MAT 615 PROBLEM SET 3

Homework Policy. This problem set explores some transversality results for maps from genus 0 curves to varieties.

Problems.

Problem 0.(**Invertible Sheaves on Projective Bundles**) Let $(S, \mathcal{O}_S)$ and $(C, \mathcal{O}_C)$ be connected schemes. Let

$$(\pi, \pi^{\#}) : (C, \mathcal{O}_C) \to (S, \mathcal{O}_S)$$

be a flat, proper morphism such that every geometric fiber C_s is nonempty, reduced, connected, and has vanishing $H^q(C_s, \mathcal{O}_{C_s})$ for every $q > 0$.

(a) Prove that $R^q\pi_*\mathcal{O}_C$ is zero for every $q > 0$, and prove that the homomorphism $\pi^{\#} : \mathcal{O}_S \to \pi_*\mathcal{O}_C$ is an isomorphism, compatibly with arbitrary base change of S. For every invertible $\mathcal{O}_S$-module $\mathcal{A}$, prove that $R^1\pi_*(\pi^*\mathcal{A})$ is zero for every $q > 0$, and the adjointness map $\mathcal{L} \to \pi_*(\pi^*\mathcal{A})$ is an isomorphism.

(b) Let $\mathcal{L}$ be an invertible sheaf on C whose restriction to a geometric fiber C_s is isomorphic to the structure sheaf. After restricting S to an appropriate open neighborhood of s, prove that $R^q\pi_*\mathcal{L}$ is zero for every $q > 0$, and prove that $\mathcal{A} := \pi_*\mathcal{L}$ is an invertible $\mathcal{O}_S$-module, compatibly with arbitrary base change of S. Prove that the adjointness map $\pi^*(\pi_*\mathcal{L}) \to \mathcal{L}$ is an isomorphism, so that $\mathcal{L}$ is isomorphic to the inverse image of an invertible sheaf $\pi^*\mathcal{A}$.

(c) Now consider the special case that S is a DVR, and assume that every geometric fiber of π is irreducible. For every invertible sheaf $\mathcal{L}$ on C whose restriction to the geometric generic fiber is isomorphic to the structure sheaf, prove that also the restriction to the closed fiber is isomorphic to the structure sheaf.

(d) In the general case of a connected scheme S, but still assuming that every geometric fiber of π is irreducible, use the previous parts to prove that if the restriction of $\mathcal{L}$ to one geometric fiber is isomorphic to the structure sheaf, then the restriction to every geometric fiber is isomorphic to the structure sheaf, and $\mathcal{L}$ is isomorphic to the pullback of a unique invertible sheaf, namely $\mathcal{A} = \pi_*\mathcal{L}$. In summary, the following sequence is exact,

$$0 \to \mathrm{Pic}(S) \xrightarrow{\pi^*} \mathrm{Pic}(C) \to \mathrm{Pic}(C_s).$$

(e) In the special case that every geometric fiber of π is isomorphic to projective space $\mathbb{P}^n$, use the degree isomorphism,

$$\deg : \mathrm{Pic}(\mathbb{P}^n) \to \mathbb{Z},$$

to rewrite the exact sequence as

$$0 \to \mathrm{Pic}(S) \xrightarrow{\pi^*} \mathrm{Pic}(C) \to \mathbb{Z}.$$

Show that this sequence is right exact if and only if there exists an invertible sheaf $\mathcal{L}$ on C whose restriction to every geometric fiber is an ample generator of the Picard group. Any such invertible sheaf is usually called a **relative Serre twisting sheaf** and is denote by $\mathcal{O}(1)$.

(f) With hypotheses as above, prove that there exists a relative Serre twisting sheaf if and only if there exist Zariski local sections of π at every point of S.

(g) Assuming that there exists a Serre twisting sheaf $\mathcal{O}(1)$, prove that for every integer $d \geq -n$, for every integer $q > 0$, the higher direct image $R^q\pi_*\mathcal{O}(d)$ is zero and $\pi_*\mathcal{O}(d)$ is a locally free sheaf of rank $\binom{n+d}{n}$ compatibly with arbitrary base change. For every integer $d \geq 0$, prove that the multiplication map,

$$\mathrm{Sym}^d_{\mathcal{O}_S}(\pi_*\mathcal{O}(1)) \to \pi_*\mathcal{O}(d),$$

is an isomorphism.

(h) Assumptions as above, prove that $\omega_\pi = \pi^*(\bigwedge^{n+1}_{\mathcal{O}_S}(\pi_*\mathcal{O}(1))) \otimes_{\mathcal{O}_S} \mathcal{O}(-n-1)$ is a relative dualizing sheaf for π. Conclude that for every integer $d \geq -n$, for every integer $q > 0$, the higher direct image $R^{n-1}\pi_*\omega_\pi(-d)$ is zero and $R^n\pi_*\omega_\pi(-d)$ is a locally free sheaf of rank $\binom{n+d}{n}$ compatibly with arbitrary base change.

Problem 1(**Free and Very Free Rational Curves**) For a smooth, morphism $X \to S$, for a field-valued point of S, say Spec $k \to S$, and for an S-morphism $u : \mathbb{P}^1_k \to X$, recall that u is **free**, resp. **very free**, if $u^*T_{X/S}$ is globally generated, resp. ample.

(a) Let $d \geq 1$ be an integer. For an invertible sheaf $\mathcal{O}(e)$ on $\mathbb{P}^1_k$, prove that the following are equivalent.

(i) There exists an effective divisor $D \subset \mathbb{P}^1_k$ of degree d such that the following morphism is surjective,

$$r_D : H^0(\mathbb{P}^1_k, \mathcal{O}(e)) \to H^0(D, \mathcal{O}(e)|_D).$$

(ii) The cohomology $H^1(\mathbb{P}^1_k, \mathcal{O}(e-d))$ is zero.
(iii) The integer e is at least as positive as $d-1$.
(iv) For every effective divisor $D \subset \mathbb{P}^1_k$ of degree $\leq d$, the morphism r_D is surjective.

Also discuss what happens when d equals 0.

(b) Let $d \geq 1$ be an integer. For a locally free sheaf $\mathcal{E}$ of finite positive rank, prove that the following are equivalent.

(i) There exists an effective divisor $D \subset \mathbb{P}^1_k$ of degree d such that the following morphism is surjective,

$$r_D : H^0(\mathbb{P}^1_k, \mathcal{E}) \to H^0(D, \mathcal{E}|_D).$$

(ii) The cohomology $H^1(\mathbb{P}^1_k, \mathcal{E}(-d))$ is zero.
(iii) The degree e of every invertible quotient of $\mathcal{E}$ is at least as positive as $d-1$.
(iv) For every effective divisor $D \subset \mathbb{P}^1_k$ of degree $\leq d$, the morphism r_D is surjective.

(c) As above, assume that X is smooth over S so that the fiber $X_k = \mathrm{Spec}\ k \times_S X$ is smooth over Spec k. Let $(M, \mathcal{O}_M)$ be a finite type, integral k-scheme. Let

$$(\pi, \pi^\#) : (C, \mathcal{O}_C) \to (M, \mathcal{O}_M),$$

be a smooth, projective morphism whose geometric fibers are all isomorphic to $\mathbb{P}^1$. Let

$$(u, u^\#) : (C, \mathcal{O}_C) \to (X, \mathcal{O}_X)$$

be a morphism of S-schemes. Let $d \geq 1$ be an integer. Assume the induced morphism of d-fold fiber products is dominant,

$$u^d : C \times_M \cdots \times_M C \to X_k \times_{\text{Spec } k} \cdots \times_{\text{Spec } k} X_k.$$

Also assume either that the characteristic of k is 0 or that there exists a subvariety of the domain that is dominant and generically finite over the target with degree prime to the characteristic. Conclude that there exists a dense open subset $U \subset M$ such that every map u_m parameterized by a geometric point m of U has pullback tangent bundle $\mathcal{E} = u_m^* T_{X/S}$ satisfying the equivalent conditions of the previous part. (**Hint**. Apply generic smoothness to u^d, and use the hypothesis on the characteristic to prove that u^d induces a separable extension of function fields.) Such a morphism u_m is called $(d-1)$-**free**. Thus, free maps are 0-free, and very free maps are 1-free.

(d) Assume now that k is algebraically closed of Repeat the previous part with M replaced by the finitely many irreducible components of $M \setminus U$ (with their reduced structures). Combined with a Noetherian induction argument, conclude that there exists a dense open subscheme V of X^d such that u_m is $(d-1)$-free whenever $u_m^d(C_m^d)$ intersects V. In particular, for every curve class β, conclude that there exists a dense open subscheme X_β of X such that the morphism

$$\text{ev} : \mathcal{M}_{0,1}(X/S, \beta) \to X$$

is smooth over X_β.

(e) Let C be a proper, connected, reduced, at-worst-nodal curve that is a tree of genus 0 curves, i.e., every irreducible component is a smooth, genus 0 curve and the arithmetic genus equals 0. A **leaf** of the tree is an irreducible component that contains precisely one node of C. Use induction on the number of leaves to prove that for every locally free $\mathcal{O}_C$-module $\mathcal{E}$ whose restriction to every component is globally generated, for every smooth k-point p of C, the cohomology group $H^1(C, \mathcal{E}(-\underline{p}))$ is zero, and the following restriction map is surjective,

$$r_p : H^0(C, \mathcal{E}) \to \mathcal{E}|_p.$$

In fact, even for p a node of C, show that r_p is surjective.

(f) As above, let C be a tree of smooth, genus 0 curves. Let $u : C \to X$ is a k-morphism whose irreducible components are free maps. Apply the previous part to $\mathcal{E} = u^* T_{X/S}$ to conclude that for every smooth k-point p of C, the evaluation morphism

$$\text{ev} : \overline{\mathcal{M}}_{0,1}(X/S, \beta) \to X$$

is smooth at the point (C, p, u). By taking p to be a node (or by using some other deformation theory), also conclude that there are deformation of (C, u) that smooth the node, so that the open subset $\mathcal{M}_{0,0}(X/S, \beta)$ is locally dense in $\overline{\mathcal{M}}_{0,0}(X/S, \beta)$ near (C, u).

(g) A smooth, projective morphism $X \to S$ is **convex** if for every tree of smooth, genus 0 curves C and for every S-morphism $u : C \to X$, the pullback $\mathcal{E} = u^* T_X$ is globally generated. In this case, prove that

$$\text{ev} : \overline{\mathcal{M}}_{0,1}(X/S, \beta) \to X$$

is everywhere smooth, and the open $\mathcal{M}_{0,0}(X/S, \beta)$ is everywhere dense in $\overline{\mathcal{M}}_{0,1}(X/S, \beta)$.

Nota bene. A smooth projective variety in characteristic 0 is convex if there exists a transitive algebraic action of an algebraic group on the variety, i.e., if the variety is a **projective homogeneous variety**. It is an open conjecture that every rationally connected convex variety is a projective homogeneous variety.

Problem 2(**Irreducibility of Spaces of Genus 0 Stable Maps to Projective Space**) Let k be an algebraically closed field of characteristic 0. Let $r \geq 1$ be an integer.

(a) Use the previous exercise to prove that for every integer $e \geq 1$, the following evaluation morphism is everywhere smooth,

$$\text{ev} : \overline{\mathcal{M}}_{0,1}(\mathbb{P}^r_k, e) \to \mathbb{P}^r_k,$$

with relative dimension equal to the "expected dimension" ("naive dimension", "virtual dimension"),

$$m_e := \langle c_1(T_{\mathbb{P}^r_k/k}), e[\text{line}]\rangle - 2 = (r+1)e - 2,$$

and $\mathcal{M}_{0,0}(\mathbb{P}^r_k, e)$ is dense in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r_k, e)$. Thus, also $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^r_k, e)$ is smooth of the expected dimension

$$m_e + r = \langle c_1(T_{\mathbb{P}^r_k/k}), e[\text{line}]\rangle + (\dim(\mathbb{P}^r) - 3) + 1 = (r+1)e + r - 2.$$

(b) For every integer $n \geq 1$, the morphism forgetting all marked points,

$$\Phi_{0,n\to 0,e} : \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r_k, e) \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r_k, e)$$

is flat of relative dimension n with connected geometric fibers. Apply this when $n = 1$ to conclude that also $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r_k, e)$ is smooth of the expected dimension $m_e + r - 1$. Next, apply this for arbitrary n to prove that every $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r_k, e)$ has pure dimension equal to the expected dimension $m_e + r - 1 + n$, and every irreducible component dominates some irreducible component of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r_k, e)$.

(c) Use the model of $\text{Hom}^e_k(\mathbb{P}^1_k, \mathbb{P}^r_k)$ as a dense Zariski open in the projective space $\mathbb{P}\text{Hom}_k(H^0(\mathbb{P}^r_k, \mathcal{O}(1)), H^0(\mathbb{P}^1_k, \mathcal{O}(e)))$ to prove that $\mathcal{M}_{0,0}(\mathbb{P}^r_k, e)$ is irreducible, and even unirational (in fact, Clemens proved that it is rational). Conclude that also $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r_k, e)$ is irreducible. Combined with the previous part, and using that $\Phi_{0,n\to 0,e}$ has irreducible fibers over $\mathcal{M}_{0,0}(\mathbb{P}^r, e)$, conclude that every $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r_k, e)$ is irreducible of the expected dimension

$$\langle c_1(T_{\mathbb{P}^r_k/k}), e[\text{line}]\rangle + \dim(\mathbb{P}^r_k) - 3 + n = (r+1)e + r - 3 + n.$$

Problem 3(**Transversality in Kontsevich's Recursion for Plane Curves of Geometric Genus** 0) Now let n equal $3e - 1$, and consider the evaluation morphism

$$\text{ev}_{0,n,e} : \overline{\mathcal{M}}_{0,n}(\mathbb{P}^2_k, e) \to (\mathbb{P}^2_k)^n.$$

Both the domain and target are irreducible of dimension $2n = 2(3e - 1)$.

(a) Conclude that there exists a dense open subscheme V_n of $(\mathbb{P}^2_k)^n$ over which every fiber of $\text{ev}_{0,n,e}$ is finite (possibly empty). Up to shrinking, arrange that the fibers are reduced. Since $\mathcal{M}_{0,n}(\mathbb{P}^2_k, e)$ is a dense open in $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2_k, e)$, up to shrinking further, arrange that the fiber over every point of V_e is contained in $\mathcal{M}_{0,n}(\mathbb{P}^2_k, e)$.

(b) The claim, to be proved by induction on e, is that $\mathrm{ev}_{0,3e-1,e}$ is generically finite to its image; equivalently, that $\mathrm{ev}_{0,3e-1,e}$ is dominant. When e equals 1 and n equals 2, prove this directly. In fact, prove that V_1 is simply the complement of the diagonal in $(\mathbb{P}^2_k)^2$ and every fiber is one reduced point (parameterizing a line).

(c) By way of induction, assume that $\mathrm{ev}_{0,3e-1,e}$ is dominant. Denote by $W \subset (\mathbb{P}^2_k)^3$ the Cartier divisor parameterizing ordered triples of collinear points. By considering genus 0 stable maps of degree $e+1$ that are a union of a line and a degree e stable map (attached at one intersection point of the line and the degree e plane curve), prove that for every $1 \le \lambda < \mu < \mu \le 3(e+1) - 2$, the image of

$$\mathrm{ev}_{0,3(e+1)-1,e+1} : \overline{\mathcal{M}}_{0,3(e+1)-1}(\mathbb{P}^2_k, e+1) \to (\mathbb{P}^2_k)^{3(e+1)-1}$$

contains the inverse image $W_{\lambda,\mu,\nu}$ of W under the projection,

$$\mathrm{pr}_{\lambda,\mu,\nu} : (\mathbb{P}^2_k)^{3(e+1)-1} \to (\mathbb{P}^2_k)^3.$$

(d) Since $\overline{\mathcal{M}}_{0,3(e+1)-1}(\mathbb{P}^2_k, e+1)$ is irreducible, the image of $\mathrm{ev}_{0,3(e+1)-1,e+1}$ is irreducible. Since the image contains the $\binom{3e+1}{3}$ distinct irreducible Cartier divisors $W_{\lambda,\mu,\nu}$, conclude that the image is all of $(\mathbb{P}^2_k)^{3(e+1)-1}$. Thus, conclude the claim by induction on e.

(e) Finally, conclude that there exists a dense open subscheme V_e of $(\mathbb{P}^2_k)^{3e-1}$ such that the fiber of $\mathrm{ev}_{0,3e-1,e}$ over every geometric point of V_e is reduced, is contained in $\mathcal{M}_{0,3e-1}(\mathbb{P}^2_k, e)$, and has length equal to N_e, the integer from Kontsevich's recursion formula.

Problem 4(**Transversality for Low Degree Genus 0 Curves on General Projective K3 Surfaces**) Let k be an algebraically closed field of characteristic 0. This exercise works through a few low degree cases of Xi Chen's Transversality Theorem for genus 0 curves on projective K3 surfaces.

(a) Let $\Pi \subset \mathbb{P}H^0(\mathbb{P}^2_k, \mathcal{O}(3))$ be a general pencil of plane curves. Denote by $C \subset \Pi \times_k \mathbb{P}^2_k$ the restriction over Π of the universal family of plane cubics. Prove that the projection $\nu : C \to \mathbb{P}^2_k$ is a projective, birational morphism that is a blowing up at 9 reduced, k-points. In particular, since the Euler characteristic of $\mathbb{P}^2_k$ equals 3, conclude that the Euler characteristic of C equals $3 + 9 = 12$. Use excision for Euler characteristics to conclude that there are 12 singular fibers of $C \to \Pi$, each of which is a nodal plane cubic with Euler characteristic 1 (smooth plane cubics have Euler characteristic 0).

(b) Compute that the relative dualizing sheaf $\omega_{C/\Pi}$ is isomorphic to the pullback of $\mathcal{O}_\Pi(1)$. For a degree 2 morphism $h : \mathbb{P}^1_k \to \Pi$ that is branched over none of the 12 discriminant points, conclude that the fiber product $S = \mathbb{P}^1_k \times_\Pi C$ is a smooth surface with a fibration over $\mathbb{P}^1_k$ have $2 \times 12 = 24$ singular fibers (thus with Euler characteristic equal to 24) and with relative dualizing sheaf isomorphic to the pullback of $\mathcal{O}_{\mathbb{P}^1_k}(2)$. Since the dualizing sheaf of $\mathbb{P}^1_k$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1_k}(-2)$, conclude that the dualizing sheaf of S is isomorphic to the structure sheaf of S. Thus, S is a K3 surface. Therefore, there exist elliptic K3 surfaces with precisely 24 singular fibers.

(c) Read about the Plücker identities among the number of flex lines, bitangents, nodes, and cusps of a plane curve and its dual plane curve. Use these to prove that a general plane sextic curve has precisely 28 bitangent lines. Conclude that for a double cover of $\mathbb{P}^2_k$ branched over a general plane sextic curve, there are precisely $2 \times 28 = 56$ genus 0 curves that map 1-to-1 to a line in $\mathbb{P}^2_k$ under the covering map.

(d) Prove that for every integer $d \geq 3$, for a sufficiently general hypersurface S_d of degree d in $\mathbb{P}^3_k$, there are finitely many tritangent 2-planes to S_d, each of which intersects S_d in a plane curve with 3 ordinary double points and no other singularities. For $d \geq 4$, count parameters to prove that a general degree d surface S_d contains no lines. Thus, for d equal to 4, conclude that a general quartic surface S_4 has finitely many hyperplane sections that are irreducible, nodal curves of geometric genus 0, and it contains no hyperplane sections that are reducible, that are nonreduced, or that have geometric genus 0 with worse singularities than 3 ordinary double points.

Problem 5(**Explicit Kollár-Ruan Theorem**.) Recall that for every uniruled compact Kähler manifold (X, J, ω), the Kollár-Ruan Theorem proves that there exists a homology class $\beta \in H_2(X, \mathbb{Z})$ of a free rational curve that has minimal pairing against $[\omega]$, and for every such class β, the primary Gromov-Witten invariant,

$$\langle \eta_X, [\omega] \smile [\omega], \dots, [\omega] \smile [\omega] \rangle^{(X,J,\omega)}_{0,m+1,\beta}$$

is strictly positive, where η_X is the Poincaré dual of the homology class of a point p (a generator of the top degree cohomology of X), and where $m = m_\beta$ equals the nonnegative integer $\langle c_1(T^{1,0}_{(X,J)}), \beta \rangle - 2$. In fact, if ω is the first Chern class of a very ample divisor, then for a general m-tuple of pairs $(H'_1, H''_1), \dots, (H'_m, H''_m)$, this primary Gromov-Witten invariant is a positive integer equal to the number of immersed, genus-0 stable maps with smooth domain of homology class β whose image contains a specified general point p and whose image intersects each of the m specified codimension 2 subvarieties $H_1 = H'_1 \cap H''_1, \dots, H_m = H'_m \cap H''_m$. This problem computes this explicitly in some cases.

(a) For X equal to projective space $\mathbb{P}(\mathbb{C}^{\oplus(n+1)})$ or a ("classical", "A_n-type", "$\mathbf{SL}_{n+1}$") Grassmannian, $\mathrm{Grass}(r, \mathbb{C}^{\oplus(n+1)})$, $1 \leq r \leq n$, prove that for every symplectic form, the homology class of the form is a positive real multiple of the first Chern class $[\omega]$ of the Plücker line bundle (the pullback of $\mathcal{O}(1)$ with respect to the Plücker embedding). Conclude that the the unique minimal free β is the homology class of a line (with respect to the Plücker embedding).

(b) For X a Grassmannian, for $[\omega]$ the Plücker class, for β equal to the class of a line, prove that for every $p \in X$, the fiber of the evaluation map is isomorphic to a product of projective spaces $\mathbb{P}(\mathbb{C}^{\oplus r}) \times \mathbb{P}(\mathbb{C}^{\oplus n+1-r}) \cong \mathbb{CP}^{r-1} \times \mathbb{CP}^{n-r}$. Also prove that the ample divisor class of lines that intersect $H = H' \cap H''$ is the sum of the pullbacks via the two projections of the first Chern class of the respective Serre twisting sheaves $\mathcal{O}(1)$ on $\mathbb{CP}^{r-1}$ and $\mathbb{CP}^{n-r}$. Deduce that the primary Gromov-Witten invariant above equals the binomial coefficient $\binom{n-1}{r-1}$.

(c) For X a product of two projective spaces, say $\mathbb{CP}^{n_1} \times \mathbb{CP}^{n_2}$, $n_1, n_2 \geq 1$, show that the cohomology classes of symplectic forms are all linear combinations with strictly positive coefficients of the two classes $A_1 = \mathrm{pr}_1^* c_1(\mathcal{O}(1))$ and $A_2 = \mathrm{pr}_2^* c_1(\mathcal{O}(1))$, i.e., $[\omega] = a_1 A_1 + a_2 A_2$ for $a_1, a_2 > 0$. If $a_1 > a_2$, resp. if $a_2 > a_1$, prove that the unique minimal free homology class β equals $\beta_2 = [\{p_1\} \times L_2]$, resp. $\beta_1 = [L_1 \times \{p_2\}]$

where $L_1 \subset \mathbb{CP}^{n_1}$, resp. $L_2 \subset \mathbb{CP}^{n_2}$ is a line. If a_1 equals a_2, prove that both β_1 and β_2 are minimal free classes.

(d) Prove that the fiber of the evaluation map for β_1, resp. for β_2, is a projective space $\mathbb{P}^{n_1-1}$, resp. $\mathbb{P}^{n_2-1}$. Prove that the cohomology class of $(a_1A_1 + a_2A_2) \smile (a_1A_1 + a_2A_2)$ on this fiber is $a_1^2 c_1(\mathcal{O}(1))$, resp. $a_2^2 c_1(\mathcal{O}(1))$. Thus, in cases that $a_2 \geq a_1$, resp. $a_2 \geq a_1$, prove that the primary Gromov-Witten invariant equals $a_1^{2(n_1-1)}$, resp. $a_2^{2(n_2-1)}$.

(e) Let d and n be positive integers with $1 \leq d \leq n-1$ and $n \geq 4$. For every smooth, degree d hypersurface $X \subset \mathbb{CP}^n$, prove that every homology class of a symplectic form is a positive real multiple of $[\omega] = c_1(\mathcal{O}(1))|_X$. Prove that the unique minimal free class β is the homology class of a line L contained in X.

(f) Continuing the previous case, for a general point p in X, prove that the fiber of the evaluation map for X, considered as a subvariety of the fiber $\mathbb{CP}^{n-1}$ of the fiber of the evaluation map for $\mathbb{CP}^n$, is a complete intersection of $d-1$ hypersurfaces of respective degrees $2, 3, \ldots, d-1, d$. Moreover, show that $[\omega] \smile [\omega]$ gives the restriction to the complete intersection of $c_1(\mathcal{O}(1))$ from $\mathbb{CP}^{n-1}$. Conclude that the primary Gromov-Witten invariant above equals $d!$, the factorial of d.

(g) For d equal to n, $n \geq 4$, every smooth hypersurface X of degree d in $\mathbb{CP}^n$ is uniruled, but the minimal homology free homology class is $\beta = 2[L]$ (lines are not free, there are free conics). What is the primary Gromov-Witten invariant?

Problem 6(**Non-Transversality for the Rational Connectedness Problem**) Recall from lecture that a conjecture of Kollár, proved in complex dimension ≤ 3 by Zhiyu Tian, predicts symplectic invariance of rational connectedness. This exercise explains one reason that the proof of the Kollár-Ruan theorem does not extend to this setting.

(a) Let $a \leq b$ be integers. For every integer c, prove that there exists a surjective $\mathcal{O}_{\mathbb{P}^1}$-module homomorphism from $\mathcal{O}_{\mathbb{P}^1}(b) \oplus \mathcal{O}_{\mathbb{P}^1}(a)$ to $\mathcal{O}_{\mathbb{P}^1}(c)$ if and only if $c \geq b$. In this case, compute that the kernel of the surjection is $\mathcal{O}_{\mathbb{P}^1}(d)$, where $a+b$ equals $c+d$.

(b) For every integer $e \geq 0$, by definition the **Hirzebruch surface** Σ_e is the $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ associated to the locally free sheaf $\mathcal{O}_{\mathbb{P}^1}$-module $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)$. Prove that this is (non-canonically) isomorphic to the $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ associated to $\mathcal{O}_{\mathbb{P}^1}(b) \oplus \mathcal{O}_{\mathbb{P}^1}(a)$ for every pair of integers $a \leq b$ such that $b-a$ equals e.

(c) Conclude from the first part that for every integer $e \geq 0$, the Hirzebruch surface Σ_{e+2} deforms to Σ_e. Thus, there are precisely two deformation classes of Hirzebruch surfaces depending only on the parity of e.

(d) For every integer $e > 0$, prove that there exists a unique section $\sigma_0 : \mathbb{P}^1 \to \Sigma_e$ of the projection $\pi : \Sigma_e \to \mathbb{P}^1$ such that the normal bundle of the section pulls back to $\mathcal{O}_{\mathbb{P}^1}(-e)$. This section is called the **directrix** of the Hirzebruch surface (it corresponds to a **maximal destabilizing subbundle**). Denote by $D_0 \in H^2(\Sigma_e, \mathbb{Z})$ the Poincaré dual cohomology class of $[\sigma_0]$, and denote by $F \in H^2(\Sigma_e, \mathbb{Z})$ the class of a fiber.

(e) For $e \geq 1$, for real numbers r, prove that $D_0 + rF \in H^2(\Sigma_e, \mathbb{R})$ is the $(1,1)$ class of a Kähler form if and only if $r > e$. In this case, deformations of this class are also Kähler on Σ_{e-2}, Σ_{e-4}, etc. Show that the homology class β of a fiber of π

is the unique minimal free class, the fiber of the evaluation map is one point, and the primary Gromov-Witten invariant equals 1.

(f) For $e \geq 1$, prove that the minimal chain of rational curves connecting 2 general points of Σ_e is a union of $\sigma_0(\mathbb{P}^1)$ and two fibers. Compute that the pairing of the homology class β of this chain agains $D_0 + rF$ equals $2 + (r - e)$.

(g) Finally, consider the case that $e \geq 4$ is even, resp. odd. Prove that the homology class β from the previous part is not represented by a genus 0 stable map on the deformation Σ_0, resp. on Σ_1. Conclude that every genus 0 Gromov-Witten invariant on Σ_e defined with respect to β equals 0. Thus, the naive extension of the Kollár-Ruan proof to the setting of genus 0 stable maps connecting 2 general points does not work for Σ_e.

Problem 7(**Kollár-Ruan Theorem in Mixed Characteristic**) This problem is for students who like to think about algebraic geometry in positive characteristic or mixed characteristic. Let R be a finitely generated $\mathbb{Z}$-algebra that is an integrally closed, integral domain and such that $\mathbb{Z} \to R$ is injective, i.e., $R \otimes_{\mathbb{Z}} \mathbb{Q}$ is nonzero. Let X_R be a smooth, projective R-scheme whose geometric fibers are connected. Let L_R be an ample invertible sheaf on X_R (this will be the stand-in for the symplectic form).

(a) Let $\eta : R \to \mathbb{C}$ be a ring homomorphism (by the Lefschetz principle, such a ring homomorphism exists), and denote by (X_η, L_η) the base change complex projective manifold together with an ample invertible sheaf. Assume that (X_η, L_η) is uniruled. Let β be a minimal free homology class on X_η. Denote by e the intersection number $\langle c_1(L_\eta), \beta \rangle$. Thus, the (polarized) Hom scheme $\mathrm{Hom}^e_{\mathbb{C}}(\mathbb{CP}^1, (X_\eta, L_\eta))$ is nonempty, and, in fact, the evaluation morphism

$$\mathrm{ev} : \mathrm{Hom}^e_{\mathbb{C}}(\mathbb{CP}^1, (X_\eta, L_\eta)) \times \mathbb{CP}^1 \to X_\eta$$

has a nonempty Zariski open subset U_η of the domain on which the morphism is smooth.

(b) Consider the relative Hom scheme over Spec R,

$$H^e_R = \mathrm{Hom}^e_R(\mathbb{P}^1_R, (X_R, L_R)),$$

together with its evaluation morphism,

$$\mathrm{ev} : H^e_R \times_{\mathrm{Spec}\ R} \mathbb{P}^1_R \to X_R.$$

Prove that there exists a maximal open subscheme U of $H^e_R \times_{\mathrm{Spec}\ R} \mathbb{P}^1_R$ (which *a priori* might be empty) on which ev is a smooth morphism. Prove that the formation of U is compatible with flat base change of R. Since the base change of U by η is nonempty, conclude that for the unique minimal open and closed subscheme U_i of U (i.e., union of connected components) whose base change by η intersects U_η in a dense open of U_η, the generic fiber of $U_i \to \mathrm{Spec}\ R$ is nonempty. Up to replacing R by the image of a finite, injective ring homomorphism $R \to R'$ and then replacing (X_R, L_R) by the base change $(X_{R'}, L_{R'})$, we may even assume that every connected component of U_i has connected base change $U_{i,\eta}$ that intersects U_η in a nonempty open subset. Thus, performing this base change, the data of U_i is a stand-in for the homology class β.

(c) Since U is smooth over X_R, and since X_R is smooth over Spec R, also U is smooth over Spec R. Since smooth morphisms are universally open, conclude that the image of U_i in Spec R is a dense Zariski open subset of Spec R. By considering the dense Zariski open subsets Spec $R[1/n]$ of Spec R, observe that this general argument gives no information about which "bad characteristics" we need to exclude to guarantee the existence of free, genus 0 curves in the base change of X_R.

(d) Read about *a priori* bounds on degrees of singular loci of projective varieties of bounded degrees (Mumford's article on varieties defined by quadratic equations is a classic). Apply these bounds to the fibers of the R-morphism,

$$\text{ev} : \overline{\mathcal{M}}_{0,1}(X_R/\text{Spec } R, U_i) \to X_R,$$

where "U_i" means that we allow all curve classes obtained from maps parameterized by U_i. Conclude that, in fact, there exists an explicit positive integer n_0 depending only on the dimension and the degree of the generic fiber of the evaluation morphism,

$$\text{ev} : \overline{\mathcal{M}}_{0,1}(X_\eta, \beta) \to X_\eta,$$

with respect to the ample invertible sheaf coming from $c_1(L_\eta) \smile c_1(L_\eta)$ such that after inverting n_0 in R, for every ring homomorphism to an algebraically closed field,

$$s : R[1/n_0] \to k,$$

the evaluation morphism of the base change

$$\text{ev} : \overline{\mathcal{M}}_{0,1}(X_s, U_{i,s}) \to X_s$$

is smooth over a dense Zariski open subset of X_s. Conclude positivity of the Kollár-Ruan primary Gromov-Witten invariant for X_s coming from each $\beta' \in U_{i,s}$ (defined now via algebraic geometry, so that the definition extends to positive characteristic).

(d) For every finitely generated $\mathbb{Z}$-algebra $\widetilde{R}$ that is an integrally closed, integral domain with $\widetilde{R} \otimes_{\mathbb{Z}} \mathbb{Q}$ nonempty, for every proper smooth $\widetilde{R}$-scheme $\widetilde{X}_{\widetilde{R}} \to \text{Spec } \widetilde{R}$ and ample line bundle $\widetilde{L}_{\widetilde{R}}$, for every ring homomorphism,

$$\widetilde{s} : \widetilde{R} \to k,$$

such that the base change $(\widetilde{X}_{\widetilde{s}}, \widetilde{L}_{\widetilde{s}})$ is isomorphic to (X_s, L_s) as polarized k-schemes, the Gromov-Witten invariant for $(\widetilde{X}_{\widetilde{s}}, \widetilde{L}_{\widetilde{s}})$ equals the Gromov-Witten invariant for (X_s, L_s). Since algebro-geometric Gromov-Witten invariants are invariant under "generization" (sometimes called, "deformations-in-the-small"), conclude that also for every ring homomorphism

$$\widetilde{\eta} : \widetilde{R} \to \mathbb{C},$$

the base change complex projective manifold $\widetilde{X}_{\widetilde{\eta}}$ is uniruled. Thus, the Kollár-Ruan Theorem not only proves symplectic deformation invariance of uniruledness, but also invariance under mixed characteristic deformations that avoid positive characteristics p that are less than an explicit integer n_0 depending only on Chern numbers of the polarized complex manifold (X_η, L_η) and Gromov-Witten invariants.

(e) To convince yourself that deformation invariance through mixed characteristic is very different from symplectic deformation invariance, read about Serre's examples of complex projective manifolds that are deformation invariant through mixed characteristic (in fact "Galois twists" for the fraction field of a ring of integers R), yet have non-isomorphic fundamental groups (thus, are not even homotopic, much less symplectically deformation equivalent).