MAT 615 PROBLEM SET 1

Homework Policy. This problem set explores ideas from [Sch68] and [LS67]. Together with Artin's Approximation Theorem, Schlessinger's Theorem is one of the key ingredients in Artin's Axioms for Algebraicity of a functor or stack (one of the key techniques for constructing parameter spaces and moduli spaces). Moreover, infinitesimal deformation theory gives properties of these moduli spaces: dimension bounds, smoothness, bounds on the dimension of Zariski tangent spaces, flatness, etc.

Problems.

Problem 1.(**The category of Artin algebras has finite products and coproducts.**) Let Λ be a local Noetherian ring. Denote the maximal ideal by μ, and denote the residue field Λ/μ by k. For simplicity, assume that Λ is complete with respect to μ. A Λ-k-**Artin algebra** is a local homomomorphism,

$$\alpha : (\Lambda, \mu) \to (A, \mathfrak{m}),$$

such that A is a finite length Λ-module and the induced field homomorphism,

$$\Lambda/\mu \to A/\mathfrak{m},$$

is an isomorphism. For Λ-k-Artin algebras (A, α) and (A', α'), a **morphism** of Λ-k-Artin algebras from (A, α) to (A', α') is an Λ-algebra homomorphism $\psi : A \to A'$.

(a) Check that every Λ-k Artin algebra has a unique morphism of Λ-k-algebras to k. Check that every morphism of Λ-k-Artin algebras is a local homomorphism. Check that the identity map on a Λ-k-Artin algebra is a morphism of Λ-k-Artin algebras. Also check that a composition of two morphisms of Λ-k-Artin algebras is a morphism of Λ-k-Artin algebra.

Thus, these notions define a category of Λ-k-Artin algebras that has a final object k. Denote this category by $C_{\Lambda,k}$. A Λ-k-**complete algebra** is a local homorphism,

$$\rho : (\Lambda, \mu) \to (R, \mathfrak{n}),$$

such that for every positive integer e, the composite local homomorphism,

$$\rho_e : (\Lambda, \mu) \to (R, \mathfrak{n}) \to (R/\mathfrak{n}^{e+1}, \mathfrak{n}/\mathfrak{n}^{e+1}),$$

is a Λ-k-Artin algebra and such that the induced local homomorphism,

$$(R, \mathfrak{n}) \to \operatorname{proj\,lim}_e (R/\mathfrak{n}^{e+1}, \mathfrak{n}/\mathfrak{n}^{e+1}),$$

is an isomorphism. A **morphism** of Λ-k-complete algebras from (R, ρ) to (R', ρ') is a Λ-algebra homomorphism $\psi : R \to R'$.

(b)(**Full embedding of $C_{\Lambda,k}$ in $\widehat{C}_{\Lambda,k}$.**) Check that these notions define a category $\widehat{C}_{\Lambda,k}$ of Λ-k-complete algebras. Check that every Λ-k-Artin algebra is a Λ-k-complete algebra. Check that the induced functor $C_{\Lambda,k} \to \widehat{C}_{\Lambda,k}$ is fully faithful.

For every covariant functor,
$$F : C_{\Lambda,k} \to \mathbf{Sets},$$
define the associated **profunctor**,
$$\widehat{F} : \widehat{C}_{\Lambda,k} \to \mathbf{Sets}, \quad \widehat{F}(R,\rho) := \operatorname{proj}\lim_e F(R/\mathfrak{m}_R^{e+1}),$$
and for every natural transformation $\theta : F \Rightarrow G$ of set-valued functons on $C_{\Lambda,k}$, define
$$\widehat{\theta} : \widehat{F} \Rightarrow \widehat{G}, \quad \widehat{\theta}_{(R,\rho)} := \operatorname{proj}\lim_e \theta_{R/\mathfrak{m}_R^{e+1}}.$$
For every object (R,ρ) of $\widehat{C}_{\Lambda,k}$, denote by $h^{(R,\rho)}$ the **Yoneda covariant functor**,
$$h^{(R,\rho)} : \widehat{C}_{\Lambda,k} \to \mathbf{Sets}, \quad h^{(R,\rho)}(S,\sigma) = \operatorname{Hom}_{\widehat{C}_{\Lambda,k}}((R,\rho),(S,\sigma)),$$
and for every morphism $\psi : (R,\rho) \to (R',\rho')$ of $\widehat{C}_{\Lambda,k}$, denote by h^{ψ} the natural transformation,
$$h^{\psi} : h^{(R',\rho')} \Rightarrow h^{(R,\rho)}, \quad h^{\psi}(u : (R',\rho') \to (S,\sigma)) = u \circ \psi.$$
For every set-valued functor E on $\widehat{C}_{\Lambda,k}$, denote by E_C the restriction of E to $C_{\Lambda,k}$. The functor is **continuous** if the natural transformation,
$$E \Rightarrow \widehat{E}_C, \quad E(R,\rho) \to \operatorname{proj}\lim_e E(R/\mathfrak{m}_R^{e+1}),$$
is a natural isomorphism. A set-valued functor F on $C_{\Lambda,k}$ is **prorepresentable** if there exists an object (R,ρ) of $\widehat{C}_{\Lambda,k}$ such that E is naturally isomorphic to $h_C^{(R,\rho)}$.

(c)(**The pro-Yoneda lemma**.) Check that $h^{(R,\rho)}$ is continuous. Check the following pro-Yoneda lemma: for every object (R,ρ) of $\widehat{C}_{\Lambda,k}$, for every set-valued functor E on $C_{\Lambda,k}$, every natural transformation from $h_C^{(R,\rho)}$ to E arises from a unique element of $\widehat{E}(R,\rho)$. Check that k is a final object in $C_{\Lambda,k}$. Check that $(\Lambda, \mathrm{Id}_\Lambda)$ is an initial object in $\widehat{C}_{\Lambda,k}$.

For objects of $C_{\Lambda,k}$,
$$(A, \alpha : R \to A), \quad (A', \alpha' : R \to A'), \quad (A'', \alpha : R \to A''),$$
for morphisms of $C_{\Lambda,k}$,
$$\psi' : (A',\alpha') \to (A,\alpha), \quad \psi'' : (A'',\alpha'') \to (A,\alpha),$$
form the induced fiber product in the category of underlying sets,
$$p' : A' \times_A A'' \to A', \quad p'' : A' \times_A A'' \to A'', \quad \psi' \circ p' = \psi'' \circ p''.$$
Denote by
$$(\alpha', \alpha'') : \Lambda \to A' \times_A A''$$
the unique set map such that $p' \circ (\alpha',\alpha'')$ equals α' and $p'' \circ (\alpha',\alpha'')$ equals α''.

(d)(**Existence of fiber products**.) Check that there exists a unique structure of Λ-k-Artin algebra on $(A' \times_A A'', (\alpha',\alpha''))$ such that both p' and p'' are morphisms in $C_{\Lambda,k}$. Conclude that the category $C_{\Lambda,k}$ has finite fiber products. By taking inverse limits, show that also $\widehat{C}_{\Lambda,k}$ has finite fiber products, and the embedding of $C_{\Lambda,k}$ in $\widehat{C}_{\Lambda,k}$ preserves fiber products.

For morphisms of $C_{\Lambda,k}$,
$$\phi' : (A,\alpha) \to (A',\alpha'), \quad \phi'' : (A,\alpha) \to (A'',\alpha''),$$

denote by

$$q' : A' \to A' \otimes_A A'', \quad q'' : A'' \to A' \otimes_A A'',$$

the usual tensor product of A-modules.

(e)(**Existence of cofiber coproducts**.) Check that there exists a unique structure of Λ-k-Artin algebra on $A' \otimes_A A''$ such that both q' and q'' are morphisms in $C_{\Lambda,k}$. Conclude that the category $C_{\Lambda,k}$ has (finite) cofiber coproducts. By taking inverse limits of cofiber coproducts in $C_{\Lambda,k}$, show that also $\widehat{C}_{\Lambda,k}$ has cofiber coproducts, and the embedding of $C_{\Lambda,k}$ in $\widehat{C}_{\Lambda,k}$ preserves cofiber coproducts.

Please note, for morphisms of $\widehat{C}_{\Lambda,k}$,

$$\phi' : (R, \rho) \to (R', \rho'), \quad \phi'' : (R, \rho) \to (R'', \rho''),$$

for the cofiber coproduct $R' \widehat{\otimes}_R R''$ in $\widehat{C}_{\Lambda,k}$, the natural homomorphism,

$$R' \otimes_R R'' \to R' \widehat{\otimes}_R R'',$$

is not necessarily an isomorphism, since $R' \otimes_R R''$ might not be complete. This homomorphism is the completion with respect to the maximal ideal $\mathfrak{m}' \otimes_R R'' + R' \otimes_R \mathfrak{m}''$.

Problem 2.(**Adjoints and Vector Space Objects.**) For the completion $\Lambda\llbracket t \rrbracket$ of the polynomial ring $\Lambda[t]$ with repsect to the maximal ideal $\mu\Lambda[t] + t\Lambda[t]$, prove that the Yoneda functor $\widetilde{T}_0 = h^{\Lambda\llbracket t \rrbracket}$ associates to every object (R, ρ) of $\widehat{C}_{\Lambda,k}$ the maximal ideal or R. This is a functor from $\widehat{C}_{\Lambda,k}$ to the Abelian category $\Lambda - \mathbf{mod}$ of Λ-modules. Also, denote by T_0 the covariant functor,

$$T_0 : \widehat{C}_{\Lambda,k} \to k - \mathbf{mod}, \quad T_0(R, \rho) = \mathfrak{m}_R / \mathfrak{m}_R^2.$$

Denote by θ the natural transformation

$$\theta : \widetilde{T}_0 \Rightarrow T_0, \quad \mathfrak{m}_R \mapsto \mathfrak{m}_R / \mathfrak{m}_R^2.$$

Since T_0 is a functor, for every morphism in $\widehat{C}_{\Lambda,k}$,

$$\chi : (R, \rho) \to (R', \rho'),$$

there is an induced morphism of k-vector spaces,

$$T_0(R, \rho) \to T_0(R', \rho').$$

Denote the cokernel k-vector space by $T_0(\chi)$.

(a)(**Functoriality of T_0**.) Prove that T_0 is covariant in (R', ρ') with (R, ρ) held fixed, and prove that T_0 is contravariant in (R, ρ) with (R', ρ') held fixed. For every pair of morphisms in $\widehat{C}_{\Lambda,k}$,

$$\chi : (R, \rho) \to (R', \rho'), \quad \chi' : (R', \rho') \to (R'', \rho''),$$

check that the following is a right exact sequence of k-vector spaces,

$$T_0(\chi) \to T_0(\chi' \circ \chi) \to T_0(\chi') \to 0.$$

(b)(**The functor of maximal ideals**.) Prove that the restriction of $\widetilde{T}_0$ to $C_{\Lambda,k}$ is a functor to the Abelian category $\Lambda - \mathbf{mod}_0$ of finite length Λ-modules. Also, check that a morphism in $C_{\Lambda,k}$,

$$\phi : (A, \alpha) \to (A', \alpha'),$$

is surjective, resp. injective, bijective, if and only if the induced set map

$$h^{\Lambda[\![t]\!]}(\phi) : h^{\Lambda[\![t]\!]}(A, \alpha) \to h^{\Lambda[\![t]\!]}(A', \alpha'),$$

is surjective, resp. injective, bijective.

(c)(The left adjoint of $\widetilde{T}_0$.) Denote by $\Lambda-\mathbf{mod}_{\text{fg}}$ the full subcategory of $\Lambda-\mathbf{mod}$ of finitely generated Λ-modules (with the natural complete μ-adic topology). Prove that for every finitely generated Λ-module M, there is an object $\Lambda[\![M]\!]$ in $\widehat{C}_{\Lambda,k}$ such that the functor $h^{\Lambda[\![M]\!]}(R, \rho)$ is naturally isomorphic to

$$\text{Hom}_{\Lambda-\mathbf{mod}_{\text{fg}}}(M, \mathfrak{m}_R).$$

Prove that this is functorial in M, and the induced functor

$$\Lambda[\![-]\!] : \Lambda - \mathbf{mod}_{\text{fg}} \to \widehat{C}_{\Lambda,k},$$

is "essentially" a left adjoint of $\widetilde{T}_0$ in the sense that there is a natural equivalence for every (A, α) an object of $C_{\Lambda,k}$,

$$\text{Hom}_{\Lambda-\mathbf{mod}_{\text{fg}}}(M, \widetilde{T}_{0,C}(A, \alpha)) \cong \text{Hom}_{\widehat{C}_{\Lambda,k}}(\Lambda[\![M]\!], (A, \alpha)).$$

In particular, deduce isomorphisms,

$$\Lambda[\![M \oplus N]\!] \cong \Lambda[\![M]\!] \widehat{\otimes}_\Lambda \Lambda[\![N]\!],$$

and use the addition map on M to deduce a morphism in $\widehat{C}_{\Lambda,k}$,

$$\Sigma_M^* : \Lambda[\![M]\!] \to \Lambda[\![M \oplus M]\!] = \Lambda[\![M]\!] \widehat{\otimes}_\Lambda \Lambda[\![M]\!].$$

inducing the addition map on the Yoneda functor $Hom_{\Lambda-\mathbf{mod}}(M, h^{\Lambda[\![t]\!]}(A))$. Since the Yoneda functor is endowed with a structure of functor to $\Lambda - \mathbf{mod}$, the object $\Lambda[\![M]\!]$ is a Λ-module object in the opposite category $\widehat{C}_{\Lambda,k}^{\text{opp}}$.

On the other hand, for every k-vector space V, denote by $k \oplus V$ the quotient of $\Lambda[\![V]\!]$ by the sum of $\mu \cdot \Lambda[\![V]\!]$ and the square of the maximal ideal. Thus, $k \oplus V$ is a k-algebra whose maximal ideal is the k-vector space V, and the square of this maximal ideal is zero.

(d)(The right adjoint of T_0.) For the contravariant functor,

$$h_{k\oplus V} : \widehat{C}_{\Lambda,k}^{\text{opp}} \to \mathbf{Sets}, \quad h_{k\oplus V}(R, \rho) = \text{Hom}_{\widehat{C}_{\Lambda,k}}((R, \rho), k \oplus V),$$

check that there is a natural equivalence

$$\text{Hom}_{\widehat{C}_{\Lambda,k}}((R, \rho), k \oplus V) \cong \text{Hom}_{k-\mathbf{mod}}(T_0(R, \rho), V) = \text{Der}_\Lambda((R, \rho), V).$$

In particular, deduce isomorphisms,

$$k \oplus (V \oplus W) \cong (k \oplus V) \times_k (k \oplus W),$$

and use the addition map on V to deduce a morphism in $C_{\Lambda,k}$,

$$\Sigma_V : (k \oplus V) \times_k (k \oplus V) \to k \oplus V,$$

inducing the addition map on the Yoneda functor $\text{Hom}_{k-\mathbf{mod}}(T_0(-), V)$. This addition map together with the natural endomorphisms of $k \oplus V$ obtained by scaling V by elements of k make $h_{k\oplus V}$ into a functor with values in k-vector spaces, i.e., $k \oplus V$ is a k-vector space object in $C_{\Lambda,k}$.

Problem 3.(**Formal smoothness**.) For covariant functors,

$$F, G : C_{\Lambda,k} \to \mathbf{Sets},$$

a natural transformation $\eta : F \Rightarrow G$ is **formally smooth** if for every surjective morphism ϕ' in $C_{\Lambda,k}$,

$$\phi' : (A', \alpha') \to (A, \alpha),$$

also the set map

$$(F(\phi'), \eta_{(A',\alpha')}) : F(A', \alpha') \to F(A, \alpha) \times_{G(A,\alpha)} G(A', \alpha'),$$

is surjective. Since $\mathfrak{m}_{A'} \to \mathfrak{m}_A$ and $\mathfrak{m}_{A'}^2 \to \mathfrak{m}_A^2$ are both surjective, the natural transformation $\theta : \widetilde{T}_0 \Rightarrow T_0$ from **Problem 1(b)** is smooth, i.e., the object $\Lambda [\![t]\!]$ of $\widehat{C}_{\Lambda,k}$ together with the element t in its maximal ideal is a hull for T_0 (sometimes also called a "miniversal formal deformation").

(a)(**Formally smooth objects and projective objects**.) For every M in $\Lambda - \mathbf{mod}_{\text{fg}}$, for the unique morphism in $\widehat{C}_{\Lambda,k}$,

$$\psi : \Lambda \to \Lambda [\![M]\!],$$

prove that the induced natural transformation of Yoneda functors,

$$h^\psi : h^{\Lambda [\![M]\!]} \Rightarrow h^\Lambda,$$

is formally smooth if and only if M is a projective Λ-module, i.e., if and only if M is isomorphic to $\Lambda^{\oplus r}$ for some nonnegative integer r. (**Hint.** Consider the special case of a surjective homomorphism $s : N' \to N$ in $\Lambda - \mathbf{mod}_{\text{fg}}$, the associated morphism $\Lambda [\![s]\!] : \Lambda [\![N']\!] \to \Lambda [\![N]\!]$, and let ϕ'_e be the associated morphism in $C_{\Lambda,k}$ obtained by forming the quotient for the domain and target by the $(e+1)^{\text{st}}$ power of each respective maximal ideal.) Deduce that for every object (R, ρ) of $\widehat{C}_{\Lambda,k}$ and every finitely generated, projective Λ-module M, for the induced morphism in $\widehat{C}_{\Lambda,k}$,

$$q : (R, \rho) \to (R, \rho) \widehat{\otimes}_\Lambda \Lambda [\![M]\!],$$

the natural transformation of Yoneda functors is formally smooth.

For every morphism in $\widehat{C}_{\Lambda,k}$,

$$\chi : (R, \rho) \to (R', \rho'),$$

with induced map of k-vector spaces

$$T_0(R, \rho) \to T_0(R', \rho'),$$

let M be a free Λ-module of rank r equal to the k-vector space dimension of the cokernel $T_0(\chi)$, and let

$$s : M \to \widetilde{T}_0(R', \rho')$$

be a Λ-module homomorphism whose composition to $T_0(\chi)$ is a surjection. Let

$$\psi : (R, \rho) \widehat{\otimes}_\Lambda \Lambda [\![M]\!] \to (R', \rho')$$

be the induced morphism in $\widehat{C}_{\Lambda,k}$, i.e., ψ is a surjection from a power series algebra over R to the R-algebra R'. Let I denote the kernel of I, and let $T_1(\chi)$ denote the k-vector space $I/\mathfrak{m} \cdot I$, where $\mathfrak{m}$ denotes the maximal ideal of $(R, \rho) \widehat{\otimes}_\Lambda \Lambda [\![M]\!]$.

(b)(**The functor T_1 and formal smoothness**.) Prove that $T_1(\chi)$ is independent of the choice of (M, s, ψ). Prove that $T_1(\chi)$ is functorial in (R', ρ') with (R, ρ) held fixed, and in (R, ρ) with (R', ρ') held fixed. Use Nakayama's Lemma to prove that

I is zero if and only if $T_1(\chi)$ is zero. Finally, prove that there exists a splitting of the surjection ψ in the category of (R,ρ)-algebras if and only if I is zero. Conclude that χ is formally smooth if and only if $T_1(\chi)$ is zero. Thus, every formally smooth (R,ρ)-algebra arises as in part (a).

Problem 4.(**Schlessinger's Theorem**.) A covariant functor F from $C_{\Lambda,k}$ to **Sets** is **pointed** if $F(k)$ is a singleton set, say $\{0\}$. If also the natural set map

$$F((k\oplus V)\times_k (k\oplus W)) \to F(k\oplus V)\times_{F(k)} F(k\oplus W),$$

is a bijection for every pair (V,W) of finite dimensional k-vector spaces, then the maps Σ_V and the natural k-algebra endomorphisms of $k\oplus V$ make each set $F(k\oplus V)$ into a k-vector space in such a way that the composite functor

$$F(k\oplus -) : k-\mathbf{mod}_0 \to k-\mathbf{mod}, \quad V\mapsto F(k\oplus V),$$

is a k-linear, exact functor. For the free, 1-dimensional k-vector space k, denote by $T^0(F)$ the image of this functor on k.

A morphism in $C_{\Lambda,k}$ that is a surjection on underlying sets,

$$\phi' : (A',\alpha'') \to (A,\alpha),$$

is an **infinitesimal extension**, resp. a **small extension**, if the kernel of ϕ is annihilated by the maximal ideal $\mathfrak{m}_{A'}$, resp. if it is an infinitesimal extension and the kernel is isomorphic to k as a k-vector space. In particular, every $k\oplus V$ is an infinitesimal extension of k that is a finite iterated sequence of small extensions.

For every object (R,ρ) of $\widehat{C}_{\Lambda,k}$, check that the Yoneda functor $F = h^{R,\rho}$ is a pointed functor that satisfies all of the following conditions relative to every pair of morphisms in $C_{\Lambda,k}$,

$$\phi' : (A',\alpha') \to (A,\alpha), \quad \phi'' : (A'',\alpha'') \to (A,\alpha),$$

and the induced map

$$F(\phi',\phi'') : F(A'\times_A A'') \to F(A')\times_{F(A)} F(A'').$$

- (H1) The set map $F(\phi',\phi'')$ is surjective whenever ϕ' is a small extension.
- (H2) The set map $F(\phi',\phi'')$ is a bijection whenever A equals k and A'' equals $k\oplus V$. (Because of (H1), it suffices to check when V is 1-dimensional.)
- (H3) The natural k-vector space structure on each $F(k\oplus V)$ is finite dimensional.
- (H4) The set map $F(\phi',\phi')$ is a bijection for every small extension ϕ'.

Also check that $T^0(h^{(R,\rho)})$ is the dual k-vector space of $T_0(R,\rho)$.

Theorem 0.1. [Sch68, Theorem 2.11] *Every pointed functor*

$$F : C_{\Lambda,k} \to \textit{\textbf{Sets}}$$

that satisfies (H1) – (H3), resp. that satisfies (H1) – (H4), has a hull, resp. is naturally isomorphic to $h^{(R,\rho)}$ for some (R,ρ) in $\widehat{C}_{\Lambda,k}$.

Let X_Λ be a scheme that is projective and flat over over Spec Λ. Let Z_0 be a closed subscheme of the fiber $X_0 = X\times_{\text{Spec }\Lambda}\text{Spec }k$. Denote by $\text{Hilb}_{X_\Lambda/\Lambda,Z_0}$ the (covariant) functor on $C_{\Lambda,k}$ that associates to every (A,α) the set of closed subschemes Z_A of $X\times_{\text{Spec }\Lambda}\text{Spec }A$ that are A-flat and whose base change $Z_A\times_{\text{Spec }A}\text{Spec }k$ equals Z_0. Try to directly verify the hypotheses of Schlessinger's theorem for pro-representability of F.

Problem 4.(**The Functor T^1 and Obstructions**.) The notation here is as in **Problem 3(b)**. For every finite dimensional k-vector space N, denote by $T^1(\chi, N)$ the k-vector space,
$$T^1(\chi, N) = \text{Hom}_{k-\mathbf{mod}}(T_1(\chi), N).$$
This is evidently covariant in N. For every commutative diagram in $\widehat{C}_{\Lambda,k}$,
$$\begin{array}{ccc} (R,\rho) & \xrightarrow{\chi} & (R',\rho') \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\beta'} \\ (A,\alpha) & \xrightarrow{\phi} & (A',\alpha'), \end{array}$$
where ϕ is an infinitesimal extension in $C_{\Lambda,k}$ with kernel $\text{Ker}(\phi) = N$, by formal smoothness of
$$(R,\rho) \to (R,\rho)\widehat{\otimes}_\Lambda \Lambda[\![M]\!],$$
there exists an R-algebra homomorphism,
$$b : (R,\rho)\widehat{\otimes}_\Lambda \Lambda[\![M]\!] \to (A,\alpha),$$
such that $\phi \circ b$ equals $\beta' \circ \psi$. The restriction of b to I is a $R\widehat{\otimes}_\Lambda \Lambda[\![M]\!]$-module homomorphism,
$$I \to N.$$
Since ϕ is an infinitesimal extension, this factors uniquely through the surjection,
$$I \to I/\mathfrak{m}\cdot I.$$
Denote by
$$o_{\chi,\phi,\beta,\beta'} \in T^1(\chi, N)$$
the induced k-linear transformation,
$$T_1(\chi) \to N.$$

(a)(**Functoriality**.) Check that $o_{\chi,\phi,\beta,\beta'}$ is independent of the choice of lift b. Also check that this is functorial for diagrams $(\chi, \phi, \beta, \beta')$ with χ held fixed.

(b)(**Liftings and obstructions**.) Check that there exists a lift of β' to a morphism $(R',\rho') \to (A,\alpha)$ making all diagrams commute if and only if the obstruction element $o_{\chi,\phi,\beta,\beta'}$ is zero. Finally, by considering the diagrams
$$\begin{array}{ccc} (R,\rho) & \xrightarrow{\chi} & (R',\rho') \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\beta'} \\ R[\![M]\!]/(\mathfrak{m}\cdot I + \mathfrak{m}^{e+1}) & \longrightarrow & R[\![M]\!]/(I + \mathfrak{m}^{e+1}), \end{array}$$
as the nonnegative integer e grows, conclude that there exists a diagram such that the obstruction
$$o_{\chi,\phi,\beta,\beta'} : T_1(\chi) \to N$$
is injective. Thus, the maximal rank of $o_{\chi,\phi,\beta,\beta'}$ over all diagrams equals the k-vector space dimension of $T_1(\chi)$.

For pointed functors,
$$F, F' : C_{\Lambda,k} \to \mathbf{Sets},$$
and a natural transformation $\eta : F' \Rightarrow F$, an **infinitesimal deformation** over η is a datum
$$\zeta = (\phi : (A,\alpha) \to (A',\alpha'), \beta, \beta')$$

of an infinitesimal extension ϕ in $C_{\Lambda,k}$ with kernel $\text{Ker}(\phi) = N$, an element $\beta \in F(A,\alpha)$, and an element $\beta' \in F'(A',\alpha')$ such that the images of β and β' are equal in $F(A',\alpha')$. For an infinitesimal deformation over η,

$$\widetilde{\zeta} = (\widetilde{\phi} : (\widetilde{A}, \widetilde{\alpha}) \to (\widetilde{A}', \widetilde{\alpha}'), \widetilde{\beta}, \widetilde{\beta}'),$$

a **morphism** of infinitesimal extension over η from ζ to $\widetilde{\zeta}$ is a commutative diagram in $C_{\Lambda,k}$,

$$\begin{array}{ccc} (A,\alpha) & \xrightarrow{\phi} & (A',\alpha') \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u'} \\ (\widetilde{A},\widetilde{\alpha}) & \xrightarrow{\widetilde{\phi}} & (\widetilde{A}',\widetilde{\alpha}'), \end{array}$$

such that $F(u)$ maps β to $\widetilde{\beta}$ and such that $F'(u')$ maps β' to $\widetilde{\beta}'$. Note that (u, u') defines an induced map,

$$\text{Ker}(u, u') : \text{Ker}(\phi) \to \text{Ker}(\widetilde{\phi}).$$

These operations define a category Inf_η of infinitesimal extensions over η. There is a functor,

$$\text{Ker} : \text{Inf}_\eta \to k - \mathbf{mod}_0, \quad \zeta \mapsto \text{Ker}(\phi), \quad (u, u') \mapsto \text{Ker}(u, u').$$

There is also a constant functor,

$$\underline{k} : \text{Inf}_\eta \to k - \mathbf{mod}_0, \quad \zeta \mapsto k, \quad (u, u') \mapsto \text{Id}_k.$$

A **preobstruction theory** for η is a k-linear functor,

$$O : k - \mathbf{mod}_0 \to k - \mathbf{mod}$$

and a natural transformation of functors $\text{Inf}_\eta \to k - \mathbf{mod}$,

$$o : \underline{k} \Rightarrow O \circ \text{Ker}.$$

Every k-linear functor O is additive, and thus is of the form

$$O(N) \cong O(k) \otimes_k N,$$

for a k-vector space $O(k)$. Since every k-linear transformation from k to a k-vector space is uniquely determined by the image of $1 \in k$, the natural transformation is equivalent to a functorial assignment to every infinitesimal deformation ζ over η of an element,

$$o_\zeta \in O(\text{Ker}(\phi)).$$

A preobstruction theory for η is an **obstruction theory** if for every infinitesimal extension ζ over η, the element o_ζ vanishes if and only if there exists $\widehat{\beta} \in F'(A,\alpha)$ that maps to both $\beta \in F(A,\alpha)$ and $\beta' \in F'(A',\alpha')$.

(c)(**The T^1 obstruction theory**.) For a morphism $\chi : (R,\rho) \to (R',\rho')$ in $\widetilde{C}_{\Lambda,k}$, for the associated natural transformation $h_C^\chi : h_C^{(R',\rho')} \Rightarrow h_C^{(R,\rho)}$, check that $T^1(\chi, N)$ and the elements $o_{\chi,\phi,\beta,\beta'}$ define an obstruction theory for h_C^χ. Moreover, using the last part, prove that every obstruction theory O for h_C^χ is induced by a k-linear transformation $T^1(k) \to O(k)$ that is **injective**.

(d)(**Criterion for flatness**.) Read in a commutative algebra book about the **Local Flatness Theorem**, e.g., [Mat89, Theorem 22.5 and Corollary, pp. 176–177]. For the maximal ideal $\mathfrak{m}_R$ of R, conclude that the k-k-complete algebra $R'/\mathfrak{m}_R \cdot R'$

has Krull dimension at least $\dim_k T_0(\chi) - \dim_k T^1(\chi, k)$. Also conclude that when equality holds, the local homorphism χ is flat (even a formally LCI morphism). Combined with the previous part, conclude that for every obstruction theory O for h_C^χ, the Krull dimension is at least $\dim_k T_0(\chi) - \dim_k O(k)$, and that when equality holds, the local homomorphism χ is flat (even a formally LCI morphism).

Problem 5(**The Standard Obstruction Theory for the Hilbert Scheme**.) As in **Problem 3**, let X_Λ be a scheme that is projective and flat over Spec Λ. Let Z_0 be a closed subscheme of the fiber $X_0 = X \times_{\text{Spec }\Lambda} \text{Spec }k$. Denote by $\text{Hilb}_{X_\Lambda/\Lambda,Z_0}$ the pointed functor on $C_{\Lambda,k}$ that associates to every (A,α) the set of closed subschemes Z_A of $X \times_{\text{Spec }\Lambda} \text{Spec }A$ that are A-flat and whose base change $Z_A \times_{\text{Spec }A} \text{Spec }k$ equals Z_0. Denote by

$$\eta : \text{Hilb}_{X_\Lambda/\Lambda,Z_0} \to h_C^\Lambda$$

the tautological natural transformation of pointed functors.

Denote by $\mathcal{I}_0$ the ideal sheaf of Z_0 on X_0, and denote by $\mathcal{O}_{Z_0}$ the quotient by this ideal sheaf, i.e., the structure sheaf of Z_0 considered as a coherent $\mathcal{O}_{X_0}$-module. The k-linear functor of the **standard obstruction theory** is

$$O : k - \mathbf{mod} \to k - \mathbf{mod}, \quad O(N) = \text{Ext}^1_{\mathcal{O}_{X_0}}(\mathcal{I}_0, N \otimes_k \mathcal{O}_{Z_0}).$$

Every infinitesimal extension ζ over η is an infinitesimal extension in $C_{\Lambda,k}$,

$$\phi : (A,\alpha) \twoheadrightarrow (A',\alpha'), \quad N := \text{Ker}(\phi),$$

and an ideal sheaf,

$$\mathcal{I}_{A'} \subset \mathcal{O}_{X_{A'}},$$

of a closed subscheme $Z_{A'}$ of $X_{A'}$ that is A'-flat. Denote by $\phi_X^{\text{pre}}(\mathcal{I}_{A'})$ the inverse image in $\mathcal{O}_{X_A}$ of $\mathcal{I}_{A'}$ with respect to the surjective homomorphism of sheaves of A-algebras,

$$\phi_X : \mathcal{O}_{X_A} \to A' \otimes_A \mathcal{O}_{X_A} = \mathcal{O}_{X_{A'}}.$$

Denoting by $\mathfrak{m}_A$ the maximal ideal of A, there is a short exact sequence of $\mathcal{O}_{X_0}$-modules,

$$o_\zeta : 0 \to N \otimes_k \mathcal{O}_{Z_0} \to \phi_X^{\text{pre}}(\mathcal{I}_{A'})/\mathfrak{m}_A \cdot \phi_X^{\text{pre}}(\mathcal{I}_{A'}) \to \mathcal{I}_0 \to 0.$$

This defines an element,

$$o_\zeta \in \text{Ext}^1_{\mathcal{O}_{X_0}}(\mathcal{I}_0, N \otimes_k \mathcal{O}_{Z_0}) = O(N).$$

(a)(**Preobstruction Theory**.) Check that this defines a preobstruction theory for η, i.e., the elements o_ζ are covariant for morphisms of infinitesimal extensions over η.

(b)(**Obstruction Theory**.) For every ideal sheaf $\mathcal{I}_A \subset \mathcal{O}_{X_A}$ of an A-flat closed subscheme of X_A extending $\mathcal{I}_{A'}$, check that

$$\mathfrak{m}_A \cdot \phi_X^{\text{pre}}(\mathcal{I}_{A'}) \subset \mathcal{I}_A \subset \phi_X^{\text{pre}}(\mathcal{I}_{A'}).$$

Check that the image of $\mathcal{I}_A$ in the quotient $\phi_X^{\text{pre}}(\mathcal{I}_{A'})/\mathfrak{m}_A \cdot \phi_X^{\text{pre}}(\mathcal{I}_{A'})$ gives a splitting of the short exact sequence o_ζ. Show that this defines a bijection between ideal sheaves $\mathcal{I}_A$ and splittings of o_ζ. Conclude that the preobstruction theory is an obstruction theory, i.e., o_ζ is split if and only if there exists an ideal sheaf $\mathcal{I}_A$ as above.

(c)(Lower bound on the dimension.) In the special case that A' equals k and A equals $k \oplus V$ for a 1-dimensional k-vector space V, conclude that the k-vector space $T^0(\eta, V)$ of ideal sheaves $\mathcal{I}_A$ is naturally isomorphic to

$$T^0(\eta, V) \cong \mathrm{Hom}_{\mathcal{O}_{X_0}}(\mathcal{I}_0, \mathcal{O}_{Z_0}) \otimes_k V.$$

Combined with **Problem 4(d)**, conclude that the fiber ring of $\mathrm{Hilb}_{X_\Lambda/\Lambda, Z_0}$ modulo μ has Krull dimension at least equal to

$$\dim_k(\mathrm{Hom}_{\mathcal{O}_{X_0}}(\mathcal{I}_0, \mathcal{O}_{Z_0})) - \dim_k(\mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\mathcal{I}_0, \mathcal{O}_{Z_0})),$$

and when equality holds, the Hilbert scheme is flat over Λ near $[Z_0]$. If the obstruction group vanishes, then the Hilbert scheme is formally smooth over Λ near $[Z_0]$.

(c)(The Local-Global Sequence.) Read about the local-global spectral sequence for Ext, and conclude the following long exact sequence of low degree terms,

$$0 \to H^1(X_0, \mathit{Hom}_{\mathcal{O}_{X_0}}(\mathcal{I}_0, \mathcal{O}_{Z_0}) \to \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\mathcal{I}_0, \mathcal{O}_{Z_0}) \to H^0(X_0, \mathit{Ext}^1_{\mathcal{O}_{X_0}}(\mathcal{I}_0, \mathcal{O}_{Z_0})) \to H^2(X_0, \mathit{Hom}_{\mathcal{O}_{X_0}}(\mathcal{I}_0, \mathcal{O}_{Z_0}))$$

If the image of o_ζ in $\mathit{Ext}^1_{\mathcal{O}_{X_0}}(\mathcal{I}_0, \mathcal{O}_{Z_0}) \otimes_k N$ is always zero, then Z_0 is called **locally unobstructed**. In this case, the **reduced obstruction groups** is

$$O_{\mathrm{red}}(N) := H^1(X_0, \mathit{Hom}_{\mathcal{O}_{X_0}}(\mathcal{I}_0, \mathcal{O}_{Z_0}) \otimes_k N.$$

For this reduced obstruction theory, the lower bound on the Krull dimension of the fiber ring equals

$$\dim_k H^0(X_0, \mathit{Hom}_{\mathcal{O}_{X_0}}(\mathcal{I}_0, \mathcal{O}_{Z_0})) - \dim_k H^1(X_0, \mathit{Hom}_{\mathcal{O}_{X_0}}(\mathcal{I}_0, \mathcal{O}_{Z_0})).$$

If H^1 vanishes, the Hilbert scheme is smooth over Λ near $[Z_0]$. When Z_0 is 1-dimensional, so that H^q vanishes for all $q \geq 0$, interpret the difference of dimensions as the (sheaf cohomology) Euler characteristic of the sheaf $\mathit{Hom}_{\mathcal{O}_{X_0}}(\mathcal{I}_0, \mathcal{O}_{Z_0})$ on Z_0, which can be computed by Riemann-Roch.

(d)(Regular Embeddings.) The closed subscheme Z_0 of X_0 is a **regular embedding** if at every point of Z_0 the ideal sheaf $\mathcal{I}_0$ is generated by a regular sequence. In this case, prove that Z_0 is locally unobstructed, by proving that there is always locally a lift of the regular sequence. Moreover, show that the sheaf $\mathit{Hom}_{\mathcal{O}_{X_0}}(\mathcal{I}_0, \mathcal{O}_{Z_0})$ is locally free of rank equal to the codimension of Z_0 in X_0. This sheaf is usually called the *normal sheaf*, N_{Z_0/X_0}. When both Z_0 and X_0 are smooth over k, this sheaf is canonically isomorphic to the cokernel of the derivative map,

$$T_{Z_0/k} \to T_{X_0/k} \otimes_{\mathcal{O}_{X_0}} \mathcal{O}_{Z_0}.$$

Problem 6(The Standard Obstruction Theory for the Flag Hilbert Scheme.) With notation as above, let $W_0 \subset Z_0$ be a closed subscheme. The **pointed flag Hilbert functor** $\mathrm{fHilb}_{X_\Lambda/\Lambda, Z_0, W_0}$ is the pointed functor that associates to every (A, α) the set of pairs (Z_A, W_A) of an A-flat closed subscheme $Z_A \subset X_A$ that reduces to Z_0 over k and an A-flat closed subscheme $W_A \subset Z_A$ that reduces to W_0 over k. Modify the previous exercises in this context. In particular, if Z_0 is a regular embedding, and if W_0 is an effective Cartier divisor in Z_0 (i.e., a regular embedding of codimension 1), show that the pair is locally unobstructed and the reduced obstruction theory for the natural transformation,

$$\xi : \mathrm{fHilb}_{X_\Lambda/\Lambda, Z_0, W_0} \to \mathrm{Hilb}_{X_\Lambda/\Lambda, W_0},$$

has

$$T^0(k) = H^0(Z_0, \mathcal{N}_{Z_0/X_0}(-\underline{W}_0)), \quad O_{\text{red}}(k) = H^1(Z_0, \mathcal{N}_{Z_0/X_0}(-\underline{W}_0)).$$

If Z_0 is 1-dimensional, conclude that ξ is smooth if H^1 vanishes, and there is a lower bound on the dimension at $([Z_0],[W_0])$ of the fiber of $[W_0]$ of the flag Hilbert scheme given by the Euler characteristic,

$$\chi(Z_0, \mathcal{N}_{Z_0/X_0}(-\underline{W}_0)).$$

Problem 7(**Grothendieck's Π functor**.) Let B_Λ be a projective, flat Λ-scheme. Let $\pi_\Lambda : X_\Lambda \to B_\Lambda$ be a projective morphism such that X_Λ is flat over Λ. Let $s_0 : B_0 \to X_0$ be a k-morphism that is a section of π_0. The pointed **Grothendieck Π functor**, $\Pi_{X_\Lambda/B_\Lambda/\Lambda,[s_0]}$ is the pointed functor that associates to every (A,α) the set of sections $s_A : B_A \to X_A$ of the A-morphism $\pi_A : X_A \to B_A$. By associating to each section the closed image, interpret this in terms of the Hilbert functor $\text{Hilb}_{X_\Lambda/\Lambda,[s_0(B_0)]}$. When π_0 is smooth at every point in $s_0(B_0)$, conclude that s_0 is a regular embedding with normal bundle $N_{s_0(B_0)/X_0}$ isomorphic to $s_0^*T_{\pi_0}$ (where T_{π_0} is the dual of the locally free sheaf of relative differentials $\Omega^1_{\pi_0}$). Finally, if also B_0 is a curve and if W_Λ is an effective Cartier divisor in B_Λ, conclude that there is a lower bound on the fiber dimension at $[s_0]$ of the restriction functor,

$$\Pi_{X_\Lambda/B_\Lambda/\Lambda,[s_0]} \to \Pi_{X_\Lambda \times_{B_\Lambda} W_\Lambda/W_\Lambda/\Lambda,[s_{W,0}]},$$

given by

$$\chi(B_0, s_0^*T_{\pi_0}(-\underline{W}_0)),$$

and the restriction is smooth at $[s_0]$ if $h^1(B_0, s_0^*T_{\pi_0}(-\underline{W}_0))$ equals 0.

Problem 8(**The Hom functor**.) Let B_Λ and Y_Λ be projective, flat Λ-schemes. Define X_Λ to be the fiber product of these, and let $\pi_\Lambda : X_\Lambda \to B_\Lambda$ be the projection. Let $u_0 : B_0 \to Y_0$ be a k-morphism, and let s_0 be the graph of u_0. In this case, show that Grothendieck's Π functor equals the Hom scheme $\text{Hom}_\Lambda(B_\Lambda, Y_\Lambda)$. Conclude the lower bound used in lecture for the fiber dimension of the flag Hilbert scheme over the Hilbert scheme of Y_Λ,

$$\chi(B_0, u_0^*T_{Y_0/k}(-\underline{W}_0)) = \deg_{B_0}(u_0^*T_{Y_0/k}) + \dim(Y_0)(1 - p_a(B_0) - \text{length}(W_0)).$$

Finally, if B_0 equals $\mathbb{P}^1_k$, if $u_0^*T_{Y_0/k}$ is globally generated, resp. ample, and if $\text{length}(W_0) \leq 1$, resp. $\text{length}(W_0) \leq 2$, conclude that the restriction morphism is smooth near $[u_0]$. In this case, the morphism u_0 is called **free**, resp. **very free**.

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