MAT 123 Introduction to Calculus — **Jason Starr**
Midterm 1 — **Spring 2015**
Thursday 2/26 8:30pm

MAT 123 Practice for Midterm 1 with Solutions

Remark. If you are comfortable with all of the following problems, you will be well prepared for Midterm 1.

Exam Policies. You must show up on time for all exams. Please bring your student ID card: ID cards may be checked, and students may be asked to sign a picture sheet when turning in exams. Other policies for exams will be announced / repeated at the beginning of the exam.

If you have a university-approved reason for taking an exam at a time different than the scheduled exam (because of a religious observance, a student-athlete event, etc.), please contact your instructor as soon as possible. Similarly, if you have a documented medical emergency which prevents you from showing up for an exam, again contact your instructor as soon as possible.

All exams are closed notes and closed book. Once the exam has begun, having notes or books on the desk or in view will be considered cheating and will be referred to the Academic Judiciary.

It is not permitted to use cell phones, calculators, laptops, MP3 players, Blackberries or other such electronic devices at any time during exams. If you use a hearing aid or other such device, you should make your instructor aware of this before the exam begins. You must turn off your cell phone, etc., prior to the beginning of the exam. If you need to leave the exam room for any reason before the end of the exam, it is still not permitted to use such devices. Once the exam has begun, use of such devices or having such devices in view will be considered cheating and will be referred to the Academic Judiciary. Similarly, once the exam has begun any communication with a person other than the instructor or proctor will be considered cheating and will be referred to the Academic Judiciary.

Mastery Review Topics.
In addition to the Core Competency Exam, each midterm and final will include mastery questions. Students who have already passed the Core Competency Exam need not repeat the Core Competency Exam: those students may devote all of their time and energy to the mastery questions.
For Midterm 1, these are the mastery topics.

(1) Determine the domain of a function given as an equation, including those equations involving radical, denominators (that might be zero), and absolute values.

(2) Understand horizontal and vertical linear transformations. Given an equation, find the new equation whose graph achieves a specified transformation of the original graph. Conversely, given an original equation and a transformed equation, recognize the transformation.

(3) Understand basic operations involving two specified functions: addition, subtraction, multiplication, division and function composition. Understand the domain of the function obtained by such an operation.

(4) Understand what it means for a function to be one-to-one, i.e., invertible (as a function to its range). Know the Horizontal Line Test. Find the inverse of an invertible function. Understand the relationship between the domain, respectively range, of an invertible function and the domain, resp. range, of the inverse function.

(5) Understand the various equations of a line (slope-intercept, slope-point, point-point) and how to translate between these equations. Understand when lines are parallel, respectively perpendicular.

(6) Find the roots of a quadratic equation both by the quadratic formula and by completing the square. Use completing the square to find the vertex of a given parabola. Understand when a parabola opens up, resp. opens down.

(7) Recognize the equations of an ellipse, resp. of a hyperbola. Use completing the square to find the center of a given ellipse (resp. hyperbola).

Practice Problems.

(1) In each of the following cases, determine the maximal domain of the given expression. Write your answer in interval notation.

$$(a)\ \sqrt{x-3},\quad (b)\ \frac{1}{x+5},\quad (c)\ \frac{1}{|x+2|-3},\quad (d)\ \frac{\sqrt{x+4}}{x},\quad (e)\ \sqrt{4-x^2},\quad (f)\ \frac{1+(1/x)}{1-(1/x)}.$$

Solution to 1

$$(a)\ \boxed{[3,\infty)},\quad (b)\ \boxed{(-\infty,-5)\cup(-5,\infty)},\quad (c)\ \boxed{(-\infty,-5)\cup(-5,1)\cup(1,\infty)},$$

$$(d)\ \boxed{[-4,0)\cup(0,\infty)},\quad (e)\ \boxed{[-2,2]},\quad (f)\ \boxed{(-\infty,0)\cup(0,1)\cup(1,\infty)}.$$

(2) In each of the following cases, for the given functions $f(x)$ and $g(x)$, compute the given operation. Also specify the domain.

$$(a) f(x)=\sqrt{x},\ [0,\infty),\quad g(x)=x^2-1,\quad f(x)/g(x)=?,$$

$$(b) f(x)=\frac{x+1}{1-x},\ (1,\infty),\quad g(x)=\frac{1-x}{x+1},(-1,\infty),\quad f(x)+g(x)=?$$

$$(c) f(x) = \sqrt{x},\ [0,\infty),\quad g(x) = x^2 - 1,\quad g(f(x)) = ?,$$

$$(c) f(x) = \sqrt{x},\ [0,\infty),\quad g(x) = x^2 - 1,\quad f(g(x)) = ?,$$

$$(d) f(x) = x^2 - 9,\quad g(x) = \frac{16+x}{16-x}, (16,\infty),\quad g(f(x)) = ?$$

Solution to 2

$$(a)\ \boxed{\tfrac{\sqrt{x}}{x^2-1},\ [0,1)\cup(1,\infty)},\quad (b)\ \boxed{2\tfrac{1+x^2}{1-x^2},\ (1,\infty)},\quad (c)\ \boxed{x-1,\ [0,\infty)},$$

$$(c')\ \boxed{\sqrt{x^2-1},\ (-\infty,-1]\cup[1,\infty)},\quad (d)\ \boxed{\tfrac{7+x^2}{25-x^2},\ (-\infty,-5)\cup(5,\infty)}.$$

(3) In each of the following cases, for the specified function and specified domain, determine the range. Say whether the function on the specified domain is one-to-one, i.e., satisfies the Horizontal Line Test. If the function is one-to-one, find the inverse function (defined on the range of the original function); simplify your answer as much as possible.

$$(a)\ x^2+1,\ (0,\infty),\quad (b)\ x^2+1,\ (-\infty,0),\quad (c)\ \sqrt{x-3},\ [3,\infty),\quad (d)\ x^2-2x+1,\ (0,\infty),$$

$$(e)\ \frac{x+1}{x-1},\ (1,\infty),\quad (f)\ \frac{1+(1/x)}{1-(1/x)},\ (1,\infty),\quad (g)\ \frac{x^2+2x+1}{x^2-2x+1},\ (1,\infty),\quad (h)\ x^3-x,\ [0,\infty),$$

$$(i)\ x^3,\ [0,\infty),\quad (j)\ \frac{2x+3}{x-2},\ (2,\infty),\quad (k)\ \frac{2x+3}{x-2},\ (-\infty,2).$$

Solution to 3

$$(a)\ \boxed{\text{Range} = (1,\infty).}\ \boxed{\text{One-to-one.}}\ \boxed{f^{-1}(x) = \sqrt{x-1}.}$$

$$(b)\ \boxed{\text{Range} = (1,\infty).}\ \boxed{\text{One-to-one.}}\ \boxed{f^{-1}(x) = -\sqrt{x-1}.}$$

$$(c)\ \boxed{\text{Range} = [0,\infty).}\ \boxed{\text{One-to-one.}}\ \boxed{f^{-1}(x) = x^2+3.}$$

$$(d)\ \boxed{\text{Range} = [0,\infty).}\ \boxed{\text{Not one-to-one; } f(1/2) = f(3/2).}$$

$$(e)\ \boxed{\text{Range} = (1,\infty).}\ \boxed{\text{One-to-one.}}\ \boxed{f^{-1}(x) = \tfrac{x+1}{x-1}.}$$

$$(f)\ \boxed{\text{Range} = (1,\infty).}\ \boxed{\text{One-to-one.}}\ \boxed{f^{-1}(x) = \tfrac{x+1}{x-1}.}$$

$$(g)\ \boxed{\text{Range} = (1,\infty).}\ \boxed{\text{One-to-one.}}\ \boxed{f^{-1}(x) = \sqrt{(x+1)/(x-1)}.}$$

$$(h)\ \boxed{\text{Range} = [-2\sqrt{3}/9, \infty).}\ \boxed{\text{Not one-to-one; } f(0) = f(1).}$$

$$(i)\ \boxed{\text{Range} = [0, \infty).}\ \boxed{\text{One-to-one.}}\ \boxed{f^{-1}(x) = \sqrt[3]{x}.}$$

$$(j)\ \boxed{\text{Range} = (2, \infty).}\ \boxed{\text{One-to-one.}}\ \boxed{f^{-1}(x) = (2x+3)/(x-2).}$$

$$(k)\ \boxed{\text{Range} = (-\infty, 2).}\ \boxed{\text{One-to-one.}}\ \boxed{f^{-1}(x) = (2x+3)/(x-2).}$$

(4) In each of the following cases, for the specified function and the specified linear transformation, find a new function whose graph is obtained from the original graph by the specified linear transformation. Simplify your answer as much as possible.

$$(a)\ f(x) = x,\ \text{vertical scale by 2 followed by horizontal shift by } -3.$$

$$(b)\ f(x) = x^2,\ \text{horizontal scale by 3 followed by horizontal shift by } +1.$$

$$(c)\ f(x) = \sqrt{1-x^2},\ \text{vertical and horizontal scale by 2 followed by vertical shift by 2.}$$

$$(d)\ f(x) = x^2,\ \text{horizontal reflection, horizontal scale by } 1/2, \text{ followed by horizontal shift by } -1/2.$$

Solution to 4

$$(a)\ g(x) = 2f(x-(-3)) = 2(x+3) = \boxed{2x+6.}$$

$$(b)\ g(x) = f\left(\frac{x-1}{3}\right) = \left(\frac{x-1}{3}\right)^2 = \boxed{(x^2-2x+1)/9.}$$

$$(c)\ g(x) = 2f\left(\frac{x}{2}\right) + 2 = 2\sqrt{1-(x/2)^2} + 2 = \boxed{2+\sqrt{4-x^2}.}$$

$$(d)\ g(x) = f\left(\frac{-(x-(-1/2))}{1/2}\right) = f(-2x-1) = (-2x-1)^2 = \boxed{4x^2+4x+1.}$$

(5) In each of the following cases, describe a linear transformation that transforms the graph of $y = f(x)$ to the graph of $y = g(x)$.

$$(a)\quad f(x) = x,\quad g(x) = 3x+2,\quad (b)\quad f(x) = x^2,\quad g(x) = 3x^2+2,$$

$$(c)\quad f(x) = \sqrt{1-x^2},\quad g(x) = \sqrt{16-12x-4x^2},\quad (d)\quad f(x) = \frac{1}{x},\quad g(x) = \frac{3x-1}{x-1}.$$

Solution to 5 Some of these have more than one correct solution.

$$(a)\ \boxed{\text{Vertical scale by 3 followed by vertical shift by 2.}}$$

(b) $\boxed{\text{Vertical scale by 3 followed by vertical shift by 2.}}$

(c)Use completing the square to rewrite $g(x) = \sqrt{25 - 4(x + (3/2))^2} = 5\sqrt{1 - \left(\frac{x + (3/2)}{5/2}\right)^2}$.

$\boxed{\text{Vertical scale by 5, horizontal scale by 5/2, followed by horizontal shift by } -3/2.}$

(d) Proper fraction equals $g(x) = 3 + \dfrac{2}{x-1}$.

$\boxed{\text{Vertical scale by 2, followed by horizontal shift by } +1\text{, vertical shift by } +3.}$

(6) In each of the following cases, say whether the function is even, odd or neither.

$$(a)\ x + \frac{1}{x},\quad (-\infty, 0) \cup (0, \infty)\quad (b)\ \frac{x^3 - x}{x^3 + x},\quad (-\infty, 0) \cup (0, \infty)\quad (c)\ |x|,$$

$$(d)\ \frac{x}{|x|},\ (-\infty, 0) \cup (0, \infty)\quad (e)\ \sqrt{x^4 + x^2 + 1}.$$

Solution to 6

$$(a)\ f(-x) = -f(x),\ \boxed{\text{Odd.}}\quad (b)\ f(-x) = f(x),\ \boxed{\text{Even.}}\quad (c)\ f(-x) = f(x),\ \boxed{\text{Even.}}$$

$$(d)\ f(-x) = -f(x),\ \boxed{\text{Odd.}}\quad (e)\ f(-x) = f(x),\ \boxed{\text{Even.}}$$

(7) For each parabola below, say whether it opens up or opens down, find the vertex, and find the x-coordinates of any intersection points with the x-axis (assuming there are any). Sketch the graph of your parabola labelling all of the above.

$$(a)\ y = 2x^2 + 2x + 1,\ (b)\ y = -x^2 + x + 1,\ (c)\ y = 7x^2 + 14x + 1,\ (d)\ y = 3x^2 + 5x + 1,\ (e)\ y = -x^2 - x - \frac{1}{4}.$$

Solution to 7

(a) $y = 2\left(x + \frac{1}{2}\right)^2 + \frac{1}{2}$. $\boxed{\text{Opens up. Vertex at } (-1/2, 1/2)\text{. No intersection with } x\text{-axis.}}$

(b) $y = -\left(x - \frac{1}{2}\right)^2 + \frac{5}{4}$. $\boxed{\text{Opens down. Vertex at } (1/2, 5/4)\text{. Intersects } x\text{-axis at } x = (1 \pm \sqrt{5})/2.}$

(c) $y = 7(x + 1)^2 - 6$. $\boxed{\text{Opens up. Vertex at } (-1, -6)\text{. Intersects } x\text{-axis at } x = (-7 \pm \sqrt{42})/7.}$

(d) $y = 3\left(x + \frac{5}{6}\right)^2 - \frac{13}{12}$. $\boxed{\text{Opens up. Vertex at } (-5/6, -13/12)\text{. Intersects } x\text{-axis at } x = (-5 \pm \sqrt{13})/6.}$

(e) $y = -\left(x + \frac{1}{2}\right)^2$. $\boxed{\text{Opens down. Vertex at } (-1/2, 0)\text{. Intersects } x\text{-axis only at } x = -1/2.}$

(8) For each of the following equations of a circle, find the center and the radius of the circle.

(a) $x^2+x+y^2+y = 4$, (b) $2x^2-x+2y^2+3y = 14$, (c) $x^2+2x+y^2-2y+2 = 0$, (d) $3x^2+6x+3y^2-3y = 15$.

Solution to 8

$$(a)\ \left(x+\frac{1}{2}\right)^2 + \left(y+\frac{1}{2}\right)^2 = \frac{18}{4}.$$

$$\boxed{\text{Center at } (x,y) = (-1/2, -1/2)\text{. Radius } r = 3\sqrt{2}/2.}$$

$$(b)\ 2\left(x-\frac{1}{4}\right)^2 + 2\left(y+\frac{3}{4}\right)^2 = \frac{138}{8}.$$

$$\boxed{\text{Center at } (x,y) = (1/4, -3/4)\text{. Radius } r = \sqrt{138}/4.}$$

$$(c)\ (x+1)^2 + (y-1)^2 = 0.$$

$$\boxed{\text{Center at } (x,y) = (-1, +1)\text{. Radius } r = 0.}$$

$$(d)\ 3(x+1)^2 + 3\left(y-\frac{1}{2}\right)^2 = \frac{75}{4}.$$

$$\boxed{\text{Center at } (x,y) = (-1, 1/2)\text{. Radius } r = 5/2.}$$

(9) In each of the followin cases, say whether the given conic section is a parabola, a circle, an ellipse, or a hyperbola.

$$(a)\ x^2+2x-y^2+2y = 4,\quad (b)\ 9x^2+16y^2 = 25,\quad (c)\ 4y^2+4y-x^2-6x^2 = 17,$$

$$(d)\ 2x^2+3x+4y = 3,\quad (e)\ x^2+x-y+y^2 = 5.$$

Solution to 9

(a) $\left(\frac{x+1}{2}\right)^2 - \left(\frac{y-1}{2}\right)^2 = 1$, $\boxed{\text{Hyperbola.}}$ Center at $(-1, 1)$. Asymptotes $(y-(1/2)) = \pm(x+1/2)$.

(b) $\left(\frac{x}{5/3}\right)^2 + \left(\frac{y}{5/4}\right)^2 = 1$, $\boxed{\text{Ellipse.}}$ Center at $(0, 0)$. Special points $(\pm 5/3, 0)$, $(0, \pm 5/4)$.

(c) $\left(\frac{y+(1/2)}{3/2}\right)^2 - \left(\frac{x-3}{3}\right)^2 = 1$, $\boxed{\text{Hyperbola.}}$ Center at $(3, -1/2)$. Asymptotes $(y+(1/2)) = \pm\frac{1}{2}(x-3)$.

$$(d)\ 4\left(y-\left(-2\left(x-\frac{-3}{4}\right)^2+\frac{31}{32}\right)\right)=0,\ \boxed{\text{Parabola.}}\ \text{Vertex at } (-3/4, 31/32).$$

$$\text{Opens down. } x-\text{Intercepts at } x=(-6\pm\sqrt{31})/8.$$

$$(e)\ \left(\frac{x-(-1/2)}{\sqrt{5}}\right)^2-\left(\frac{y-(1/2)}{\sqrt{5}}\right)^2=1,\ \boxed{\text{Hyperbola.}}\ \text{Center at } (-1/2, 1/2).$$

$$\text{Asymptotes } (y-(1/2))=\pm(x-(-1/2)).$$