MAT 131 FINAL EXAM – SOLUTIONS
FALL 2010

(1) (a) The limit is the indeterminate form 0/0. Using L'Hospital's Rule:

$$\lim_{x\to 0}\frac{\sin(2x)}{\sin(3x)} = \lim_{x\to 0}\frac{2\cos(2x)}{3\cos(3x)} = \frac{2\cdot 1}{3\cdot 1} = \boxed{\frac{2}{3}}$$

(b) The limit is the indeterminate form 0/0. Using L'Hospital's Rule and the known limit $\lim_{x\to 0}\sin(x)/x = 1$:

$$\lim_{x\to 0}\frac{1-\cos(x)}{x^2} = \lim_{x\to 0}\frac{\sin(x)}{2x} = \frac{1}{2}\lim_{x\to 0}\frac{\sin x}{x} = \boxed{\frac{1}{2}}$$

(c) Using the factorizations for $a^3 - b^3$ and $a^2 - b^2$:

$$\lim_{x\to 1}\frac{x^3 - x^{-3}}{x^2 - x^{-2}} = \lim_{x\to 1}\frac{(x - x^{-1})(x^2 + xx^{-1} + x^{-2})}{(x - x^{-1})(x + x^{-1})} = \lim_{x\to 1}\frac{x^2 + 1 + x^{-2}}{x + x^{-1}} = \boxed{\frac{3}{2}}$$

(d) First we find the limit of $\ln x^{1/(x-1)} = (\ln x)/(x-1)$, using L'Hospital's Rule:

$$\lim_{x\to 1}\frac{\ln x}{x-1} = \lim_{x\to 1}\frac{1/x}{1} = 1.$$

Then, because $x^{1/(x-1)} = e^{(\ln x)/(x-1)}$ and the exponential function is continuous,

$$\lim_{x\to 1} x^{1/(x-1)} = \lim_{x\to 1} e^{(\ln x)/(x-1)} = e^1 = \boxed{e}$$

(2) $y = f(x)$, with

$$y = \frac{6x^2 - 8}{x^3} = 6x^{-1} - 8x^{-3}.$$

(a) Computation of $f'(x)$ and $f''(x)$, using the power rule and the sum rule:

$$f'(x) = \boxed{-6x^{-2} + 24x^{-4}}$$

$$f''(x) = \boxed{12x^{-3} - 96x^{-5}}$$

(b) $f(x)$ is $\boxed{\textbf{odd}}$, because

$$f(-x) = \frac{6(-x)^2 - 8}{(-x)^3} = \frac{6x^2 - 8}{-x^3} = -\frac{6x^2 - 8}{x^3} = -f(x)$$

(c) Vertical asymptotes occur where the denominator of the rational expression defining y equals zero. The equation of the only vertical asymptote is $\boxed{x = 0}$.

(d) Horizontal asymptotes are found by looking at the limit of the function at ±∞.

$$\lim_{x\to\infty}\frac{6x^2 - 8}{x^3} = \lim_{x\to -\infty}\frac{6x^2 - 8}{x^3} = 0$$

because the degree of the denominator is greater than the degree of the numerator. The only horizontal asymptote is $\boxed{y = 0}$.

(e) $f'(x)$ can be written in the form $g(x)/x^4$ with $g(x)$ a polynomial as

$$\boxed{f'(x) = \frac{-6x^2+24}{x^4}} \qquad \text{or} \qquad \boxed{f'(x) = \frac{-6(x^2-4)}{x^4}}.$$

(f) $f'(x)$ is defined everywhere $f(x)$ is defined, so the only critical points occur where $f'(x) = 0$. This happens exactly when the numerator of one of the expressions from (e) is zero, that is when $x^2 - 4 = 0$, or $\boxed{x = \pm 2}$.

(g) $f(x)$ is increasing when $f'(x) > 0$ and decreasing when $f'(x) < 0$. From part (f), we know that $f'(x) = 0$ when $x = \pm 2$. $f(x)$ is note defined at $x = 0$. Checking $f'(\pm 1)$ and $f'(\pm 3)$, we see that

$$\boxed{\begin{array}{c} f(x) \text{ is increasing on } (-2,0) \text{ and } (0,2). \\ f(x) \text{ is decreasing on } (-\infty,-2) \text{ and } (2,\infty). \end{array}}$$

(h) $f''(x)$ can be written in the form $h(x)/x^5$ with $h(x)$ a polynomial as

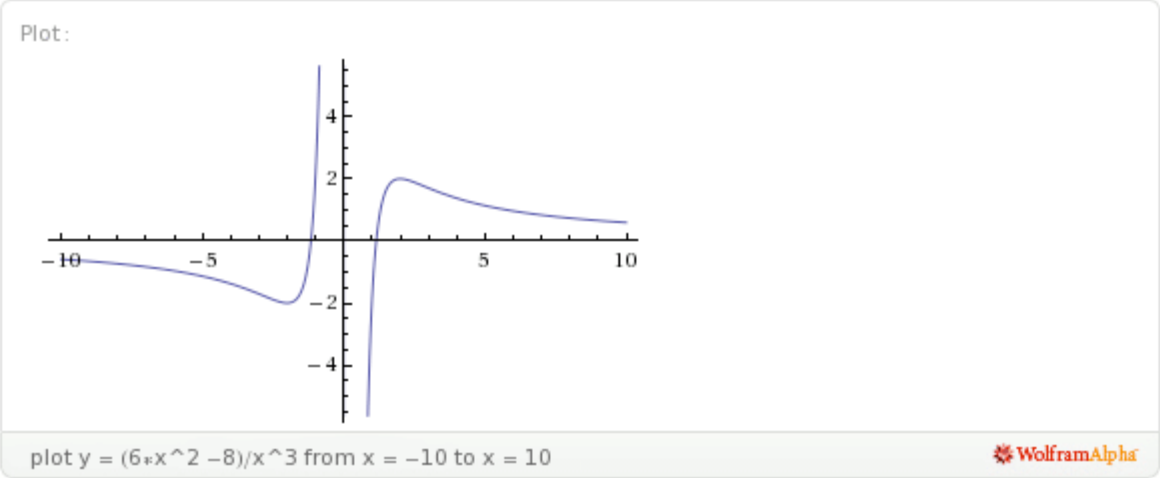
$$\boxed{f'(x) = \frac{12x^2-96}{x^5}} \qquad \text{or} \qquad \boxed{f'(x) = \frac{12(x^2-8)}{x^5}}.$$

(i) From part (h) we know that $f''(x) = 0$ when $x^2 - 8 = 0$ or $\boxed{x = \pm 2\sqrt{2}}$. These are indeed inflection points, because $f''(x)$ changes sign at both of them.

(j) $f(x)$ is concave up when $f''(x) > 0$ and concave down when $f''(x) < 0$. From part (h) we know that $f''(x) = 0$ when $x = \pm 2\sqrt{2}$. When $|x| > 2\sqrt{2}$, $f''(x) > 0$ and when $|x| < 2\sqrt{2}$ (with $x \neq 0$), $f''(x) < 0$. Thus

$$\boxed{\begin{array}{c} f(x) \text{ is concave up on } (-\infty,-2\sqrt{2}) \text{ and } (2\sqrt{2},\infty). \\ f(x) \text{ is concave down on } (-2\sqrt{2},0) \text{ and } (0,2\sqrt{2}). \end{array}}$$

(k) Here is the graph of $y = f(x)$:



plot y = (6*x^2 −8)/x^3 from x = −10 to x = 10

(3) (a) $\displaystyle\int_1^{\sqrt{2}} (x^3 - x)\,dx = \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_1^{\sqrt{2}} = \left(\frac{(\sqrt{2})^4}{4} - \frac{(\sqrt{2})^2}{2}\right) - \left(\frac{1^4}{4} - \frac{1^2}{2}\right) = \boxed{\frac{1}{4}}$

(b) $\displaystyle\int_{-\pi/6}^{\pi/6} 2\cos(x)\,dx = 2\sin(x)\Big|_{-\pi/6}^{\pi/6} = 2\left(\sin\frac{\pi}{6} - \sin\frac{-\pi}{6}\right) = 4\sin\frac{\pi}{6} = 4\cdot\frac{1}{2} = \boxed{2}$

(c) $\displaystyle\int \frac{1}{1+t^2}\,dt = \boxed{\arctan(t) + C}$

(d) $\displaystyle\int_{-e^5}^{-e^2} \frac{1}{x}\,dx = \ln|x|\Big|_{-e^5}^{-e^2} = \ln|-e^2| - \ln|-e^5| = 2\ln e - 5\ln e = 2 - 5 = \boxed{-3}$

(e) $\displaystyle\int_{\pi/4}^{\pi/3} \frac{\sec^2(x)}{\tan(x)}\,dx$. Setting $u = \tan(x)$, we have $du = \sec^2(x)\,dx$ and the new limits of integration $\tan\frac{\pi}{4} = 1$ and $\tan\frac{\pi}{3} = \sqrt{3}$. The integral becomes

$$\int_1^{\sqrt{3}} \frac{1}{u}\,du = \ln u\Big|_1^{\sqrt{3}} = \ln\sqrt{3} - \ln 1 = \boxed{\frac{1}{2}\ln 3}.$$

(We do not need to include the absolute value in the antiderivative $\ln u$ here because the entire domain over which the integral is taken consists of positive values of u.)

(f) $\displaystyle\int \frac{x^5 + x^2}{(x^3+1)^2}\,dx = \int \frac{x^5 + x^2}{x^6 + 2x^3 + 1}\,dx$

We make the substitution $u = x^6 + 2x^3 + 1$, so that $du = 6(x^5 + x^2)\,dx$:

$$\int \frac{x^5 + x^2}{x^6 + 2x^3 + 1}\,dx = \frac{1}{6}\int \frac{1}{u}\,du = \frac{1}{6}\ln|u| + C = \boxed{\frac{1}{6}\ln|x^6 + 2x^3 + 1| + C} \quad \text{or} \quad \boxed{\frac{1}{3}\ln|x^3 + 1| + C}$$

(4) The (rectangular) box has two equal dimensions of length x and one longer dimension of length y. Its volume is therefore $V = x^2 y$. We seek to minimize the length of three edges of length x on the top and bottom each, plus one edge of length y, which gives the expression $L = 6x + y$. From the constraint $V = x^2 y$ we have $y = V/x^2$, so $L = 6x + V/x^2$. This function is continuous at all positive values of x, and it clearly approaches ∞ as $x \to 0$ or $x \to \infty$, so a global minimum occurs at a critical point. The derivative is

$$L' = 6 - \frac{2V}{x^3}$$

which equals zero when $6x^3 = 2V$, or $x^3 = V/3$. Because V is positive, this has a unique positive solution. Plugging in the given value $V = 1000$, we find

$$x = \sqrt[3]{\frac{1000}{3}} = \boxed{\frac{10}{3^{1/3}}\text{ ft}}, \qquad \text{from which} \qquad y = \frac{1000}{(10/3^{1/3})^2} = \boxed{10\cdot 9^{1/3}\text{ ft}}.$$

(These values are approximately $x \approx 6.9$ and $y \approx 20.8$, but the approximations are not needed to answer the question.)

(5) We recall that the limit definition of the derivative of f at x is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

For $f(x) = \dfrac{1}{1+x^2}$, this becomes

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\frac{1}{1+(x+h)^2} - \frac{1}{1+x^2}}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{1+x^2}{(1+x^2)(1+(x+h)^2)} - \frac{1+(x+h)^2}{(1+x^2)(1+(x+h)^2)}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{1+x^2-(1+x^2+2xh+h^2)}{(1+x^2)(1+(x+h)^2)}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{-2xh-h^2}{(1+x^2)(1+(x+h)^2)}\right) \\
&= \lim_{h\to 0} \frac{-2x-h}{(1+x^2)(1+(x+h)^2)} \\
&= \boxed{\frac{-2x}{(1+x^2)^2}}
\end{aligned}$$

(6) Let r be the distance from a point (x, y) to the origin. Our goal is to find dr/dt under the given conditions. The Pythagorean theorem implies that $r^2 = x^2 + y^2$. Differentiating both sides of this equation with respect to t, we obtain $2r(dr/dt) = 2x(dx/dt) + 2y(dy/dt)$, or (after dividing both sides by 2)

$$r\,\frac{dr}{dt} = x\,\frac{dx}{dt} + y\,\frac{dy}{dt}.$$

We are told that the point (x, y) is traveling on the curve $3x^2 - 2xy + 3y^2 = 12$, which means that

$$6x\,\frac{dx}{dt} - 2\left(x\,\frac{dy}{dt} + y\,\frac{dx}{dt}\right) + 6y\,\frac{dy}{dt} = 0$$

at all times. Substituting in the coordinates of the point $(0, 2)$, we have that

$$-2(2)\,\frac{dx}{dt} + 6(2)\,\frac{dy}{dt} = 0 \qquad \rightsquigarrow \qquad \frac{dy}{dt} = \frac{1}{3}\,\frac{dx}{dt}$$

at the moment we are interested in. At this moment, we are told, $dx/dt = 6$, and so $dy/dt = 6/3 = 2$. Now we just need to observe that $(0, 2)$ is $r = 2$ units from the origin, and we have all of the pieces needed to find dr/dt:

$$2\,\frac{dr}{dt} = (0)(6) + (2)(2) = 4, \qquad \text{and so} \qquad \boxed{\frac{dr}{dt} = 2}.$$

(7) We consider the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{i}{n}\left(1 - \frac{i^2}{n^2}\right)$$

(a) We are asked to find an integral of the form $\int_0^1 f(x)\,dx$ equal to the above limit. Because the limits of integration are taken to be 0 and 1, our "step size" is $\Delta x = (1-0)/n = 1/n$. The values i/n, from $i = 1$ to $i = n$, are then the right-hand endpoints of the subintervals of length Δx, and so we take $x_i = i/n$. This means that the function we are looking for is $x(1-x^2)$, since this is the function evaluated at each of the x_i. The desired integral is

$$\boxed{\int_0^1 x(1-x^2)\,dx}.$$

(b) To find the value of the limit, we can just compute the integral from part (a) using the FTC:

$$\int_0^1 (x - x^3)\,dx = \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \left(\frac{1^2}{2} - \frac{1^4}{4}\right) - (0-0) = \boxed{\frac{1}{4}}.$$