MAT131 Fall 2008 Final Review Sheet

The final exam will be **cumulative**. Please look at the review sheets for Midterms 1 and 2 to review the material from earlier in the semester. (We may choose to write up a few more problems from earlier in the semester. You can find many more such problems in the textbook and on exams from previous semesters.)

Particular emphasis on the final will be placed on the material which has not been tested on Midterms 1 or 2. Among this new material is the following.

(i) Related rates problems. Given several dependent variables which are differentiable functions of the same independent variable, and given constraints among these dependent variables, solve for an unknown rate of change from given rates of change. Combine this with problem-solving skills to handle related rates word problems.

(ii) Absolute maxima and minima; local maxima and minima; inflection points. Know how to find the absolute maximum and absolute minimum value of a differentiable function on a closed, bounded interval. Know how to find local maxima and minima and inflection points of functions, and use this to help graph the function.

(iii) L'Hôpital's rule. Recognize indeterminate forms. Simplify limits leading to $0/0$ and ∞/∞ indeterminate forms using l'Hôpital's rule. Know how to transform other indeterminate forms into one of these two types.

(iv) Optimization problems. Given a word problem attempting to maximize or minimize some quantity given a collection of constraints, turn this into a calculus problem for finding an absolute maximum or absolute minimum. Solve this calculus problem.

(v) Antiderivatives. Recognize the most common antiderivatives: those arising as the derivatives of x^n, trigonometric functions, exponential functions, logarithmic functions and inverse trigonometric functions.

(vi) Know how to set up a Riemann sum associated to a given integrand and a given interval. Be able to evaluate the limit of Riemann sums to compute the Riemann integral in the case of some simple integrands.

(vii) Know the statement of the Fundamental Theorem of Calculus. Understand how to use this to evaluate definite integrals when you can find a simple form for the antiderivative. Understand how the fundamental theorem always gives an antiderivative of a continuous function, where the antiderivative is defined in terms of the Riemann integral/definite integral.

(viii) Given a limit of sums, recognize when this is a limit of Riemann sums. Be able to use the fundamental theorem of calculus to evaluate this limit of Riemann sums.

(ix) Simplify antiderivatives using direct substitution.

(x) Evaluate definite integrals using direct substitution and the fundamental theorem of calculus.

Following are some practice problems. More practice problems are in the textbook as well as on the practice midterm.

Problem 1. A calculus instructor of height 170 cm is moving at speed 5.2 km/hr away from a street lamp of height 3m. What is the speed of the shadow of the instructor's head?

Problem 2. A long, straight piece of wood of length 7 meters rests with its foot on the ground and its midpoint on the top of a 3 meter tall fence. The foot of the plank begins to slide straight away from the fence with the plank still touching the top of the fence. At the moment when the foot of the plank is 4 meters from the base of the fence, the distance between the top of the plank and the ground is decreasing at 0.3 meters per second. At what speed is the foot of the plank moving away from the base of the fence?

Problem 3 In a flat ceiling two hooks are fastened 21 cm apart. A length of 27 cm of inextensible wire is suspended between the hooks. A heavy weight is hung from the wire near the first hook and slides along the wire toward a point equidistant from both hooks, pulling the wire taut at each moment. At the moment when the weight is 10 cm from the first hook, it is moving away from the first hook at a speed of 1 cm/sec.

(a) With what speed is the weight moving towards the second hook at this moment?

(b) What is distance between the weight and the ceiling at this moment?

(c) With what speed is the weight moving away from the ceiling (i.e., what is the rate of change of the distance between the weight and the ceiling)?

Problem 4 A car approaches a large hotel at night, driving along a semicircular driveway which becomes tangent to the front wall of the hotel at the entrance. The headlights of the car illuminate a spot on the front wall. If the radius of the driveway is 10 meters, and if the car is moving at a speed of 10 km/hour, with what speed is the spot moving when the angle between the entrance, the center of the circle and the car is $\pi/3$ radians, i.e., 60 degrees?

Problem 5 A cube of ice rests on a hot plate. It melts in such a way that its shape at every moment is a cube and the rate of decrease of the volume is a constant multiple of the area of the face resting on the hot plate. If after 5 minutes the volume of the cube is one eighth of its initial volume, how much longer is required before the cube melts entirely?

Problem 6. In each of the following cases, sketch the graph of the given function. On your graph state whether the function is even, odd or neither. State whether or not the function is periodic, and state the period if it is periodic. Label all discontinuities and the type. Label all vertical and horizontal asymptotes.

Label *both* the x and y-coordinates of all local maxima and minima and state where the function is increasing and where decreasing. Label *both* the x and y-coordinates of all inflection points and state where the function is concave up and where concave down.

(a)

$$y = x^2 - 4.$$

(b)

$$y = \frac{1}{x^2 - 4}$$

(c)

$$y = \frac{1}{x+1} - 2 + \frac{1}{x-1}.$$

(d)

$$y = \sin(x)$$

(e)

$$y = \tan(x)$$

(f)

$$y = \sin(x)\cos(x)$$

(g)

$$y = e^{-x^2/2}$$

(h)

$$y = \ln((x+1)^2/(x-1)^2).$$

(i)

$$y = \frac{e^x}{1+e^x}.$$

(j)

$$y = e^{-1/x^2}.$$

Problem 7 Let S be the square in the xy-plane centered at the origin, of edge length $\sqrt{2}$, and with diagonal edges of slopes $+1$ respectively -1. The equation for this square is

$$|x| + |y| = 1.$$

Let R be a square in the xy-plane centered at the origin whose horizontal and vertical edges are parallel to the x-axis and y-axis respectively. Among squares R which intersect S, what is the maximum possible area of the region lying outside the inner square and inside the outer square?

Problem 8 Compute the maximum volume of a right circular cone whose surface area (just of the cone, not of the "bottom" disk of the cone) is a fixed constant A. The surface area of a right circular cone is $A = \pi r s$ where r is the radius of the bottom disk and s is the *slant height*, i.e., the distance from the vertex of the cone to a point on the bottom circle of the cone. What is the ratio of radius to height for such a cone?

Problem 9 You will build a box in a corner of a room using the floor and two walls as sides of the box. To do this, remove a square from one corner of a square sheet of metal of edge length 10 feet. Fold the edges of the sheet meeting the missing square to form two sides of a box. The remaining square of the sheet forms the top of the box. Slide these three sides into the corner to form a box with square top. What is the maximum volume of this box?

Problem 10 Compute each of the following limits.

(a)

$$\lim_{x \to 0^+} \frac{1 - \cos(x)}{\sin^2(x)}$$

(b)

$$\lim_{x \to 0^+} \ln(1 - \cos(x)) - 2\ln(\sin(x))$$

(c)

$$\lim_{x \to \infty} \frac{1 + x}{x^2}$$

(d)

$$\lim_{x \to \pi/2} \tan(x) - \sec(x)$$

(e)

$$\lim_{x \to \infty} x \ln\left(\frac{x}{x+1}\right).$$

Problem 11 By thinking about areas compute an antiderivative of $\sqrt{1 - x^2}$. (Hint: Sketch the region whose area is the definite integral of this function from 0 to x.)

Problem 12 Consider the Riemann integral

$$\int_1^4 (2x + 1)dx.$$

Partition the interval into n subintervals of equal length. Compute the Riemann sum S_n for this partition using right endpoints. Write down the value of this Riemann sum. Directly compute the limit as n goes to infinity to find the Riemann integral. Double-check your answer against the Fundamental Theorem of Calculus.

Problem 13 Consider the Riemann integral

$$\int_0^1 3^x dx.$$

Partition the interval into n subintervals of equal length. Compute the Riemann sum S_n for this partition using left endpoints. Write down the value of this Riemann sum. Directly compute the limit as n goes to infinity to find the Riemann integral (you may use L'Hôpital's rule to evaluate the limit). Double-check your answer against the Fundamental Theorem of Calculus.

Problem 14 Compute each of the following definite and indefinite integrals.

(a)

$$\int x^{-1/2} dx$$

(b)

$$\int (x + \sin(x)) dx$$

(c)

$$\int \frac{1}{\sqrt{1-x^2}} dx$$

(d)

$$\int \sec(\theta) \tan(\theta) d\theta$$

(e)

$$\int_{-2}^{2} x^4 dx.$$

(f)

$$\int_{-\pi/2}^{\pi/2} \cos(2x) - \cos(x) dx$$

(g)

$$\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}dx$$

(h)

$$\int_{\pi/4}^{\pi/3} \sec^2(\theta)d\theta$$

Problem 15 Evaluate each of the following limits.

(a)

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} 1.$$

(b)

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}(1-\frac{2i}{n})$$

(c)

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{n-2i}{n^2}.$$

(d)

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{2^{i/n}}{n}$$

(e)

$$\lim_{m\to\infty} \sum_{i=1}^{m} \frac{m}{m^2+i^2}$$

Problem 16 Compute each of the following indefinite and definite integrals.

(a)

$$\int_a^b f'(cx)dx$$

(b)

$$\int x\sin(x^2)dx$$

(c)

$$\int_0^{\pi/4} \tan(x)dx$$

(d)

$$\int_0^{\pi/2} \sin^3(x)dx$$

(e)

$$\int \frac{\ln(x)}{x}dx$$

(f)

$$\int_1^2 \frac{e^{\ln(x)}}{x}dx$$

(g)

$$\int_0^{x^2} \frac{f'(\sqrt{t})}{\sqrt{t}}dt$$