infty} - \infty = \infty - \infty \to \underline{\text{Indeterminate}}$.

$$\sqrt{x^2-2x} - x = \frac{\sqrt{x^2-2x} - x}{1} \times \frac{\sqrt{x^2-2x}+x}{\sqrt{x^2-2x}+x} = \frac{(x^2-2x)-x^2}{\sqrt{x^2-2x}+x} = \frac{-2x}{\underline{\sqrt{x^2-2x}+x}} \longrightarrow \frac{-2x}{\sqrt{x^2}+x} = \frac{-2x}{x+x}$$

$$\frac{-2x}{2x} = \underline{\underline{-1}}$$

$\lim_{x\to+\infty} f(x) = \boxed{-1}$

$\underline{x < 0}$. $f(x) = \frac{|x|-2}{|x|-1} = \frac{-x-2}{-x-1} = \frac{x+2}{x+1} \longrightarrow \frac{x}{x} = \underline{\underline{1}}$

$\lim_{x\to-\infty} f(x) = \boxed{1}$

Equation of each horiz. asymptote. $\boxed{y = -1 \text{ and } y = 1}$

Name: ______________________ **Problem 5:** ____________ /10

Problem 5(10 points) The following function is everywhere continuous.

$$f(x) = \begin{cases} x^2, & x > 1 \\ 2\mid x \mid -1, & x \leq 1. \end{cases}$$

(a)(5 points) At each of the following points, say whether the function is differentiable by circling the correct answer: **Diff.** if it is differentiable and **Nondiff.** if it is nondifferentiable. **Show your work and give reasons for your answers.** In this problem, you may use any formula you know for the derivatives; you need not use the limit definition.

$\underline{x>1}$. $f(x)=x^2$, $f'(x)=2x$, $\lim_{x\to1^+} f(x) = \lim_{x\to1^+} 2x = \underline{2}$; $\underline{0<x<1}$, $f(x) = 2|x|-1 = 2x-1$, $f'(x)=2$, $\lim_{x\to1^-} f'(x) = \underline{2}$.

Since $\lim_{x\to1^+} f'(x)$ and $\lim_{x\to1^-} f'(x)$ exist and are equal, $f(x)$ is differentiable at $x=1$. And $f'(1)$ equals 2.

$x = 2.$ $\boxed{\text{Diff.}}$ or Nondiff.

$\underline{0<x<1}$, $f(x)=2|x|-1 = 2x-1$, $f'(x)=2$, $\lim_{x\to0^+} f'(x) = \underline{2}$; $\underline{x<0}$, $f(x) = 2|x|-1 = 2(-x)-1 = -2x-1$, $f'(x)=-2$, $\lim_{x\to0^-} f'(x) = \underline{-2}$

Since $\lim_{x\to0^+} f'(x)$ does not equal $\lim_{x\to0^-} f'(x)$, $f(x)$ has a "cusp" or "corner" at $x=0$. So $f(x)$ is nondiff. at $x=0$.

$x = 0.$ Diff. or $\boxed{\text{Nondiff.}}$

Name: ______________________________ **Problem 5, continued**

(b)(5 points) On the grid below, sketch the graph of $y = f(x)$. Carefully label the points on the graph where $x = 0$ and where $x = 2$. Make certain your sketch matches your answer from (a) at each of these points.