THE SNAKE DIAGRAM

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ABSTRACT. This is a test. This is only a test. Do not attempt to adjust your set. Do not contact emergency personnel.

1. STATEMENT OF THE LEMMA

Let $\mathcal{C}$ be an Abelian category. In particular, every image in $\mathcal{C}$ equals the coimage. Thus we make no distinction between images and coimages in what follows. One of the fundamental notions of homological algebra is the following.

Definition 1.1. A *short exact sequence*

$$\Sigma_A: \ 0 \longrightarrow A' \xrightarrow{q_A} A \xrightarrow{p_A} A'' \longrightarrow 0$$

is a pair of morphisms in $\mathcal{C}$

$$\Sigma_A = (q_A : A' \to A, p_A : A \to A)$$

such that all of the following hold:

(i) q_A is a monomorphism,
(ii) p_A is an epimorphism, and
(iii) the image of q_A equals the kernel of p_A.

There is a category whose objects are short exact sequences in $\mathcal{C}$. Here is the notion of morphism in this category.

Definition 1.2. Let $\Sigma_A = (q_A, p_A)$ and $\Sigma_B = (q_B, p_B)$ be short exact sequences in $\mathcal{C}$. A *morphism* Σ_f from Σ_A to Σ_B,

$$\begin{array}{ccccccccc}
\Sigma_A: & 0 & \longrightarrow & A' & \xrightarrow{q_A} & A & \xrightarrow{p_A} & A'' & \longrightarrow & 0 \\
\Sigma_f\big\downarrow & & & f'\big\downarrow & & \big\downarrow f & & \big\downarrow f'' & & \\
\Sigma_B: & 0 & \longrightarrow & B' & \xrightarrow[q_B]{} & B & \xrightarrow[p_B]{} & B'' & \longrightarrow & 0
\end{array},$$

is a triple of morphisms in $\mathcal{C}$

$$\Sigma_f = (f' : A' \to B', f : A \to B, f'' : A'' \to B'')$$

such that every square commutes, i.e., both of the following hold:

(i) $q_B \circ f'$ equals $f \circ q_A$, and
(ii) $p_B \circ f$ equals $f'' \circ p_A$.

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In the category of short exact sequences the identity morphisms and the compositions are the obvious notions. The category of short exact sequences is an additive category.

Let Σ_f be a morphism of short exact sequences as above. Denote the kernels of f', respectively f, f'' by,

$$i' : K'_{\Sigma_f} \to A', \text{ resp. } \ i : K_{\Sigma_f} \to A, \ i'' : K''_{\Sigma_f} \to A''.$$

Similarly, denote the cokernels of f', respectively f', f'' by,

$$s' : B' \to C'_{\Sigma_f}, \text{ resp. } \ s : B \to C_{\Sigma_f}, \ s'' : B'' \to C''_{\Sigma_f}.$$

Because $q_B \circ f'$ equals $f \circ q_A$, also $f \circ (q_A \circ i')$ equals $q_B \circ (f' \circ i')$, which equals $q_B \circ 0 = 0$. Thus, by the universal property of the kernel, there is a unique morphism

$$q_K : K'_{\Sigma_f} \to K_{\Sigma_f}$$

such that $i \circ q_K$ equals $q_A \circ i'$. For a similar reason, there is a unique morphism

$$p_K : K_{\Sigma_f} \to K''_{\Sigma_f}$$

such that $i'' \circ p_K$ equals $p_A \circ i$. And by analogous arguments there are unique morphisms

$$q_C : C'_{\Sigma_f} \to C_{\Sigma_f}, \quad p_C : C_{\Sigma_f} \to C''_{\Sigma_f}$$

such that $q_C \circ s'$ equals $s \circ q_B$, and $p_C \circ s$ equals $s'' \circ p_B$. To summarize, we have that the following diagram is commutative.

$$
\begin{array}{ccccccccc}
 & & K'_{\Sigma_f} & \xrightarrow{q_K} & K_{\Sigma_f} & \xrightarrow{p_K} & K''_{\Sigma_f} & & \\
 & & {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i''} & & \\
\Sigma_A :\ 0 & \longrightarrow & A' & \xrightarrow{q_A} & A & \xrightarrow{p_A} & A'' & \longrightarrow & 0 \\
{\scriptstyle \Sigma_f}\downarrow & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\
\Sigma_B :\ 0 & \longrightarrow & B' & \xrightarrow[q_B]{} & B & \xrightarrow[p_B]{} & B'' & \longrightarrow & 0 \\
 & & {\scriptstyle s'}\downarrow & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s''} & & \\
 & & C'_{\Sigma_f} & \xrightarrow[q_C]{} & C_{\Sigma_f} & \xrightarrow[p_C]{} & C''_{\Sigma_f} & &
\end{array}
$$

By hypothesis, both $f'' \circ p_A$ and $p_B \circ f$ are equal. Denote by t this common morphism

$$t : A \to B''.$$

Denote the kernel of t by

$$j : K_t \to A.$$

Now $f'' \circ (p_A \circ j)$ equals $t \circ j$, which is 0. By the universal property of the kernel of f'', there is a unique morphism

$$\widetilde{p_A} : K_t \to K''_{\Sigma_f}$$

such that $i'' \circ \widetilde{p_A}$ equals $p_A \circ j$. Similarly, $p_B \circ (f \circ j)$ equals $t \circ j$, which is 0. By the universal property of the kernel of p_B, there is q unique morphism

$$\widetilde{f} : K_t \to B'$$

such that $q_B \circ \widetilde{f}$ equals $f \circ j$.

Lemma 1.3 (The Snake Lemma)**.** *For a morphism Σ_f of commutative diagrams as above, all of the following hold.*

(i) *The morphism q_K is a monomorphism, and the morphism p_C is an epimorphism.*

(ii) *The image of q_K equals the kernel of p_K, and the kernel of p_C equals the image of q_C.*

(iii) *There is a unique morphism $\delta_{\Sigma_f} : K''_{\Sigma_f} \to C'_{\Sigma_f}$ such that $\delta_{\Sigma_f} \circ \widetilde{p_A}$ equals $s' \circ \widetilde{f}$ as morphisms $K_t \to C'_{\Sigma_f}$.*

(iv) *The image of p_K equals the kernel of δ_{Σ_f}, and the kernel of q_C equals the image of δ_{Σ_f}.*

In summary, the following long sequence is exact,

$$0 \longrightarrow K'_{\Sigma_f} \xrightarrow{q_K} K_{\Sigma_f} \xrightarrow{p_K} K''_{\Sigma_f} \xrightarrow{\delta_{\Sigma_f}} \dots$$

$$\dots \xrightarrow[\delta_{\Sigma_f}]{} C'_{\Sigma_f} \xrightarrow[q_C]{} C_{\Sigma_f} \xrightarrow[p_C]{} C''_{\Sigma_f} \longrightarrow 0.$$

This entire situation is often summarized with the following large diagram.

$$\begin{array}{cccccccccc}
 & & & 0 & & 0 & & 0 & & \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 & \longrightarrow & K'_{\Sigma_f} & \xrightarrow{q_K} & K_{\Sigma_f} & \xrightarrow{p_K} & K''_{\Sigma_f} & \xrightarrow{\delta_{\Sigma_f}} & \dots \\
 & & & {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i''} & & \\
\Sigma_A : & 0 & \longrightarrow & A' & \xrightarrow{q_A} & A & \xrightarrow{p_A} & A'' & \longrightarrow & 0 \\
{\scriptstyle \Sigma_f}\downarrow & & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\
\Sigma_B : & 0 & \longrightarrow & B' & \xrightarrow[q_B]{} & B & \xrightarrow[p_B]{} & B'' & \longrightarrow & 0 \\
 & & & {\scriptstyle s'}\downarrow & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s''} & & \\
 & \dots & \xrightarrow[\delta_{\Sigma_f}]{} & C'_{\Sigma_f} & \xrightarrow[q_C]{} & C_{\Sigma_f} & \xrightarrow[p_C]{} & C''_{\Sigma_f} & \longrightarrow & 0 \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & 0 & & 0 & & 0 & &
\end{array}$$

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