The Snake Lemma

A Beamer Presentation

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Beware of Snakes!

Problems from Topology

HOMOLOGICAL ALGEBRA ORIGINATED FROM COMPUTATIONAL ISSUES WITH THE HOMOLOGY / COHOMOLOGY OF SPACES.

1. The Künneth decomposition does not work "on the nose" in the presence of torsion, e.g., $\mathbb{R}\mathbf{P}^2 \times \mathbb{R}\mathbf{P}^2$.
2. Again in the presence of torsion, the Universal Coefficients Theorem for homology / cohomology with coefficients has an extra term.
3. The first computations of group homology / cohomology were *ad hoc* (they were also topological, being the homology / cohomology of a classifying space of the group).

Problems from Topology

Homological algebra originated from computational issues with the homology / cohomology of spaces.

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Ad Hoc Constructions

Motto

Homological algebra systematizes *ad hoc* constructions from topology and algebra.

- Group cohomology: original definitions all in terms of explicit cocycles.
- Tor: MacLane's explicit description.
- Ext: The Yoneda approach.
- MacLane's *Homological Algebra* develops these operations **without** the language of derived functors.
- For readers who know derived functors, the original constructions are now difficult to read.

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Formal Development

THE FORMAL DEVELOPMENT OF HOMOLOGICAL ALGEBRA BEGAN IN THE POST-WAR PERIOD AND CONTINUES TO TODAY.

Highpoints.

1. *Homological Algebra* by Cartan and Eilenberg, 1956.
2. Grothendieck's "Tohoku paper", *Sur quelques points d'algèbre homologique*, 1957: **Abelian categories**, **delta functors**, **sheaf cohomology**, **Grothendieck spectral sequence**.
3. Verdier's thesis on derived categories, early 1960s.
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5. Giraud's non-Abelian cohomology, Quillen's model categories, Grothendieck's topoi, operads, higher categories, stacks, etc., up to the present.

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Abelian Categories

Additive Categories

An *additive category* is a category where Hom sets are Abelian groups, composition is biadditive, there is a zero object, and there are finite products.

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In an Abelian category, most categorical facts familiar from the category of modules hold, e.g., image and coimage are the same. Frequently Abelian categories are assumed to satisfy additional "compatibility axioms" with arbitrary products and coproducts.

Examples.

- The category of (left) modules over an associative, unital ring.
- The category of sheaves of Abelian groups on a space.

Freyd-Mitchell Theorem

Every *small* Abelian category is equivalent to a full subcategory of the category of modules over a ring.

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Short Exact Sequences

For what follows, fix an Abelian category. All objects and morphisms are in this Abelian category.

Short Exact Sequences

A *short exact sequence*

$$\Sigma_A : \qquad 0 \longrightarrow A' \xrightarrow{q_A} A \xrightarrow{p_A} A'' \longrightarrow 0$$

is a pair of morphisms

$$\Sigma_A = (q_A : A' \to A, p_A : A \to A)$$

such that all of the following hold:

(i) q_A is a monomorphism,

(ii) p_A is an epimorphism, and

(iii) the image of q_A equals the kernel of p_A.

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Morphisms of Short Exact Sequences

Let $\Sigma_A = (q_A, p_A)$ and $\Sigma_B = (q_B, p_B)$ be short exact sequences. A *morphism* Σ_f from Σ_A to Σ_B,

$$\begin{array}{ccccccccccc}
\Sigma_A : & & 0 & \longrightarrow & A' & \xrightarrow{q_A} & A & \xrightarrow{p_A} & A'' & \longrightarrow & 0 \\
\Sigma_f \big\downarrow & & & & \big\downarrow f' & & \big\downarrow f & & \big\downarrow f'' & & \\
\Sigma_B : & & 0 & \longrightarrow & B' & \xrightarrow[q_B]{} & B & \xrightarrow[p_B]{} & B'' & \longrightarrow & 0
\end{array}$$

is a triple of morphisms

$$\Sigma_f = (f' : A' \to B', f : A \to B, f'' : A'' \to B'')$$

such that every square commutes, i.e., both of the following hold:

(i) $q_B \circ f'$ equals $f \circ q_A$, and

(ii) $p_B \circ f$ equals $f'' \circ p_A$.

Short Exact Sequences Form a Category

The short exact sequences in a fixed Abelian category form their own category.

- Composition of morphisms of short exact sequences is term-by-term composition for f', f and f'' separately.
- The identity morphism of a short exact sequence Σ_A is just $(\mathrm{Id}_{A'}, \mathrm{Id}_A, \mathrm{Id}_{A''})$.
- The category of short exact sequences is additive, but it is *not* Abelian.

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Kernels and Cokernels

For a morphism of short exact sequence,

The morphism Σ_f

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denote the kernels of f', respectively f, f'' by,

$$i' : K'_{\Sigma_f} \to A', \text{ resp. } i : K_{\Sigma_f} \to A, \; i'' : K''_{\Sigma_f} \to A''.$$

Similarly, denote the cokernels by

$$s' : B' \to C'_{\Sigma_f}, \text{ resp. } s : B \to C_{\Sigma_f}, \; s'' : B'' \to C''_{\Sigma_f}.$$

Morphisms of the Kernels

There are induced morphisms of the kernels.

Morphisms of Kernels

$$K'_{\Sigma_f} \xrightarrow{q_K} K_{\Sigma_f} \xrightarrow{p_K} K''_{\Sigma_f}$$

- Since $q_B \circ f'$ equals $f \circ q_A$, also $f \circ (q_A \circ i')$ equals $q_B(f' \circ i') = 0$. This gives q_K.
- The morphism p_K is similar.

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There are also induced morphisms of the cokernels.

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$$C'_{\Sigma_f} \xrightarrow{q_C} C_{\Sigma_f} \xrightarrow{p_C} C''_{\Sigma_f}$$

- The proof of the existence of the morphisms of cokernels is similar to the proof of the existence of the morphisms of the kernels.

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To summarize, given a morphism of short exact sequences,

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The Associated Large Diagram

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 & & & & \downarrow{\scriptstyle s'} & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s''} & & \\
 & & & & C'_{\Sigma_f} & \xrightarrow[q_C]{} & C_{\Sigma_f} & \xrightarrow[p_C]{} & C''_{\Sigma_f} & &
\end{array}
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The Common Composition

By hypothesis, both $f'' \circ p_A$ and $p_B \circ f$ are equal. Denote by t this common composition.

The common composition

$$\begin{array}{ccc} A & \xrightarrow{p_A} & A'' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f''} \\ B & \xrightarrow[p_B]{} & B'' \end{array}$$

And denote the kernel of t by

The kernel of t

$$j : K_t \to A.$$

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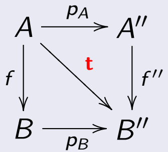

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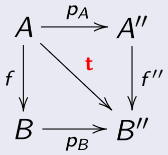

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The Key Morphisms

Now $f'' \circ (p_A \circ j)$ equals $t \circ j$, which is 0. So by the universal property of the kernel of f'', there is a unique morphism

The first key morphism

$$\widetilde{p_A} : K_t \to K''_{\Sigma_f}$$

such that $i'' \circ \widetilde{p_A}$ equals $p_A \circ j$.
similarly, $p_B \circ (f \circ j)$ equals $t \circ j$, which is 0. By the universal property of the kernel of p_B, there is a unique morphism

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Statement of the Snake Lemma

Finally this brings us to the statement of the snake lemma for a morphism Σ_f of short exact sequences.

The Snake Lemma

All of the following hold.

(i) The morphism q_K is a monomorphism, and the morphism p_C is an epimorphism.

(ii) The image of q_K equals the kernel of p_K, and the kernel of p_C equals the image of q_C.

(iii) There is a unique morphism $\delta_{\Sigma_f} : K''_{\Sigma_f} \to C'_{\Sigma_f}$ such that $\delta_{\Sigma_f} \circ \widetilde{p_A}$ equals $s' \circ \widetilde{f}$ as morphisms $K_t \to C'_{\Sigma_f}$.

(iv) The image of p_K equals the kernel of δ_{Σ_f}, and the kernel of q_C equals the image of δ_{Σ_f}.

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(iii) There is a unique morphism $\delta_{\Sigma_f} : K''_{\Sigma_f} \to C'_{\Sigma_f}$ such that $\delta_{\Sigma_f} \circ \widetilde{p_A}$ equals $s' \circ \widetilde{f}$ as morphisms $K_t \to C'_{\Sigma_f}$.

(iv) The image of p_K equals the kernel of δ_{Σ_f}, and the kernel of q_C equals the image of δ_{Σ_f}.

Statement of the Snake Lemma

Finally this brings us to the statement of the snake lemma for a morphism Σ_f of short exact sequences.

The Snake Lemma

All of the following hold.

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