Math 18.726 Lecture Summaries

Homework. These are the problems from the assigned Problem Set which can be completed using the material from that date's lecture.

Lecture	**1.**	Feb.	8	Sheaves
Lecture	**2.**	Feb.	13	Locally ringed spaces
Lecture	**3.**	Feb.	15	Affine schemes and Proj
Lecture	**4.**	Feb.	21	Properties of morphisms
Lecture	**5.**	Feb.	22	Fiber products and separatedness
Lecture	**6.**	Feb.	27	Valuative criteria
Lecture	**7.**	Mar.	1	Proper morphisms
Lecture	**8.**	Mar.	6	Chow's lemma
Lecture	**9.**	Mar.	8	Quasi-coherent sheaves
Lecture	**10.**	Mar.	10	Modules and quasi-coherent sheaves
Lecture	**11.**	Mar.	15	Quotients and Proj
Lecture	**12.**	Mar.	20	Projective morphisms and divisors
Lecture	**13.**	Mar.	22	More projective morphisms and divisors
Lecture	**14.**	Apr.	3	Ample and very ample divisors
Lecture	**15.**	Apr.	5	Homological algebra
Lecture	**16.**	Apr.	10	Compatibility of derived functors
Lecture	**17.**	Apr.	12	Cohomology of quasi-coherent sheaves on an affine scheme
Lecture	**18.**	Apr.	19	Čech cohomology
Lecture	**19.**	Apr.	24	Comparison of Čech and sheaf cohomology
Lecture	**20.**	Apr.	26	Cohomology of invertible sheaves on projective space
Lecture	**21.**	May	1	Duality for projective space
Lecture	**22.**	May	3	Ext and Serre duality I
Lecture	**23.**	May	8	Serre duality II
Lecture	**24.**	May	10	Lefschetz theorems; relative differentials
Lecture	**25.**	May	15	Dualizing and canonical sheaves; the theorem on formal functions
Lecture	**26.**	May	17	Zariski's Main Theorem

Lecture 1. February 8, 2006

Homework. Problem Set 1 Part I: (a), (b), (c), (e); Part II: Problems 2 and 3.

Stated "highlights", i.e., the most important theorems, from the first half of the semester. Discussed glueing lemma and how it leads to the notion of sheaves. Defined sheaves. Stated problem of sheafification. Defined stalks of sheaves of sets. Used stalks to construct sheafification of presheaves of sets. Proved Proposition II.1.1: a morphism sheaves of sets is an isomorphism if and only if every associated map of stalks is an isomorphism.

Fun problem 1. Let k be an algebraically closed field whose characteristic is not 2. Let (u, v, w) be a general triple of elements in k, i.e., $uvw(u+v+w)$ is nonzero. In $\mathbb{P}^2$, how many conics are tangent to the 5 lines $L_1 = \mathbb{V}(x)$, $L_2 = \mathbb{V}(y)$, $L_3 = \mathbb{V}(z)$, $L_4 = \mathbb{V}(x+y+z)$ and $L_5 = \mathbb{V}(ux+vy+wz)$? What is the equation of this line? Use the equation to find the number of conics tangent to $L_1, \dots, L_4$ and containing $[x_0, y_0, z_0]$ for any triple such that $x_0y_0z_0(x_0 + y_0 + z_0)$ is nonzero.

Lecture 2. February 13, 2006

Homework. Problem Set 1 Part I: (d); Part II: Problem 1.

Sheaves of Abelian groups form an Abelian category. Associated to a continuous map $f : X \to Y$, there are functors,

$$f_* : \text{Sheaves}_X \to \text{Sheaves}_Y,$$
$$f^{-1} : \text{Sheaves}_Y \to \text{Sheaves}_X.$$

These functors form an *adjoint pair*.

Defined locally ringed spaces. Associated to every commutative ring A a locally ringed space Spec A. Stated the universal property of Spec A.

Partial answer to Fun Problem 1. Using duality between conics in $\mathbb{P}^2$ and conics in the dual $\mathbb{P}^2$ of lines, observed the tangency conditions correspond to linear conditions on the dual conic. Thus there exists a unique conic tangent to $L_1, \dots, L_5$. At request of students, left the remaining parts of the problem until next lecture.

Fun Problem 2. Let k be a finite field and let $f(x, y, z)$ be a quadratic, homogeneous polynomial with coefficients in k. Show that f has a nonzero solution in k^3. This is tricky! It is best to start with $k = \mathbb{F}_2$, $\mathbb{F}_3$, and maybe $\mathbb{F}_5$. Given a point p in $\mathbb{P}^2$, how many conics with coefficients in k contain p? How many smooth conics contain p? How many smooth conics are there in total in $\mathbb{P}^2$? How many points are contained on a smooth conic? What happens when you compare these numbers?

Lecture 3. February 15, 2006

Homework. Problem Set 2

The scheme Spec A associated to a commutative ring has a *universal property*. There is a ring isomorphism

$$i : A \to \mathcal{O}_{\text{Spec } A}(\text{Spec } A).$$

For every locally ringed space $(T, \mathcal{O}_T)$, this determines a map of sets

$$\theta_T : \text{Hom}_{LRS}(T, \text{Spec } A) \to \text{Hom}_{\text{Rings}}(A, \mathcal{O}_T(T))$$

by associating to a morphism $(f, f^\#) : (T, \mathcal{O}_T) \to (\text{Spec } A, \mathcal{O}_{\text{Spec } A})$ the composite map of commutative rings

$$A \xrightarrow{i} \mathcal{O}_{\text{Spec } A}(\text{Spec } A) \xrightarrow{f^\#} \mathcal{O}_T(T).$$

The universal property is that θ_T is a bijection for every $(T, \mathcal{O}_T)$.

One of the key steps in the proof is the observation that for locally ringed spaces $(S, \mathcal{O}_S)$ and $(T, \mathcal{O}_T)$, a morphism of ringed spaces $(f, f^\#) : (S, \mathcal{O}_S) \to (T, \mathcal{O}_T)$ is a morphism of *locally ringed spaces* if and only if for every open subset U of T and every element $a \in \mathcal{O}_T(U)$, the inverse image $f^{-1}(D(a))$ equals $D(f^\# a)$.

Defined Proj of a graded ring S. Sketched why it is a scheme. Discussed generic points and the full embedding of the category of varieties into the category of schemes.

Lecture 4. February 21, 2006

Homework. Problem Set 2

Discussed open subschemes and quasi-compactness. Proved affine schemes are quasi-compact. Defined the property of being locally Noetherian. Proved for every Noetherian ring A, Spec A is locally Noetherian. Proved a scheme is locally Noetherian if and only if it has a covering by open affine schemes Spec A with A Noetherian.

Defined a number of properties of morphisms: affine, quasi-affine, locally finite type, finite type, locally finitely presented, finitely presented, finite and quasi-finite. Proved $f : X \to Y$ is affine if and only if there exists a covering of Y by open affine schemes Y_i such that each $f^{-1}(Y_i)$ is also affine. Thus, affineness is *local on the target*. The same is true for quasi-compact, locally finite type, finite type, locally finitely presented, finitely presented, finite and quasi-finite.

Defined fiber products. Asserted fiber products exist in the category of locally ringed spaces. Proved a fiber product of affine schemes exists and is affine.

Fun Problem 3. From János Kollár's colloquium. Compute the group of automorphisms of the *Klein quartic curve*, i.e., the zero set in $\mathbb{P}^2$ of the quartic polynomial,

$$x^3 y + y^3 z + z^3 x.$$

Lecture 5. February 22, 2006

Homework. Problem Set 3

Finished the proof that fiber products of schemes exist in the category of locally ringed spaces and are schemes.

Defined closed immersions and the associated ideal sheaves. Mentioned the quasi-coherence property of this sheaf. Explained the universal property of open immersions and closed immersions. Defined locally closed immersions.

Stated that each of the following properties is stable under base change: quasi-compact, affine, quasi-affine, open, closed and locally closed immersions, locally finite type, finite type, locally finitely presented, finitely presented, finite, and quasi-finite.

Defined the diagonal morphism, quasi-separated and separated morphisms. Proved the diagonal morphism is a locally closed immersion. Proved affine morphisms are separated.

Stated properties of quasi-separatedness and separatedness: Immersions are separated. (Quasi-)separatedness is stable under base change. (Quasi-)separatedness is local on the target. A composition of (quasi-)separated morphisms is (quasi-)separated. If a composition $g \circ f$ is (quasi-)separated, then f is (quasi-)separated. Proved a few of these properties.

Fun Problem 4. Find the equations of all lines contained in the quadric Q in $\mathbb{P}^3$ with homogeneous quadratic equation,

$$X_3X_0 - X_1X_2 = 0.$$

How many such lines intersect a general line L in $\mathbb{P}^3$ not contained in Q?

Lecture 6. February 27, 2006

Homework. Problem Set 3

Notion of specialization and generization. The image of a quasi-compact morphism is closed if and only if it is stable under specialization. Definition of valuation rings. Basic existence result: The set of local rings in a fixed field partially ordered by domination has maximal elements, which are valuation rings. Thus, for every scheme X and pair of points (x_η, x_0) such that x_0 is contained in the closure of $\{x_\eta\}$, there is a valuation ring R contained in the residue field $k(x_\eta)$ dominating the local ring $\mathcal{O}_{X,x_0}/\mathfrak{p}_\eta$. This defines a morphism Spec $R \to X$ whose generic point η maps to x_η and whose closed point 0 maps to x_0.

Using this, one gets a valuative criterion for closedness of $f(X)$. The image of a quasi-compact morphism $f : X \to Y$ is closed if and only if for every valuation ring R and every commutative diagram

$$\begin{array}{ccc} \text{Spec } K(R) & \xrightarrow{g_\eta} & X \\ \downarrow & & \downarrow f \\ \text{Spec } R & \xrightarrow{g_R} & Y \end{array}$$

the image of g_R is contained in $f(X)$. This in turn quickly implies the first version of the valuative criterion of separatedness. A quasi-separated morphism $f : X \to Y$ is separated if and only if for every valuation ring R and every commutative diagram

$$\begin{array}{ccc} \text{Spec } K(R) & \xrightarrow{g_K} & X \\ i \downarrow & & \downarrow \Delta_f \\ \text{Spec } R & \xrightarrow{g_R} & X \times_Y X \end{array}$$

the image of g_R is contained in $\Delta(X)$. Indeed, we already know Δ_f is a locally closed immersion, so f is separated if and only if $\Delta_f(X)$ is closed. Since f is quasi-separated, by definition Δ_f is quasi-compact. And then we apply the valuative criterion for closedness of the image.

For every local domain R, for every locally ringed space X, and for every morphism of locally ringed spaces g : Spec $R \to X$, there is an induced datum (x_η, x_0, ϕ), of the image x_η of the prime ideal (0) of R, the image x_0 of the maximal ideal $\mathfrak{m}_R$ of R, and the induced field extension $\phi : k(x_\eta) \to K(R)$. This last map comes from the induced map of stalks $g^{\#}_(0) : \mathcal{O}_{X,x_\eta} \to R$. Since this is a local homomorphism, it induces the map ϕ of residue fields. By construction the triple (x_η, x_0, ϕ) satisfies the conditions that x_0 is contained in $\{x_\eta\}$, and the image under ϕ of $\mathcal{O}_{X,x_0}/\mathfrak{p}_{x_\eta}$ is dominated by the local ring R. Altogether, this defines a set map,

$$\mathrm{Hom}_{LRS}(\mathrm{Spec}\ R, X) \to \{(x_\eta, x_0, \phi) | x_\eta \in X, x_0 \in \overline{\{x_\eta\}}, \phi : k(x_\eta) \to K(R), \phi(\mathcal{O}_{X,x_0}) \text{ dominated by } R\}.$$

If X is a scheme, this is a bijection of sets. Using this, it follows that if $g_1, g_2 : \mathrm{Spec}\ R \to X$ are two morphisms which agree as set maps and whose composition with $\mathrm{Spec}\ K(R) \to \mathrm{Spec}\ R$ agree, then g_1 equals g_2. Using this, it follows that for the map g_R in the last paragraph, if the image of g_R is contained in the image of Δ, then the two compositions $\pi_1 \circ g_R$ and $\pi_2 \circ g_R$ are equal. Therefore there is a morphism $h_R : \mathrm{Spec}\ R \to X$ such that g equals $\Delta \circ h_R$.

This gives rise to the more common version of the valuative criterion of separatedness. A quasi-compact morphism of schemes $f : X \to Y$ is separated if and only if for every valuation ring R and every commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}\ K(R) & \xrightarrow{g_K} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}\ R & \xrightarrow{g_R} & Y \end{array}$$

there is at most one morphism $h : \mathrm{Spec}\ R \to X$ such that $h \circ i$ equals g_K and $f \circ h$ equals g_R.

Fun Problem 5. Find the equations of all lines contained in the hypersurface X in $\mathbb{P}^3$ with homogeneous cubic equation,

$$X_0^3 + X_1^3 + X_2^3 + X_3^3 = 0.$$

Assume the base field does not have characteristic 2 or 3.

Lecture 7. March 1, 2006

Homework. Problem Set 4

There is an equivalent characterization of separated that is sometimes useful. A scheme X is separated if and only if for every pair of open affine U, V of X, $U \cap V$ is affine and the induced map of rings $\mathcal{O}_X(U) \otimes_{\mathcal{O}_X(X)} \mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V)$ is surjective. Using this, it is easy to see that the line with doubled origin is not separated.

Definition of properness: separated, finite type and "universally closed". Given a property $\mathcal{P}$ of morphisms, a morphism $f : X \to Y$ is said to satisfy $\mathcal{P}$ *universally* if for every base change $f' : X' \to Y'$, f' has property $\mathcal{P}$. Thus a morphism is universal closed if every base change morphism f' is closed, i.e., sends closed subset to closed subsets.

Closed immersions are proper. Composition of proper morphisms is proper. If g is separated and $g \circ f$ is proper, then f is proper. Properness is stable under base change. Properness is local on the target.

There is a valuative criterion for properness similar to the valuative criterion of separatedness. A finite type, quasi-separated morphism $f : X \to Y$ is proper if and only if for every valuation ring

R and every commutative diagram

$$\begin{array}{ccc} \text{Spec } K(R) & \xrightarrow{g_K} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ \text{Spec } R & \xrightarrow{g_R} & Y \end{array}$$

there exists one and only one morphism $h : \text{Spec } R \to X$ such that $h \circ i$ equals g_K and $f \circ h$ equals g_R.

A projective morphism is proper. There are other ways to do this, but the valuative criterion gives a very quick argument. There is a unique morphism $\pi : \mathbb{A}^{n+1}_{\mathbb{Z}} - 0 \to \mathbb{P}^n_{\mathbb{Z}}$ whose restriction to $D(x_i)$ factors through $D_+(x_i)$ and is given by $\pi^* Z_{i,j} = x_j/x_i$ (where $(Z_{i,j}|j \neq i)$ are the standard coordinates on $D_+(x_i)$). Given a valuation ring R and a morphism $g_K : \text{Spec } K \to \mathbb{P}^n_{\mathbb{Z}}$, there exists a morphism $h : \text{Spec } K \to \mathbb{A}^{n+1}_{\mathbb{Z}} - 0$ such that $g = h \circ \pi$. The induced map of rings $h^* : \mathbb{Z}[x_0, \dots, x_n] \to K$ does not map every element x_i to 0 (by construction). Thus, the minimum of the valuations $v(h^*(x_i))$ is a finite number. Up to rechoosing coordinates, assume that $v(h^*(x_0))$ is minimum. Define a ring homomorphism

$$\psi : \mathcal{O}_{\mathbb{P}^n}(D_+(x_0)) = \mathbb{Z}[Z_{0,1}, Z_{0,2} \dots, Z_{0,n}] \to K$$

by $\psi(Z_{0,j}) = h^*(x_j)/h^*(x_0)$. By construction, $v(\psi(Z_{0,j}))$ is nonnegative for every $j = 1, \dots, n$. Therefore, ψ factors through the valuation ring R. The induced morphism $g_R : \text{Spec } R \to D_+(x_0) \hookrightarrow \mathbb{P}^n_{\mathbb{Z}}$ is the unique morphism whose restriction to Spec K equals g_K. Therefore $\mathbb{P}^n_{\mathbb{Z}}$ is proper.

From the properties of proper morphisms, it follows that every projective morphism is proper. Chow's Lemma says that every separated, finite type morphism is "close" to being quasi-projective. Let $f : X \to Y$ be a separated, finite type morphism of quasi-compact schemes. Chow's Lemma says that there exists a projective, birational morphism $g : X' \to X$ such that the composition $f \circ g : X' \to Y$ is quasi-projective. The morphism f is proper if and only if the morphism $f \circ g$ is proper. On the other hand, a quasi-projective morphism is proper if and only if it is projective. Therefore, f is proper if and only if $f \circ g$ is projective. For morphisms of finite type schemes over a field, this is the basis for an "improved" version of the valuative criterion of properness that uses normal curves in place of valuation rings.

Fun Problem 6. Let d be an integer ≥ 3. Let k be a field whose characteristic is larger than d. Find the equations of all lines contained in the hypersurface X in $\mathbb{P}^3$ with homogeneous equation,

$$X_0^d + X_1^d + X_2^d + X_3^d = 0.$$

Lecture 8. March 6, 2006

Homework. Problem Set 4

We completed the proof of Chow's lemma. We used this to prove that for a separated morphism of finite type schemes over a field, $f : X \to Y$, the morphism f is proper if and only if it satisfies the following curve criterion: for every normal curve C over the field, for every dense open subset U of C, for every commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{g_U} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow{g_C} & Y \end{array}$$

there exists at least one morphism $h : C \to X$ such that $h \circ i$ equals g_U and $f \circ h$ equals g_C (which by separatedness will turn out to be unique). In other words, in the usual valuative criterion, the role of Spec R is replaced by C and the role of Spec $K(R)$ is replaced by U.

It was not discussed in class, but one can also use this to get a curve criterion for separatedness. Let $f : X \to Y$ be a morphism of finite type schemes over a field. The diagonal morphism $\Delta_{X/Y} : X \to X \times_Y X$ is a separated morphism of finite type schemes over a field. Applying the curve criterion to $\Delta_{X/Y}$, f is separated if and only if it satisfies the following curve criterion: for every normal curve C over the field, for every dense open subset U of C, for every commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{g_U} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow{g_C} & Y \end{array}$$

there exists at most one morphism $h : C \to X$ such that $h \circ i$ equals g_U and $f \circ h$ equals g_C.

This then gives a slightly better curve criterion for properness. Let $f : X \to Y$ be a morphism of finite type schemes over a field (not assumed to be separated). The morphism f is proper if and only if it satisfies the following curve criterion: for every normal curve C over the field, for every dense open subset U of C, for every commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{g_U} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow{g_C} & Y \end{array}$$

there exists one and only one morphism $h : C \to X$ such that $h \circ i$ equals g_U and $f \circ h$ equals g_C.

Defined (pre-)sheaves of $\mathcal{O}_X$-modules. Explained the functor

$$\widetilde{(-)} : \mathcal{O}_X(X) - \text{modules } \to \text{Sheaves of } \mathcal{O}_X - \text{modules}.$$

Proved the universal property of this construction: For every $\mathcal{O}_X(X)$-module M and every sheaf of $\mathcal{O}_X$-module $\mathcal{F}$, the following natural map is a bijection

$$\mathrm{Hom}_{\mathcal{O}_X-\mathrm{mod}}(\widetilde{M}, \mathcal{F}) \to \mathrm{Hom}_{\mathcal{O}_X(X)-\mathrm{mod}}(M, \mathcal{F}(X)).$$

Defined a sheaf of $\mathcal{O}_X$-modules to be *quasi-coherent* if for every point p of X there exists an open subset $p \in U$ such that the following natural map of sheaves of $\mathcal{O}_U$-modules is an isomorphism

$$\widetilde{\mathcal{F}(U)} \to \mathcal{F}|_U.$$

Stated the theorem, to be proved in the next lecture, that if X is an affine scheme and $\mathcal{F}$ is a quasi-coherent sheaf on X, then the following natural map of sheaves is an isomorphism

$$\widetilde{\mathcal{F}(X)} \to \mathcal{F}.$$

It follows that the Abelian category of quasi-coherent sheaves on X is equivalent to the Abelian category of $\mathcal{O}_X(X)$-modules.

Fun Problem 7. This problem is taken from Joe Harris's book, "Algebraic geometry, a first course". Let k be a field and let n be a positive integer. Let $P_0, \dots, P_n$ be degree d, homogeneous polynomials in the variables $X_0, \dots, X_n$ such that there is no common zero of $P_0, \dots, P_n$ in $\mathbb{P}^n_k$ and such that the following degree $(d+1)$ homogeneous polynomial is the zero polynomial

$$X_0 P_0 + X_1 P_1 + \cdots + X_n P_n.$$

(i) Working by induction on n (by setting one of the coordinates X_i equal to 0), prove that d equals 1.

(ii) For each $i = 1, \dots, n$, expand P_i as follows

$$P_i(X_0, \dots, X_n) = a_{i,0}X_0 + \cdots + a_{i,j}X_j + \cdots + a_{i,n}X_n.$$

Prove that the matrix $(a_{i,j})$ is an invertible, skew-symmetric matrix.

(iii) If $\mathrm{char}(k)$ is not 2 and n is even, prove there is no sequence of polynomials $P_0, \dots, P_n$ as above.

Bonus problem. If $\mathrm{char}(k)$ equals 2 and n is even, do there exist $P_0, \dots, P_n$ as above? Can you write one down?

Lecture 9. March 8, 2006

Homework. Problem Set 5

The category of $\mathcal{O}_X$-modules is an Abelian category. Additionally, there are 2 bifunctors associating to a pair $(\mathcal{F}, \mathcal{G})$ of $\mathcal{O}_X$-modules the $\mathcal{O}_X$-module $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$, respectively $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$. The first is called the *sheaf Hom*, the second is called the *tensor product*.

The universal property of sheaf Hom is that for every open subset U of X, there is an isomorphism of $\mathcal{O}_X(U)$-modules,

$$\mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{F}|_U, \mathcal{G}|_U) \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U).$$

Morevoer, this isomorphism is bifunctorial in $(\mathcal{F}, \mathcal{G})$, is compatible with restriction, etc.

The universal property of the tensor product is that for every triple of $\mathcal{O}_X$-modules $(\mathcal{E}, \mathcal{F}, \mathcal{G})$, there is a canonical isomorphism of $\mathcal{O}_X(X)$-modules,

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}, \mathcal{G}).$$

Moreover, this isomorphism is functorial in each of $\mathcal{E}, \mathcal{F}, \mathcal{G}$, etc.

Associated to every morphism of locally ringed spaces $f : X \to Y$, there are covariant functors,

$$f_* : \mathcal{O}_X - \text{modules} \to \mathcal{O}_Y - \text{modules}$$

$$f^* : \mathcal{O}_Y - \text{modules} \to \mathcal{O}_X - \text{modules}.$$

The functor f_* is just as before. The functor f^* is defined by $f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$. These two functors satisfy a basic adjointness condition. For every $\mathcal{O}_Y$-module $\mathcal{F}$ and every $\mathcal{O}_X$-module $\mathcal{G}$, there is a canonical isomorphism of $\mathcal{O}_X(X)$-modules,

$$\text{Hom}_{\mathcal{O}_X}(f^*\mathcal{F}, \mathcal{G}) \cong \text{Hom}_{\mathcal{O}_Y}(\mathcal{F}, f_*\mathcal{G}).$$

Moreover, this isomorphism is functorial in each of $\mathcal{F}$, $\mathcal{G}$, etc. In particular, this determines an isomorphism,

$$\mathit{Hom}_{\mathcal{O}_Y}(\mathcal{F}, f_*\mathcal{G}) \cong f_*\mathit{Hom}_{\mathcal{O}_X}(f^*\mathcal{F}, \mathcal{G}).$$

Let $f : X \to Y$ be a morphism of locally ringed spaces and let M be an $\mathcal{O}_Y(Y)$-module. The induced map $f^* : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ determines an $\mathcal{O}_X(X)$-module,

$$N := M \otimes_{\mathcal{O}_Y(Y)} \mathcal{O}_X(X).$$

The induced map $M \to N$ determines a map of $\mathcal{O}_X$-modules,

$$f^*(\widetilde{M}) \to \widetilde{N}.$$

This is an isomorphism. Moreover, this isomorphism is compatible with further pullback. It follows that for every quasi-coherent $\mathcal{O}_Y$-module $\mathcal{F}$, the pullback $f^*\mathcal{F}$ is quasi-coherent.

For every morphism of affine schemes $f : \text{Spec } A \to \text{Spec } B$ associated to a ring homomorphism $\phi : B \to A$, for every A-module N, there is a canonical isomorphism,

$$f_*(\widetilde{N}) \cong \widetilde{M},$$

where M equals N as an Abelian group, and where the B-module action is defined by $\phi : B \to A$ and the A-module action on N.

For every pair of $\mathcal{O}_X$-modules $(\mathcal{E}, \mathcal{F})$, applying adjointness of $\mathit{Hom}_{\mathcal{O}_X}(\bullet, \bullet)$ and $\bullet \otimes_{\mathcal{O}_X} \bullet$ to the identity map

$$\mathit{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \to \mathit{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}),$$

there is an associated map

$$m_{\mathcal{E},\mathcal{F}} : \mathit{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \otimes_{\mathcal{O}_X} \mathcal{E} \to \mathcal{F}.$$

This map is functorial in $\mathcal{E}$, $\mathcal{F}$, etc. In the case that $\mathcal{F} = \mathcal{O}_X$, if the map $m_{\mathcal{E},\mathcal{O}_X}$ is an isomorphism, the sheaf $\mathcal{E}$ is called an *invertible sheaf*. It is straightforward to see this holds if and only if $\mathcal{E}$ is locally isomorphic to $\mathcal{O}_X$.

Let $\mathcal{L}$ be an invertible sheaf. For every global section σ of $\mathcal{L}$, define $D(\sigma)$ to be the maximal open subscheme on which the induced map

$$\sigma|_{D(\sigma)} : \mathcal{O}_{D(\sigma)} \to \mathcal{L}|_{D(\sigma)}$$

is an isomorphism. Taking powers, for every nonnegative integer n, there is an induced isomorphism,

$$\sigma^n|_{D(\sigma)} : \mathcal{O}_{D(\sigma)} \to \mathcal{L}^{\otimes n}|_{D(\sigma)}.$$

Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Inverting this isomorphism gives an isomorphism,

$$\mathrm{Id}_{\mathcal{F}} \otimes \sigma^{-n}|_{D(\sigma)} : \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}|_{D(\sigma)} \to \mathcal{F}|_{D(\sigma)}.$$

Taking sections, this defines a morphism of $\mathcal{O}_X(X)$-modules,

$$\phi_{\mathcal{F},n} : \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}(X) \to \mathcal{F}(D(\sigma)).$$

This morphism is functorial in $\mathcal{F}$ and is compatible with σ, i.e., the composition,

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}(X) \xrightarrow{\mathrm{Id}\otimes\sigma} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes (n+1)}(X) \xrightarrow{\phi_{\mathcal{F},n+1}} \mathcal{F}(D(\sigma)),$$

equals $\phi_{\mathcal{F},n}$. Therefore the collection of maps $(\phi_{\mathcal{F},n})_{n\geq 0}$ is a compatible family of $\mathcal{O}_X(X)$-module homomorphisms. By the universal property of the colimit, there is an induced $\mathcal{O}_X(X)$-module homomorphism,

$$\phi_{\mathcal{F}} : \varprojlim_n \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}(X) \to \mathcal{F}(D(\sigma)).$$

The basic theorem is that if $\mathcal{F}$ is quasi-coherent and X is a quasi-compact scheme, then $\phi_{\mathcal{F}}$ is injective. If $\mathcal{F}$ is quasi-coherent and X is quasi-compact and quasi-separated scheme, then $\phi_{\mathcal{F}}$ is a bijection. In particular, applying this in the case when $\mathcal{L}$ equals $\mathcal{O}_X$, if X is quasi-compact and quasi-separated scheme, for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ and every element f of $\mathcal{O}_X(X)$, the following map is an isomorphism

$$\mathcal{F}(X)_f \to \mathcal{F}(D(f)).$$

Fun Problem 8. Let $\mathbb{F}_q$ be a finite field, and let X be a scheme over $\mathbb{F}_q$. If the fiber product $\mathrm{Spec}\ \overline{\mathbb{F}}_q \times_{\mathrm{Spec}\ \mathbb{F}_q} X$ is isomorphic to $\mathbb{P}^1_{\overline{\mho}_q}$ as a scheme over $\overline{\mathbb{F}}_q$, prove that X is isomorphic to $\mathbb{P}^1_{\mathbb{F}_q}$.

Lecture 10. March 13, 2006

Homework. Problem Set 5

Warning: Even if $\mathcal{E}$ and $\mathcal{F}$ are quasi-coherent $\mathcal{O}_X$-modules, the sheaf Hom $\mathit{Hom}_{\mathcal{O}_X}(\mathcal{E},\mathcal{F})$ need not be quasi-coherent. For a similar reason, given a morphism $f : Y \to X$, the natural morphism of $\mathcal{O}_Y$-modules,

$$f^*\mathit{Hom}_{\mathcal{O}_X}(\mathcal{E},\mathcal{F}) \to \mathit{Hom}_{\mathcal{O}_Y}(f^*\mathcal{E}, f^*\mathcal{F}),$$

need not be an isomorphism.

There are 2 remarks. First of all, the functor $\otimes$ is better behaved: $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}$ is quasi-coherent if $\mathcal{E}$ and $\mathcal{F}$ are, and the natural map $f^*(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) \to f^*\mathcal{E} \otimes_{\mathcal{O}_Y} f^*\mathcal{F}$ is an isomorphism. Secondly, if $\mathcal{E}$ is a quasi-coherent $\mathcal{O}_X$-module that is *locally finitely presented*, then $\mathit{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$ is quasi-coherent and $f^*\mathit{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$ is canonically isomorphic to $\mathit{Hom}_{\mathcal{O}_Y}(f^*\mathcal{E}, f^*\mathcal{F})$. This motivates the notion of locally finitely presented $\mathcal{O}_X$-modules, which are also called *coherent* in the textbook. It is clear that for every morphism $f : Y \to X$, the functor f^* sends locally finitely presented $\mathcal{O}_X$-modules to locally finitely presented $\mathcal{O}_Y$-modules.

Recall the theorem from the last lecture: if X is a quasi-compact, quasi-separated scheme, $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, and f is an element of $\mathcal{O}_X(X)$, the induced map of $\mathcal{O}_X(X)_f$-modules, $\mathcal{F}(X)_f \to \mathcal{F}(D(f))$, is a bijection. In particular, every affine scheme is quasi-compact and quasi-separated. Moreover, the basic open sets $D(f)$ form a basis for the topology. Therefore, for every affine scheme X and every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, the induced map of quasi-coherent $\mathcal{O}_X$-modules,

$$\widetilde{\mathcal{F}(X)} \to \mathcal{F},$$

is an isomorphism. This determines an equivalence of categories,

$$\{ \text{ quasi-coherent } \mathcal{O}_X - \text{modules}\} \leftrightarrow \{\mathcal{O}_X(X) - \text{modules}\}.$$

This is an equivalence of Abelian categories. But it is stronger than that. The induced functor from $\mathcal{O}_X(X)$-modules to $\mathcal{O}_X$-modules is an exact functor, i.e., it sends short exact sequences to short exact sequences. Moreover, the image category is an exact category: given a short exact sequence of $\mathcal{O}_X$-modules,

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

if any 2 of the sheaves are quasi-coherent, so is the third. In fact, each of these statements can be checked locally on X. Since every scheme is locally isomorphic to an affine scheme, it follows that the category of quasi-coherent sheaves is an exact subcategory of the category of $\mathcal{O}_X$-modules for every scheme X.

A more serious consequence is that for every short exact sequence of $\mathcal{O}_X$-modules,

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

if X is affine and $\mathcal{F}'$ is quasi-coherent, then the induced map $\mathcal{F}(X) \to \mathcal{F}''(X)$ is surjective. We will see later that for every short exact sequence of sheaves of Abelian groups, so long as X is affine and $\mathcal{F}'$ is quasi-coherent, then $\mathcal{F}(X) \to \mathcal{F}''(X)$ is surjective.

The affine scheme $\mathbb{G}_m := \text{Spec } \mathbb{Z}[t, t^{-1}]$ admits morphisms, $m : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$, $i : \mathbb{G}_m \to \mathbb{G}_m$, $e : \text{Spec } \mathbb{Z} \to \mathbb{G}_m$ defined via the following ring homomorphisms,

$$m^* : \mathbb{Z}[t, t^{-1}] \to \mathbb{Z}[t, t^{-1}] \otimes \mathbb{Z}[t, t^{-1}], \quad t \mapsto t \otimes t,$$

$$i^* : \mathbb{Z}[t, t^{-1}] \to \mathbb{Z}[t, t^{-1}], \quad t \mapsto t^{-1},$$

$$e^* : \mathbb{Z}[t, t^{-1}] \to \mathbb{Z}, \quad t \mapsto 1.$$

The datum $(\mathbb{G}_m, m, i, e)$ satisfy the analogous axioms for a group, multiplication map, inverse map, identity element. More precisely, for every scheme T, the maps m, i, e induce maps on the set,

$$\mathbb{G}_m(T) := \mathrm{Hom}_{LRS}(T, \mathbb{G}_m)$$

making this set into a group. For this reason the datum $(\mathbb{G}_m, m, i, e)$ is called a *group scheme*. To be perfectly explicit, the group $\mathbb{G}_m(T)$ is naturally isomorphic to the multiplicative group $\mathcal{O}_T(T)^*$, of invertible elements in the ring $\mathcal{O}_T(T)$. For this reason $\mathbb{G}_m$ is called the *multiplicative group*.

Given a base scheme B, a group scheme G over B and a B-scheme X, an action of G on X over B is a morphism of B-schemes, $\mu : G \times_B X \to X$ such that for every B-scheme T the induced map

$$\mu : \mathrm{Hom}_{B-\mathrm{sch}}(T, G) \times \mathrm{Hom}_{B-\mathrm{sch}}(T, X) \to \mathrm{Hom}_{B-\mathrm{sch}}(T, X)$$

is an action of the group $G(T) := \mathrm{Hom}_{B-\mathrm{sch}}(T, G)$ on the set $X(T) := \mathrm{Hom}_{B-\mathrm{sch}}(T, X)$.

For an affine scheme $X = \mathrm{Spec}\ S$, an action of $\mathbb{G}_m$ on X is equivalent to a ring homomorphism,

$$\mu^* : S \to S[t, t^{-1}],$$

satisfying the axiom above. Define S_d to be the subset of S of elements a such that $\mu^*(a) = at^d$. The axiom above is equivalent to the condition that the subgroups $(S_d)_{d \in \mathbb{Z}}$ make S into a $\mathbb{Z}$-graded ring.

Conversely, given a $\mathbb{Z}$-graded ring S, there is an induced ring homomorphism $S \to S[t, t^{-1}]$ sending a to at^d for every integer d and every a in S_d. The induced morphism $\mu : \mathbb{G}_m \times X \to X$ is an action. Therefore, an action of $\mathbb{G}_m$ on X is precisely the same thing as a $\mathbb{Z}$-grading of S.

Associated to a ring S together with a $\mathbb{Z}_{\geq 0}$-grading (which is just a particular type of $\mathbb{Z}$-grading), the Proj construction will turn out to be a very good attempt to construct the quotient of Spec S by the associated action of $\mathbb{G}_m$.

Fun Problem 9. Let X be a finite type, affine scheme over a field k. Prove that every non-constant morphism of k-schemes, $f : \mathbb{A}^1_k \to X$, is a finite morphism.

Lecture 11. March 15, 2006

Homework. Problem Set 6

Let $\mathcal{P}$ be a property of morphisms. A morphism $f : X \to Y$ satisfies $\mathcal{P}$ *universally*, respectively *uniformly*, if for every morphism $Y' \to Y$, resp. every *flat* morphism $Y' \to Y$, the pullback morphism $Y' \times_Y X \to Y'$ satisfies $\mathcal{P}$. A morphism $g : Z \to Y$ is *flat* if for every point z of Z, the induced map of stalks

$$g_z^\# : \mathcal{O}_{Y,g(z)} \to \mathcal{O}_{Z,z}$$

is flat, i.e., $\mathcal{O}_{Z,z}$ is a flat module over $\mathcal{O}_{Y,g(z)}$. Although it might seem surprising at first, this is one of the most important properties of morphisms in algebraic geometry.

Let B be a base scheme and let G be an group scheme over B. Let X and Y be B-schemes, and let $\mu_X : G \times_B X \to X$ and $\mu_Y : G \times_B Y \to Y$ be actions of G on X and Y over B. A morphism of

B-schemes $f : X \to Y$ is G-*equivariant* if the following diagram is commutative,

$$\begin{array}{ccc} G \times_B X & \xrightarrow{\mathrm{Id}_G \times f} & G \times_B Y \\ {\scriptstyle \mu_X}\downarrow & & \downarrow{\scriptstyle \mu_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

Equivalently, for every B-scheme T, the induced map $f(T) : X(T) \to Y(T)$ is compatible with the action of the group $G(T)$, where, as usual $X(T) = \mathrm{Hom}_{B-\mathrm{sch}}(T, X)$, etc.

For every B-scheme Y, there is a *trivial action* of G on Y by $\mu_Y = \mathrm{pr}_Y : G \times_B Y \to Y$. If f is G-equivariant for the trivial action of G on Y, then f is called G-*invariant*. A *categorical quotient* is a G-invariant morphism $f : X \to Q$ such that for every B-scheme Y, the following set map is a bijection,

$$\bullet \circ f : \mathrm{Hom}_{B-\mathrm{sch}}(Q, Y) \to \{h \in \mathrm{Hom}_{B-\mathrm{sch}}(X, Y) | h \text{ is } G-\text{invariant}\}.$$

The morphism $f : X \to Q$ is a *uniform categorical quotient* if for every flat morphism $Q' \to Q$, the induced map $Q' \times_Q X \to Q'$ is a categorical quotient of $Q' \times_Q X$ with its induced action of G.

Let S be a $\mathbb{Z}$-graded ring. The subgroup S_0 is a subring of S. The induced map Spec $S \to$ Spec S_0 is a uniform categorical quotient of the induced action of $\mathbb{G}_m$ on Spec S. However, this is often very far from a "good quotient". As an example, consider the graded ring $S = \mathbb{Z}[x_0, \dots, x_n]$ with $\deg(x_0) = \dots = \deg(x_n) = 1$. Then S_0 is simply $\mathbb{Z}$, so the uniform categorical quotient is the constant morphism $\mathbb{A}^n_{\mathbb{Z}} \to \mathrm{Spec}\ \mathbb{Z}$. On the other hand, there are very many distinct orbits in $\mathbb{A}^n_{\mathbb{Z}}$.

The problem is entirely caused by the closed subscheme of Spec S on which $\mathbb{G}_m$ acts as the identity. Assume that S is graded in nonnegative degrees. Define S_+ to be the ideal $\oplus_{d>0} S_d$. Then the closed subscheme on which $\mathbb{G}_m$ acts as the identity is precisely $\mathbb{V}(S_+)$, the set of prime ideals containing S_+. Define U to be Spec $S - \mathbb{V}(S_+)$. This is a $\mathbb{G}_m$-invariant open subscheme. Thus there is an induced action of $\mathbb{G}_m$ on U.

Moreover, because S_+ is generated by homogeneous elements, U is covered by basic open subsets $D(a)$ as a ranges over all homogeneous elements of S of positive degree. The open subset $D(a)$ is $\mathbb{G}_m$-invariant, thus has an induced action of $\mathbb{G}_m$. For every homogeneous element a, define $\pi_a : D(a) \to D_+(a)$, the open subset of $\mathrm{Proj}(S)$, as follows. The open subset $D_+(a)$ is isomorphic to the affine scheme of the ring $(S[1/a])_0$ and $D(a)$ is isomorphic to the affine scheme of the ring $S[1/a]$. The natural inclusion $(S[1/a])_0 \hookrightarrow S[1/a]$ induces the morphism π_a. Moreover, by the above, π_a is a uniform categorical quotient. Using this fact, it is easy to see that $\pi_a^{-1}(D_+(ab))$ equals $D(ab)$ and the restrictions of π_a and π_b to $D(ab)$ are equal. Therefore, by the glueing lemma, there is a unique morphism $\pi : U \to \mathrm{Proj}(S)$. Because locally it is a uniform categorical quotient, using the glueing lemma, π is a uniform categorical quotient of the action of $\mathbb{G}_m$ on U.

However, the map π is much better than a typical uniform categorical quotient. It is, in fact, what is known as a *good quotient*. Given a base scheme B, an affine group scheme G over B (i.e., a group scheme over B that is an affine scheme over B) and an action $\mu : G \times_B X \to X$ of G on a scheme X over B, a G-invariant B-morphism $f : X \to Q$ is a *good quotient* if it satisfies the following properties.

(i) The morphism f is a uniform categorical quotient.

(ii) For every algebraically closed field k and every morphism $q : \text{Spec } k \to Q$, the fiber product $\text{Spec } k \times_Q X$ is a single orbit of $\text{Spec } k \times_B G$.

(iii) For every morphism $Q' \to Q$ (not necessarily flat), the pullback map $Q' \times_Q X \to Q'$ is a *submersion*, i.e., a subset U of Q' is open if and only if its inverse image in $Q' \times_Q X$ is open.

(iv) The induced morphism $f^\# : \mathcal{O}_Q \to (f_*\mathcal{O}_X)^G$ is an isomorphism of sheaves.

Just to make the last condition clear, the action μ_X determines a dual map of sheaves of $\mathcal{O}_Q$-algebras,

$$\mu_X^* : f_*\mathcal{O}_X \to f_*\mathcal{O}_X \otimes_{\mathcal{O}_B} \mathcal{O}_G$$

with the obvious meaning, i.e., because G is affine over B, it comes from a quasi-coherent $\mathcal{O}_B$-algebra $\mathcal{O}_G$ and $f_*\mathcal{O}_X \otimes_{\mathcal{O}_B} \mathcal{O}_G$ is the tensor product of $f_*\mathcal{O}_X$ with the pullback to Q of $\mathcal{O}_G$. The G-invariant subsheaf $(f_*\mathcal{O}_X)^G$ is the subsheaf of sections a of $f_*\mathcal{O}_X$ such that $\mu_X^*(a) = a \otimes 1$. Because the morphism f is G-invariant, $f^\#$ factors through $(f_*\mathcal{O}_X)^G$.

Note that condition (ii) says what the points of Q must be, condition (iii) says what the open subsets of Q must be, and condition (iv) says what the sheaf of Q must be. Therefore there is a unique map of ringed spaces $(f, f^\#) : X \to Q$ roughly satisfying (ii), (iii) and (iv). However, typically Q is not even a locally ringed space, much less a scheme (much less a uniform categorical quotient, etc.).

The beautiful fact is that if S is a $\mathbb{Z}_{\geq 0}$-graded algebra, then the morphism $\pi : \text{Spec } S - \mathbb{V}(S_+) \to \text{Proj}(S)$ is a good quotient of the action of $\mathbb{G}_m$. This can be proved using a small amount of flat descent.

There is an even stronger property than being a good quotient. With notation as above, a G-invariant morphism $f : X \to Q$ is a *G-torsor over* Q if f is quasi-compact, surjective and flat and the following map is an isomorphism of schemes,

$$(\mu_X, \text{pr}_X) : G \times_B X \to X \times_Q X.$$

Using a small amount of flat descent, it turns out that f is a G-torsor over Q if there exists a quasi-compact, surjective, flat morphism $Q' \to Q$ such that $Q' \times_Q X \to Q'$ is a G-torsor over Q'. For one direction, simply take $Q' \to Q$ to be $f : X \to Q$, then it follows from the definition that $Q' \times_Q X$ is isomorphic to $G \times_B Q'$ as a Q'-torsor. In this sense, a G-torsor over Q is a scheme over Q with an action of $G \times_B Q$ over Q which is "locally" isomorphic to $G \times_B Q$, where "locally" is in the "fpqc topology".

Let S be a $\mathbb{Z}_{\geq 0}$-graded algebra. When is $\pi : \text{Spec } S - \mathbb{V}(S_+) \to \text{Proj}(S)$ a $\mathbb{G}_m$-torsor? It turns out this holds if and only if S is generated by S_1 as an S_0-algebra. In this case U is covered by sets of the form $D(a)$ for a in S_1. As above, there is a morphism $\pi_a : D(a) \to D_+(a)$ that is a uniform categorical quotient. But in fact, there is a $\mathbb{G}_m$-equivariant isomorphism $(s_a, \pi_a) : D(a) \to \mathbb{G}_m \times D_+(a)$. This is equivalent to an isomorphism of rings,

$$(s_a, \pi_a)^* : (S[1/a])_0[t, t^{-1}] \to S[1/a].$$

The restriction to the subalgebra $(S[1/a])_0$ must be the map π_a^* from above, i.e., the inclusion. The new component is that t maps to a. The inverse map sends every homogeneous element b of $S[1/a]_d$ to $(b/a^d)t^d$. It is easy to check these give inverse isomorphisms of graded algebras. Thus (s_a, π_a) is a $\mathbb{G}_m$-equivariant isomorphism. This proves that Zariski locally over $\text{Proj}(S)$, the morphism π is a $\mathbb{G}_m$-torsor. Therefore π is a $\mathbb{G}_m$-torsor.

This has some consequences for quasi-coherent sheaves on $\text{Proj}(S)$. Let M be a graded S-module. The associated quasi-coherent sheaf $\widetilde{M}$ on Spec S has an action of $\mathbb{G}_m$. To be precise, there is an isomorphism of the sheaves $\text{pr}_{\text{Spec } S}^*\widetilde{M}$ and $\mu^*\widetilde{M}$ on $\mathbb{G}_m \times \text{Spec } S$. To give such an isomorphism is equivalent to giving an isomorphism of the $S[t, t^{-1}]$-modules,

$$\psi : M[t, t^{-1}] \to M[t, t^{-1}]$$

intertwining the module structure on $\text{pr}_{\text{Spec } S}^*\widetilde{M}$ and the module structure on $\mu^*\widetilde{M}$, i.e., for every homogeneous element a_d in S_d, for every element m of M,

$$\psi(a_d m) = \psi(a_d \bullet_{\text{pr}} m) = a_d \bullet_\mu \psi(m) = a_d \psi(m) t^d.$$

As with actions of $\mathbb{G}_m$ on S, there is a bijection between the set of structures of graded module on the S-module M and the set of such isomorphisms ψ. Given a grading, define $\psi : M_e \to M[t, t^{-1}]$ by $m_e \mapsto m_e t^e$. This extends uniquely to an isomorphism $\psi : M[t, t^{-1}] \to M[t, t^{-1}]$ satisfying the intertwining condition above.

Because U is fixed by $\mathbb{G}_m$, the restriction $\widetilde{M}_U$ has an action of $\mathbb{G}_m$. Because π is $\mathbb{G}_m$-invariant, $\pi_*(\widetilde{M}|_U)$ has an action of $\mathbb{G}_m$ lifting the trivial action on $\text{Proj}(S)$. It turns out this is precisely the same thing as a $\mathbb{Z}$-grading by quasi-coherent subsheaves. The $\mathbb{G}_m$-invariant subsheaf is precisely the 0^{th} graded piece. Define $\widetilde{M}$ on $\text{Proj}(S)$ to be $(\pi_*(\widetilde{M}|_U))^{\mathbb{G}_m}$. This is a quasi-coherent sheaf that is naturally isomorphic to the sheaf constructed in the textbook.

In the other direction, for every quasi-coherent sheaf $\mathcal{G}$ on $\text{Proj}(S)$, $\pi^*\mathcal{G}$ is a quasi-coherent sheaf on U with an action of $\mathbb{G}_m$: the pullback $\mu^*\pi^*\mathcal{G}$ is canonically isomorphic to $\text{pr}_U^*\pi^*\mathcal{G}$ because $\pi \circ \mu$ equals $\pi \circ \text{pr}_U$ (because π is $\mathbb{G}_m$-invariant). Because $\pi : U \to \text{Proj}(S)$ is a $\mathbb{G}_m$-torsor, for every quasi-coherent $\mathcal{O}_U$-module $\mathcal{F}$ with an action of $\mathbb{G}_m$, the induced map,

$$\pi^*([\pi_*\mathcal{F}]^{\mathbb{G}_m}) \to \mathcal{F}$$

is an isomorphism of quasi-coherent sheaves with an action of $\mathbb{G}_m$. Define $\Gamma_*(\mathcal{G})$ to be $\pi^*\mathcal{G}(U)$. This is a graded $\mathcal{O}_X(U)$-module.

For every graded S-module M, the natural induced map $M \to \Gamma_*(\widetilde{M})$ is a map of graded S-modules that induces an isomorphism of quasi-coherent sheaves on U. Also, for every quasi-coherent sheaf $\mathcal{G}$ on $\text{Proj}(S)$, the induced map $(\Gamma_*(\mathcal{G}))^\sim \to \mathcal{G}$ is an isomorphism of quasi-coherent sheaves. Together, these operations determine an equivalence of categories,

$$\{\text{Quasi-coherent sheaves on Proj}(S)\} \leftrightarrow \{\text{Graded } \mathcal{O}_U(U) - \text{modules}\}/\text{equiv.}$$

where two morphisms of graded modules $\phi_1, \phi_2 : M \to N$ are *equivalent* if they induced the same morphism of quasi-coherent sheaves on U.

There are 2 very important theorems about these operations. If S is generated by S_1 as an S_0-algebra and S is Noetherian (which is equivalent to saying S_0 is Noetherian and S_1 is a finite S_0-module), then the equivalence above induces an equivalence between coherent sheaves on $\text{Proj}(S)$ and finitely presented $\mathcal{O}_U(U)$-modules. In particular, this implies that for every coherent sheaf $\mathcal{G}$, there is a finitely generated S_0-module M, an integer d, and a surjection of quasi-coherent sheaves,

$$(M \otimes_{S_0} S[-d])^{\sim} \to \mathcal{G}.$$

This is essentially equivalent to saying the sheaf $\widetilde{S[1]}$ is an *ample* sheaf.

The second important theorem is that, with the hypotheses above, for every coherent sheaf $\mathcal{G}$ on $\text{Proj}(S)$, the S_0-module $\mathcal{G}(\text{Proj}(S))$ is a finite S_0-module.

Lecture 12. March 20, 2006

Homework. Problem Set 6

Given a scheme B and a quasi-coherent sheaf of $\mathcal{O}_B$-algebras, $\mathcal{A}$, there is a *relative Spec*, $\pi : \text{Spec}_{\ B}(\mathcal{A}) \to B$. This is an affine morphism together with an isomorphism of quasi-coherent sheaves of $\mathcal{O}_B$-algebras, $\mathcal{A} \to \pi_*\mathcal{O}_{\text{Spec}_{\ B}(\mathcal{A})}$. By adjointness, this determines a map of quasi-coherent sheaves of algebras, $\phi : \pi^*\mathcal{A} \to \mathcal{O}_{\text{Spec}_{\ B}(\mathcal{A})}$. The pair

$$(\pi : \text{Spec}_{\ B}(\mathcal{A}), \phi)$$

has a universal property. For every pair $(f : T \to B, \psi)$ of a morphism of locally ringed spaces $f : T \to B$ and an map of quasi-coherent sheaves of algebras $\psi : f^*\mathcal{A} \to \mathcal{O}_T$, there is a unique morphism $g : T \to \text{Spec}_{\ B}(\mathcal{A})$ such that $\pi \circ g = f$ and $g^*\phi$ equals ψ (in the obvious sense). This is the universal property of the relative Spec. Observe that when B is itself an affine scheme, this reduces to the usual universal property of the affine scheme $\text{Spec}_{\ B}(\mathcal{A})$.

An important special case is when $\mathcal{A}$ is the symmetric algebra $\text{Sym}_{\mathcal{O}_B}(\mathcal{F})$ of a quasi-coherent sheaf $\mathcal{F}$ on B. Then, by the universal property of the symmetric algebra, a map of quasi-coherent sheaves of algebras $\psi : f^*\mathcal{A} \to \mathcal{O}_T$ is equivalent to map of quasi-coherent sheaves $\psi_1 : f^*\mathcal{F} \to \mathcal{O}_T$. Therefore the pair $(\text{Spec}_{\ B}(\mathcal{A}), \phi_1)$ has the following universal property. For every pair $(f : T \to B, \alpha)$ of a morphism f of locally ringed spaces and a map of quasi-coherent sheaves $\alpha : f^*\mathcal{F} \to \mathcal{O}_T$, there exists a unique morphism $g : T \to \text{Spec}_{\ B}(\mathcal{A})$ such that $\pi \circ g = f$ and $g^*\phi_1$ equals α (in the obvious sense).

In particular, for every open subset U of B, the set of morphisms $s : U \to \text{Spec}_{\ B}(\mathcal{A})$ such that $\pi \circ s$ equals the inclusion is canonically bijective to the set of maps of quasi-coherent sheaves $\mathcal{F}_U \to \mathcal{O}_U$. In other words, the set is canonically bijective to the $\mathcal{O}_U(U)$-module $\mathit{Hom}_{\mathcal{O}_B}(\mathcal{F}, \mathcal{O}_B)(U)$. Thus the sheaf of sections of π is canonically bijective to the sheaf $\mathit{Hom}_{\mathcal{O}_B}(\mathcal{F}, \mathcal{O}_B)$. This is sufficiently close to the original definition of the vector bundle associated to a sheaf to motivate the notation $\mathbb{E}(\mathcal{F})$ for $\text{Spec}_{\ B}(\mathcal{A})$. However, this certainly need not be a vector bundle! It has become conventional to call $\mathbb{E}(\mathcal{F})$ the *Abelian cone* associated to $\mathcal{F}$.

It must be said that the original convention for the vector bundle, i.e., preceding EGA, Hartshorne, etc., was such that the sheaf of sections of $\mathbb{E}(\mathcal{F})$ is isomorphic to the sheaf $\mathcal{F}$. However, there need not be any Abelian cone with this property. There is an Abelian cone with this property precisely when $\mathcal{F}$ is locally free of finite rank. In this case, the *classical vector bundle* associated to $\mathcal{F}$ is the Abelian cone,

$$\mathbb{E}^{\text{classical}}(\mathcal{F}) := \mathbb{E}(Hom_{\mathcal{O}_B}(\mathcal{F}, \mathcal{O}_B)).$$

In fact, the Abelian cone $\mathbb{E}(\mathcal{F})$ is so much more useful than the classical vector bundle, that the new convention is used more often than the original convention.

For the discussion from the previous lecture, a very important case is $\mathbb{E}(\mathcal{L})$ where $\mathcal{L}$ is an invertible sheaf. Then locally $\mathbb{E}(\mathcal{L})$ is isomorphic to $\mathbb{A}^1 \times B$. There is a natural scaling action of $\mathbb{G}_m$ on this rank 1 vector bundle. The zero section $0 : B \to \mathbb{E}(\mathcal{L})$ is the unique section pulling back ϕ_1 to the zero map $\mathcal{L} \xrightarrow{0} \mathcal{O}_B$. Locally on B, this just corresponds to $\{0\} \times B$ in $\mathbb{A}^1 \times B$. The complement of the zero section is an open subscheme $T \subset \mathbb{E}(\mathcal{L})$ which is preserved by the $\mathbb{G}_m$-action. Zariski locally on B, this action is isomorphic to the left regular action of $\mathbb{G}_m$ on $\mathbb{G}_m \times B$. Therefore $T \to B$ is a $\mathbb{G}_m$-torsor. The algebra $\pi_*\mathcal{O}_T$ is the quasi-coherent sheaf of $\mathcal{O}_B$-algebras,

$$\pi_*\mathcal{O}_T = \oplus_{d\in\mathbb{Z}}\mathcal{L}^{\otimes d},$$

where we define $\mathcal{L}^{\otimes(-d)} := Hom_{\mathcal{O}_B}(\mathcal{L}^{\otimes d}, \mathcal{O}_B)$ for d positive. Clearly this is locally isomorphic to $\mathcal{O}_B[t, t^{-1}]$ as a graded algebra of quasi-coherent sheaves. This is another way to see that T is a $\mathbb{G}_m$-torsors.

Conversely, given a $\mathbb{G}_m$-torsor $\pi : T \to B$, the algebra $\mathcal{A} := \pi_*\mathcal{O}_T$ is a graded algebra of quasi-coherent sheaves. Define $\mathcal{L}$ to be the degree 1 graded subsheaf. In the case above, this will precisely recover the invertible sheaf we started with. In the general case, one can prove by flat descent that $\mathcal{L}$ is an invertible sheaf, the induced map $\mathcal{L} \to \mathcal{A}$ extends uniquely to a map of graded algebras of quasi-coherent sheaves,

$$\oplus_{d\in\mathbb{Z}}\mathcal{L}^{\otimes d} \to \mathcal{A},$$

and the induced map $T \to \mathbb{E}(\mathcal{L})$ is an isomorphism onto the complement of the zero section.

Together these 2 maps determine a bijection between the category of $\mathbb{G}_m$-torsors over B and the category of invertible sheaves on B. In particular, the $\mathbb{G}_m$-torsor $\pi : \text{Spec } S - \mathbb{V}(S_+) \to \text{Proj}(S)$ determines an invertible sheaf $\mathcal{O}(1)$ on $\text{Proj}(S)$. This is isomorphic to the sheaf $\widetilde{(}S[1])$.

For a scheme X, a morphism $f : X \to \text{Proj}(S)$ determines a $\mathbb{G}_m$-torsor $T = X \times_{\text{Proj}(S)} U \to X$ and a morphism from T to the affine scheme Spec S factoring through the open subset U. This may sound convoluted, but using the universal property of $\mathbb{E}(\mathcal{L})$, the universal property of Spec S and the universal property of an open immersion, it translates into a very natural condition. A morphism $f : X \to \text{Proj}(S)$ is equivalent to a triple $(g, \mathcal{L}, \phi)$ of a morphism $g : X \to \text{Spec } S_0$, an invertible sheaf $\mathcal{L}$ on X, and a surjection $\phi : S_1 \otimes_{S_0} \mathcal{O}_X \to \mathcal{L}$ such that the induced map of algebras,

$$\text{Sym}_{S_0}(S_1) \otimes_{S_0} \mathcal{O}_X \to \oplus_{d\geq 0}\mathcal{L}^{\otimes d},$$

factors through the quotient algebra $S \otimes_{S_0} \mathcal{O}_X$. Such triples are up to equivalence where $(g, \mathcal{L}', \phi')$ is equivalent to a triple $(g, \mathcal{L}'', \phi'')$ if and only if there exists an isomorphism $\psi : \mathcal{L}' \to \mathcal{L}''$ such that $\phi'' = \psi \circ \phi'$.

This becomes particularly simple when $S = \text{Sym}_{S_0}(S_1)$. In this case $\text{Proj}(S)$ is usually denoted $\mathbb{P}_{S_0}(S_1)$. The universal property is that a morphism $f : X \to \mathbb{P}_{S_0}(S_1)$ is equivalent to a triple

$(g, \mathcal{L}, \phi)$ as above, where there is no additional condition on ϕ. The most important case is when S_0 equals $\mathbb{Z}$ and S_1 is the finite free $\mathbb{Z}$-module with basis given by variables $X_0, \dots, X_n$. In this case S is simply the polynomial ring $\mathbb{Z}[X_0, \dots, X_n]$. The Proj scheme is denoted $\mathbb{P}^n_{\mathbb{Z}}$. And the universal property is that a morphism $f : X \to \mathbb{P}^n_{\mathbb{Z}}$ is equivalent to a pair $(\mathcal{L}, \phi)$ of an invertible sheaf $\mathcal{L}$ on X and a surjection of quasi-coherent sheaves,

$$\phi : \mathcal{O}_X^{\oplus(n+1)} \to \mathcal{L}.$$

Because of this universal property, it is imperative to understand pairs $(\mathcal{L}, \sigma)$ of an invertible sheaf $\mathcal{L}$ and a section σ of $\mathcal{L}$. Indeed, the image under ϕ of the variables $X_0, \dots, X_n$ are sections $\sigma_0, \dots, \sigma_n$. The geometric objects corresponding to such a pair are Weil divisors and Cartier divisors.

Let X be a scheme. A *prime Weil divisor* is an integral, closed subscheme D of X such that the local ring $\mathcal{O}_{X,\eta_D}$ has pure dimension 1. Equivalently, D does not equal any irreducible component of X, and the only integral closed subschemes of X strictly containing D are irreducible components of X. A *Weil divisor* is a formal finite $\mathbb{Z}$-linear combination of prime Weil divisors. It is effective if every coefficient is nonnegative. The set of all Weil divisors forms a free Abelian group generated by the set of all prime Weil divisors. The notation for a Weil divisor is usually,

$$n_1[D_1] + \cdots + n_r[D_r].$$

It turns out this is much more useful when X is an integral, separated, Noetherian scheme that is *regular in codimension 1*. This precisely says that for every prime Weil divisor D on X, the local ring $\mathcal{O}_{X,\eta_D}$ is a discrete valuation ring. The fraction field is, of course, $K(X) := \mathcal{O}_{X,\eta_X}$. Denote the valuation by v_D. This then determines a homomorphism of Abelian groups,

$$(\bullet) : K(X)^* \to \oplus_D \mathbb{Z} \cdot [D]$$

where $K(X)^*$ is the multiplicative group of nonzero elements of $K(X)$, where $\oplus_D \mathbb{Z} \cdot [D]$ is notation for the group of Weil divisors, and where

$$(f) := \sum_D v_D(f)[D].$$

It is straightforward to see that this is a finite sum, i.e., for all but finitely many D, $v_D(f)$ equals 0. A Weil divisor is called *principal* if it is in the image of this map.

It is straightforward to check that the kernel of $(\bullet)$ is $\mathcal{O}_X(X)^*$, the group of invertible elements in $\mathcal{O}_X(X)$. The cokernel is denoted by $\mathrm{Cl}(X)$, and called the *divisor class group* of X. Altogether this gives an exact sequence of Abelian groups,

$$1 \longrightarrow \mathcal{O}_X(X)^* \longrightarrow K(X)^* \longrightarrow \oplus_D \mathbb{Z} \cdot [D] \longrightarrow \mathrm{Cl}(X) \longrightarrow 0.$$

A basic result in commutative algebra states that a Noetherian integral domain A is a unique factorization domain if and only if its class group is zero, i.e., $\mathrm{Cl}(\mathrm{Spec}\ A)$ is (0).

Closely related are *Cartier divisors*. Let X be a scheme. Denote by $\mathcal{K}$ the sheaf of total rings of fractions, i.e., the sheafification of the presheaf associating to every open affine U the total ring of fractions of $\mathcal{O}_X(U)$. An *effective Cartier divisor* is a sub-$\mathcal{O}_X$-module $\mathcal{L}$ of $\mathcal{K}$ that contains the subsheaf $\mathcal{O}_X$ and is an invertible $\mathcal{O}_X$-module.

How do such things arise? Let $\mathcal{L}$ be any invertible sheaf and let σ be a section of $\mathcal{L}$ such that the open subset $D(\sigma)$ contains every generic point of X, i.e., the generic point of every irreducible component of X. The induced isomorphism $\mathcal{O}_{D(\sigma)} \to \mathcal{L}|_{D(\sigma)}$ determines an isomorphism $\mathcal{K}|_{D(\sigma)} \to \mathcal{K} \otimes_{\mathcal{O}_D} \mathcal{L}_| D(\sigma)$. The inverse naturally extends to an injective sheaf map $\mathcal{L} \hookrightarrow \mathcal{K}$ such that the composition $\mathcal{O}_X \xrightarrow{\sigma} \mathcal{L} \to \mathcal{K}$ is the usual inclusion. This determines a 1-to-1 bijection between effective Cartier divisors on X and equivalence classes of pairs $(\mathcal{L}, \sigma)$ of an invertible sheaf $\mathcal{L}$ and a section σ such that $D(\sigma)$ contains every generic point of X.

Assume now that X is an integral, separated, Noetherian scheme that is regular in codimension 1. Given an effective Cartier divisor $\mathcal{L} \subset \mathcal{K}$ there is an associated effective Weil divisor defined as follows. For every prime Weil divisor D of X, there exists an open affine U of X intersecting D and a trivialization $\mathcal{L}|_U \cong \mathcal{O}_U$. Then the restriction of the section σ is equivalent to an element f of $\mathcal{O}_U(U)$. Define $v_D(\mathcal{L}) = v_D(f)$. Changing the choice of trivialization multiplies f by an element u that is invertible on U. Therefore $v_D(u) = 0$ so that $v_D(uf) = v_D(f)$. Therefore the integer $v_D(\mathcal{L})$ is independent of the choice of open subset U and the choice of trivialization. Moreover, it is easy to see that $v_D(\mathcal{L})$ is nonzero if and only if D is an irreducible component of the support of the cokernel of $\sigma : \mathcal{O}_X \to \mathcal{L}$. Since this is a closed subset of a Noetherian scheme, it has only finitely many irreducible components. Therefore the sum,

$$\sum_D v_D(\mathcal{L})[D]$$

is an effective Weil divisor.

We will see next time that if X is regular, i.e., all local rings $\mathcal{O}_{X,p}$ are regular, then every effective Weil divisor comes from a Cartier divisor, and the Cartier divisor is unique. Thus, the Weil divisor associated to a pair $(\mathcal{L}, \sigma)$ is a "geometric" version of the "algebraic" object $(\mathcal{L}, \sigma)$.

Lecture 13. March 22, 2006

Homework. Problem Set 7

Defined general Cartier divisors and equivalent reformulations. Proved that if X is integral, every invertible sheaf comes from a Cartier divisor. Defined the map from Cartier divisors to Weil divisors assuming condition (*). Proved that if X is normal, this map is injective (modulo a mistake corrected in the problem set). Proved that if X is locally factorial, this map is bijective. In particular this holds if X is regular. Computed the Picard group of $\mathbb{P}^n_k$ is $\mathbb{Z}$ (for $n > 0$).

Recalled the universal property of projective space. Used the computation of $\mathrm{Pic}(\mathbb{P}^n_k)$ to prove the automorphism group of $\mathbb{P}^n_k$ equals $\mathrm{PGL}_{n+1,k}$. Defined ample and very ample invertible sheaves. Recalled Serre's theorem that a very ample invertible sheaf is ample. Stated the theorem that some tensor power of an ample invertible sheaf is very ample.

Lecture 14. April 3, 2006

Homework. Problem Set 7

Hints for Problem Set 7, Part I. (a). Let R be a DVR and let $v_R : K(R)^* \to \mathbb{Z}$ be the associated valuation (in particular, $v_R(\pi)$ equals 1 for every uniformizer π). Let f be a nonzero element of R. What is the relation between the following 2 integers

$$\text{length}(R/fR) \text{ and } v_R(f)?$$

This is essentially the only thing being asked in this problem.

(b). The sketch in Hartshorne is perfectly adequate. Here is a slightly different perspective. First change coordinates as follows (this isn't really necessary)

$$\begin{cases} x_{\text{new}} & = & x+y \\ y_{\text{new}} & = & x-y \\ z_{\text{new}} & = & -8z \end{cases}$$

The equation of X in the new coordinates is,

$$f(x,y,z) = (x+y)^3 - xyz = 0.$$

The reason for the coordinate change is that this equation is a nodal plane cubic no matter what the characteristic of k (the previous equation only worked if the characteristic is not 2).

Now let $\nu : \mathbb{P}^1 \to \mathbb{P}^2$ be the morphism

$$[T_0, T_1] \mapsto [x,y,z] = [T_0^2T_1, T_0T_1^2, (T_0+T_1)^3].$$

You should check that $\nu(\mathbb{P}^1)$ equals X. Denote by Z the unique singular point $[0,0,1]$ of X. Denote by 0 the point $[1,0]$ of $\mathbb{P}^1$ and denote by ∞ the point $[0,1]$ of $\mathbb{P}^1$. As usual, denote $\mathbb{G}_m = \mathbb{P}^1 - \{0,\infty\}$. Then $\nu^{-1}(X - \{Z\})$ equals $\mathbb{G}_m$ and the induced morphism $\nu : \mathbb{G}_m \to X - \{Z\}$ is an isomorphism. Of course $\nu(0)$ and $\nu(\infty)$ equal Z. Thus ν is the normalization of X.

There is a short exact sequence of $\mathcal{O}_X$-modules,

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\nu^\#} \nu_*\nu^*\mathcal{O}_X \xrightarrow{\delta} \mathcal{O}_Z \longrightarrow 0.$$

The map δ sends a section s of $\mathcal{O}_{\mathbb{P}^1}(\nu^{-1}(U))$ to the section $s(\infty) - s(0)$ of the skyscraper sheaf $\mathcal{O}_Z$, assuming Z is in U. For every invertible sheaf $\mathcal{L}$, tensoring the sequence above by $\mathcal{L}$ gives a short exact sequence of $\mathcal{O}_X$-modules,

$$0 \longrightarrow \mathcal{L} \longrightarrow \nu_*\nu^*\mathcal{L} \xrightarrow{\delta_{\mathcal{L}}} \mathcal{L} \otimes \mathcal{O}_Z \longrightarrow 0.$$

For every k-point t of $\mathbb{P}^1$, there is an evaluation map,

$$e_t : H^0(\mathbb{P}^1, \nu^*\mathcal{L}) \to \nu^*\mathcal{L} \otimes \mathcal{O}_t.$$

Moreover, there is a canonical isomorphism,

$$f_t : \nu^*\mathcal{L} \otimes \mathcal{O}_t \to \mathcal{L} \otimes \mathcal{O}_{\nu(t)}.$$

In particular, because $\nu(0)$ equals $\nu(\infty)$ equals Z, there are 2 canonically defined maps,

$$f_0 \circ e_0, f_\infty \circ e_\infty : H^0(\mathbb{P}^1, \nu^*\mathcal{L}) \to \mathcal{L} \otimes \mathcal{O}_Z.$$

For every invertible sheaf $\mathcal{L}$, the map $H^0(\delta_{\mathcal{L}})$ induced by $\delta_{\mathcal{L}}$ on global sections is simply the difference, $\delta_{\mathcal{L}} = f_\infty \circ e_\infty - f_0 \circ e_0$.

Using our classification of invertible sheaves on $\mathbb{P}^1$, $\nu^*\mathcal{L}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(d)$ for some unique integer d, the *degree* of $\mathcal{L}$. If $\mathcal{L}$ has degree 0 then $\nu^*\mathcal{L}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}$, and thus $H^0(\mathbb{P}^1, \mathcal{L})$ is isomorphic to $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$. In other words, $\dim_k H^0(\mathbb{P}^1, \mathcal{L})$ equals 1. In this case, $f_t \circ e_t$ is an isomorphism for every k-point t of $\mathbb{P}^1$. In particular, each of $f_\infty \circ e_\infty$ and $f_0 \circ e_0$ is an isomorphism with the same 1-dimensional domain and target. Therefore, there exists a unique element λ of $k^* = \mathbb{G}_m(k)$ such that $f_\infty \circ e_\infty$ equals $\lambda \cdot (f_0 \circ e_0)$. This determines a map,

$$\Lambda : \mathrm{CaCl}^0(X) \to \mathbb{G}_m(k), \quad \mathcal{L} \mapsto \lambda.$$

The map Λ is a group homomorphism (you should check this).

Since $H^0(\delta_{\mathcal{L}})$ equals $f_\infty \circ e_\infty - f_0 \circ e_0$, which is $(\lambda - 1) \cdot f_0 \circ e_0$, the map $H^0(\delta_{\mathcal{L}})$ is an isomorphism if $\lambda \neq 1$, and is the zero map if $\lambda = 1$. Thus the kernel of $H^0(\delta_{\mathcal{L}})$, namely $H^0(X, \mathcal{L})$, is zero if $\lambda \neq 1$, and is 1-dimensional if λ equals 1. Morevoer, when λ equals 1, any nonzero global section of $\mathcal{L}$ pulls back to a generator of $\nu^*\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^1}$, and thus is already a generator of $\mathcal{L}$. Since $\Lambda(\mathcal{L})$ equals 1 if and only if $\mathcal{L}$ is isomorphic to $\mathcal{O}_X$, Λ is an injective group homomorphism.

Finally, for every λ in $\mathbb{G}_m(k)$, the invertible sheaf $\mathcal{L} = \mathcal{O}_X(\nu(\lambda) - \nu(1))$ has image $\Lambda(\mathcal{L}) = \lambda$. Therefore Λ is also surjective, thus an isomorphism of groups.

(c) Associated to every commutative diagram of locally free resolutions,

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{E}_1 \arrow[r, "\alpha"] \arrow[d, "\phi_1"'] & \mathcal{E}_0 \arrow[r, "\beta"] \arrow[d, "\phi_0"'] & \mathcal{F} \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & \mathcal{E}'_1 \arrow[r, "\alpha'"] & \mathcal{E}'_0 \arrow[r, "\beta'"] & \mathcal{F} \arrow[r] & 0 \end{tikzcd}$$

there is a short exact sequence of locally free $\mathcal{O}_X$-modules,

$$0 \longrightarrow \mathcal{E}_1 \xrightarrow{(\alpha,\phi_1)^\dagger} \mathcal{E}_0 \oplus \mathcal{E}'_1 \xrightarrow{(\phi_0, -\alpha')} \mathcal{E}'_0 \longrightarrow 0.$$

By Exercise II.5.16, there are isomorphisms,

$$\bigwedge^{r'_0} \mathcal{E}'_0 \otimes \bigwedge^{r_1} \mathcal{E}_1 \cong \bigwedge^{r_0 + r'_1} (\mathcal{E}_0 \oplus \mathcal{E}'_1) \cong \bigwedge^{r_0} \mathcal{E}_0 \otimes \bigwedge^{r'_1} \mathcal{E}'_1.$$

This gives isomorphisms,

$$\bigwedge^{r'_0} \mathcal{E}'_0 \otimes (\bigwedge^{r'_1} \mathcal{E}'_1)^\vee \cong \bigwedge^{r_0} \mathcal{E}_0 \otimes (\bigwedge^{r_1} \mathcal{E}_1)^\vee.$$

Therefore the determinant of $\mathcal{F}$ is the same for each of these resolutions.

In the general case, if there are two resolutions but not necessarily a commutative diagram of resolutions, form the third resolutions,

$$0 \longrightarrow \mathrm{Ker}(\beta,\beta') \longrightarrow \mathcal{E}_0 \oplus \mathcal{E}_0' \xrightarrow{(\beta,\beta')} \mathcal{F} \longrightarrow 0.$$

There is a commutative diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E}_1 & \xrightarrow{\alpha} & \mathcal{E}_0 & \xrightarrow{\beta} & \mathcal{F} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle (\mathrm{Id},0)^\dagger} & & \downarrow{\scriptstyle =} & & \\ 0 & \longrightarrow & \mathrm{Ker}(\beta,\beta') & \longrightarrow & \mathcal{E}_0 \oplus \mathcal{E}_0' & \xrightarrow{(\beta,\beta')} & \mathcal{F} & \longrightarrow & 0. \end{array}$$

And similarly for the second resolution. Use the previous paragraph to deduce the determinant of the first and second resolutions are each isomorphic to the determinant of the third resolution, hence isomorphic to one another.

Lecture. Spent the entire lecture proving the important theorem that for a finitely presented, quasi-separated morphism $f : X \to \mathrm{Spec}\, A$, an invertible sheaf $\mathcal{L}$ is f-very ample if and only if it is ample. Generalized this to the case that the target is quasi-compact, but not necessarily affine. This is the last lecture specifically on material from Chapter II. The students are responsible for the remaining material from Section 7. The material in Section 8 will be discussed as it arises in Chapter III.

Lecture 15. April 5, 2006

Homework. Problem Set 8

Some good references for homological algebra are "Homological algebra" by Cartan and Eilenberg, "An introduction to homological algebra" by Charles Weibel, "Sur quelques points d'algèbre homologique" by Grothendieck (in the Tohoku Math. Journal), and "Des catégories dérivées des catégories abéliennes" by J.-L. Verdier.

For every Abelian category $\mathcal{A}$ there is a notion of *cochain complexes* of objects in $\mathcal{A}$, namely a pair

$$A = ((A^p)_{p\in\mathbb{Z}}, (d_A^p)_{p\in\mathbb{Z}})$$

of a sequence $(A^p)_{p\in\mathbb{Z}}$ of objects of $\mathcal{A}$ and a sequence $d_A^p : A^p \to A^{p+1}$ of morphisms in $\mathcal{A}$ such that for every p, $d_A^{p+1} \circ d_A^p$ is the zero map. Sometimes the object A^p is called the *p-cochains of A*. The morphism d_A^p is called the *p^{th} differential of A*. A morphism of cochain complexes $u : A \to B$ is a sequence $(u^p : A^p \to B^p)_{p\in\mathbb{Z}}$ of morphisms in $\mathcal{A}$ such that for every p,

$$d_B^p \circ u^p = u^{p+1} \circ d_A^p.$$

There are obvious notions of identity morphism, composition of morphisms, addition and subtraction of morphisms (with the same domain and target), zero morphisms, etc. Together these notions form an Abelian category $\mathrm{Ch}^\bullet(\mathcal{A})$ of cochain complexes of objects in $\mathcal{A}$.

Let m be either an integer or $-\infty$, let n be either an integer or $+\infty$, and assume $m \leq n$. A complex A is said to be *concentrated in degrees $m \leq p \leq n$* if $A^p = 0$ for p outside this range.

The complexes concentrated in degree $m \leq p \leq n$ form a full subcategory of $\mathrm{Ch}^\bullet(\mathcal{A})$ denote by $\mathrm{Ch}^{[m,n]}(\mathcal{A})$, resp. $\mathrm{Ch}^{\leq n}(\mathcal{A})$ if $m = -\infty$ and $\mathrm{Ch}^{\geq m}(\mathcal{A})$ if $n = +\infty$. These categories are not necessarily preserved by $[+1]$ and $[-1]$, although there are obvious conditions on $m \leq n$ and $r \leq s$ insuring $[+1]$, resp. $[-1]$, sends $\mathrm{Ch}^{[m,n]}(\mathcal{A})$ into $\mathrm{Ch}^{[r,s]}(\mathcal{A})$.

This category $\mathrm{Ch}^\bullet(\mathcal{A})$ has many important extra structures. There is a *translation functor*

$$[+1] : \mathrm{Ch}^\bullet(\mathcal{A}) \to \mathrm{Ch}^\bullet(\mathcal{A}), \quad A[+1] = ((A^{p+1})_{p\in\mathbb{Z}}, (-d_A^{p+1})_{p\in\mathbb{Z}}); \quad u[+1] = (u^{p+1})_{p\in\mathbb{Z}}.$$

This is an exact additive functor which has an inverse functor $[-1]$ (which is also an exact additive functor). For every integer p there is a *cohomology functor*

$$h^p : \mathrm{Ch}^\bullet(\mathcal{A}) \to \mathcal{A}, \quad h^p(A) = \mathrm{Ker}(d_A^p)/\mathrm{Image}(d_A^{p-1}).$$

It is an exercise to check that for every morphism $u : A \to B$, there is a unique morphism $h^p(u) : h^p(A) \to h^p(B)$ such that the following diagram commutes,

$$\begin{array}{ccc}
\mathrm{Ker}(d_A^p) & \xrightarrow{\phi^p} & B^p/\mathrm{Image}(d_B^{p-1}) \\
{\scriptstyle\text{nat}}\downarrow & & \uparrow{\scriptstyle\text{nat}} \\
h^p(A) & \xrightarrow{h^p(u)} & h^p(B)
\end{array}$$

where nat denotes the natural monomorphism, resp. epimorphism. The cohomology functor is additive. It is typically not exact, but it is *half-exact*, i.e., for every short exact sequence of complexes,

$$0 \longrightarrow A' \xrightarrow{u} A \xrightarrow{v} A'' \longrightarrow 0,$$

the following diagram is exact in the middle,

$$h^p(A') \xrightarrow{h^p(u)} h^p(A) \xrightarrow{h^p(v)} h^p(A'').$$

There are obvious natural isomorphisms of functors,

$$h^p \circ [+1] \Rightarrow h^{p+1} : \mathrm{Ch}^\bullet(\mathcal{A}) \to \mathcal{A}, \quad h^p(A[+1]) = h^{p+1}(A), \quad h^p(u[+1]) = h^{p+1}(u).$$

In addition to the notion of isomorphism of cochain complexes, there is the notion of *quasi-isomorphism*, or *qism* for short: for every pair of integers $m \leq n$, a morphism of complexes $u : A \to B$ is a quasi-isomorphism in degrees $m \leq p \leq n$ if for every integer $m \leq p \leq n$ the induced morphism $h^p(u)$ is an isomorphism. It is a quasi-isomorphism in degrees $m \leq p \leq \infty$, resp. $-\infty \leq p \leq n$, resp. in all degrees, if for every $m \leq n$ it is a quasi-isomorphism in degrees $m \leq p \leq n$. Compositions and translations of quasi-isomorphisms are quasi-isomorphisms.

There is a strong analogy between Abelian categories and rings: the elements of the rings being analogous to the morphisms of the Abelian category: indeed, for every Abelian category $\mathcal{A}$ and every object C of $\mathcal{A}$, the set of all endomorphisms of C is a ring. For this analogy, the class of all quasi-isomorphisms is roughly an analogue of a multiplicative subset of the ring: the self-quasi-isomorphisms of $C^\bullet$ is a multiplicative subset of the ring of endomorphisms of $C^\bullet$.

Given a morphism of complexes $u : A \to B$, a *null homotopy* or *cochain contraction* of u is a sequence of morphisms $(s^p : A^{p+1} \to B^p)_{p \in \mathbb{Z}}$ such that for every integer p,

$$u^p = d_B^{p-1} \circ s^{p-1} + s^p \circ d_A^p.$$

If there exists a null homotopy of ϕ, ϕ is called *null homotopic*. For the analogy with rings, the class of all null homotopic morphisms is analogous to an ideal. Indeed, for every morphism of complexes $v : B \to B''$, the sequence $(v^p \circ s^p : A^{p+1} \to (B'')^p)_{p\in\mathbb{Z}}$ is a null homotopy of $v \circ u$. And for every morphism $t : A' \to A$, the sequence $(s^p \circ t^{p+1} : (A')^{p+1} \to B^p)_{p\in\mathbb{Z}}$ is a null homotopy of $u \circ t$. Analogous to the notion of congruence modulo an ideal, a pair of morphisms of complexes $u_1, u_2 : A \to B$ are called *homotopic* if $u_2 - u_1$ is null homotopic, and a *homotopy from* u_1 *to* u_2 is a null homotopy of $u_2 - u_1$. A pair of morphisms of complexes $u : A \to B$, $v : B \to A$ is a *homotopy equivalence* if $v \circ u$ is homotopic to Id_A and $u \circ v$ is homotopic to Id_B. Null homotopies are compatible with translation. Part of the importance of null homotopies is that if u is null homotopic, then for every integer p, the morphism $h^p(u)$ is the zero morphism. In particular, it follows that homotopy equivalences are quasi-isomorphisms.

There are a few other important notions that were not discussed in lecture. Most important is the *mapping cone*. Given a morphism of cochain complexes $u : A \to B$, the mapping cone is the complex $\mathrm{Cone}(u)$ whose terms are

$$\mathrm{Cone}(u)^p = A^{p+1} \oplus B^p$$

and whose differentials are

$$d^p_{\mathrm{Cone}} : A^{p+1} \oplus B^p \to A^{p+2} \oplus B^{p+1},$$

$$d^p_{\mathrm{Cone}} = \begin{pmatrix} -d_C^{p+1} & 0 \\ u^{p+1} & d_D^p \end{pmatrix}$$

There is a natural short exact sequence of complexes,

$$0 \longrightarrow B \xrightarrow{v} \mathrm{Cone}(u) \xrightarrow{w} A[+1] \longrightarrow 0$$

where $v = v(u)$ and $w = w(u)$ are defined by

$$v^p = (0, \mathrm{Id}_{B^p})^\dagger, \quad w^p = (\mathrm{Id}_{A^{p+1}}, 0).$$

There are several important properties of the mapping cone. First note that, term-by-term, the short exact sequence above is split. However, it is not necessarily split as a short exact sequence of complexes, i.e., the term-by-term splittings do not necessarily commute with the differentials. Indeed, a sequence of morphisms

$$\kappa : \mathrm{Cone}(u) \to B, \quad \kappa^p = (s^p, \mathrm{Id}_{B^p}) : A^{p+1} \oplus B^p \to B^p$$

commutes with the differentials if and only if $(s^p)_{p\in\mathbb{Z}}$ is a null homotopy of u, i.e., the null homotopies of u are in natural bijection with the splittings of the exact sequence above.

Second, the mapping cone Cone(u) is exact if and only if u is a quasi-isomorphism. This is an exercise in diagram-chasing.

Third, for every pair of morphisms of complexes,

$$A \xrightarrow{u} B \xrightarrow{t} C,$$

such that $t \circ u = 0$, there is a morphism of complexes $\widetilde{t} : \mathrm{Cone}(u) \to C$ given by

$$(\widetilde{t})^p : A^{p+1} \oplus B^p \to C^p, \quad (\widetilde{t})^p = (0, t^p).$$

If the pair of morphisms comes from an exact sequence,

$$0 \longrightarrow A \xrightarrow{u} B \xrightarrow{t} C \longrightarrow 0,$$

then $\widetilde{t}$ is a quasi-isomorphism.

Fourth, there is a natural isomorphism of Cone$(u)[+1]$ with Cone$(-u[+1])$. Finally, there is a morphism

$$i = i(u) : A[+1] \to \mathrm{Cone}(v(u)), \quad i^p : A^{p+1} \to B^{p+1} \oplus A^{p+1} \oplus B^p, \quad i^p = (-u^{p+1}, \mathrm{Id}_{A^{p+1}}, 0),$$

whose composition with $\widetilde{w(u)} : \mathrm{Cone}(v(u)) \to A[+1]$ is the identity $\mathrm{Id}_{A[+1]}$. And there is a homotopy (s^p) from $i \circ \widetilde{w}$ to $\mathrm{Id}_{\mathrm{Cone}(v(u))}$,

$$s^p : B^{p+2} \oplus A^{p+2} \oplus B^{p+1} \to B^{p+1} \oplus A^{p+1} \oplus B^p$$

$$s^p = \begin{pmatrix} 0 & 0 & \mathrm{Id}_{B^{p+1}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus $i(u)$ and $\widetilde{w(u)}$ give a homotopy equivalence of Cone$(v(u))$ and $A[+1]$.

Let $F : \mathcal{A} \to \mathcal{B}$ be a half-exact additive functor of Abelian categories. The basic problem of homological algebra is that F is not necessarily exact. One solution to this problem (if it exists) is the notion of a *universal (cohomological) δ-functor* (a better solution is given by δ-functors of triangulated categories). A δ-functor from $\mathcal{A}$ to $\mathcal{B}$ is a datum $((F^p)_{p\in\mathbb{Z}}, (\delta^p)_{p\in\mathbb{Z}})$ of a sequence of half-exact additive functors $F^p : \mathcal{A} \to \mathcal{B}$ together with a sequence of assignments δ^p to every short exact sequence in $\mathcal{A}$,

$$\Sigma : 0 \longrightarrow A' \xrightarrow{u} A \xrightarrow{v} A'' \longrightarrow 0,$$

of morphisms in $\mathcal{B}$,

$$\delta^p_\Sigma : F^p(A'') \to F^{p+1}(A'),$$

satisfying the following two axioms.

(i) For every short exact sequence Σ in $\mathcal{A}$, the following is a long exact sequence in $\mathcal{B}$,

$$\dots \longrightarrow F^p(A') \xrightarrow{F^p(u)} F^p(A) \xrightarrow{F^p(v)} F^p(A'') \xrightarrow{\delta^p_\Sigma} F^{p+1}(A') \longrightarrow \dots$$

(ii) For every commutative diagram of short exact sequences in $\mathcal{A}$,

$$\begin{array}{ccccccccccc} \Sigma : & 0 & \longrightarrow & A' & \xrightarrow{u_A} & A & \xrightarrow{v_A} & A'' & \longrightarrow & 0 \\ \downarrow \alpha_\Sigma & & & \downarrow \alpha' & & \downarrow \alpha & & \downarrow \alpha'' & & \\ \Theta : & 0 & \longrightarrow & B' & \xrightarrow{u_B} & B & \xrightarrow{v_B} & B'' & \longrightarrow & 0 \end{array}$$

for every integer p, the following diagram is commutative,

$$\begin{array}{ccc} F^p(A'') & \xrightarrow{\delta^p_\Sigma} & F^{p+1}(A') \\ \downarrow F^p(\alpha'') & & \downarrow F^{p+1}(\alpha') \\ F^p(B'') & \xrightarrow{\delta^p_\Theta} & F^{p+1}(B') \end{array}$$

Let $m \leq n$ be integers. A δ-functor is *concentrated in degrees* $m \leq p \leq n$, resp. $m \leq p \leq \infty$, $-\infty \leq p \leq n$, if for every p not in this range, F^p is the zero functor. A *morphism of* δ *functors*, $\theta : F \Rightarrow G$ is a collection of additive natural transformations $(\theta^p : F^p \Rightarrow G^p)_{p\in\mathbb{Z}}$ such that for every short exact sequence Σ in $\mathcal{A}$ and every integer p,

$$\delta^p_{G,\Sigma} \circ \theta^p(A'') = \theta^{p+1}(A') \circ \delta^p_{F,\Sigma}.$$

Let m be an integer and let n be either an integer $\geq m$ or else $+\infty$. A *universal cohomological* δ*-functor* concentrated in degrees $m \leq p \leq n$ is a δ-functor $F : \mathcal{A} \to \mathcal{B}$ concentrated in degrees $m \leq p \leq n$ such that for every δ-functor $G : \mathcal{A} \to \mathcal{B}$ concentrated in degrees $m \leq p \leq n$ and every additive natural transformation $\theta^m : F^m \Rightarrow G^m$, there exists a unique morphism $\theta : F \Rightarrow G$ of δ-functors whose m^{th} term is θ^m. The typical case is when $m = 0$ and $n = +\infty$; if the range is not specified, it is $0 \leq p \leq +\infty$ by default.

Similarly, let n be an integer and let m be either an integer $\leq n$ or else $-\infty$. A universal homological δ-functor concentrated in degrees $m \leq p \leq n$ is a δ-functor $G : \mathcal{A} \to \mathcal{B}$ concentrated in degrees $m \leq p \leq n$ such that for every δ-functor $F : \mathcal{A} \to \mathcal{B}$ concentrated in degrees $m \leq p \leq n$ and every additive natural transformation $\theta^n : F^n \Rightarrow G^n$, there exists a unique morphism $\theta : F \Rightarrow G$ of δ-functors whose n^{th} term is θ^n. The typical case is when $m = -\infty$ and $n = 0$; if the range is not specified, it is $-\infty \leq p \leq 0$ by default.

Here is the canonical example of a δ-functor. For every integer p, let $h^p : \mathrm{Ch}^\bullet(\mathcal{A}) \to \mathcal{A}$ be the cohomology functor from above. For every short exact sequence of complexes

$$\Sigma : 0 \longrightarrow A' \xrightarrow{u} A \xrightarrow{u''} A'' \longrightarrow 0$$

the morphism of complexes $\widetilde{u''} : \mathrm{Cone}(u) \to A''$ is a quasi-isomorphism, i.e., the following is an isomorphism,

$$h^p(\widetilde{u''}) : h^p(\mathrm{Cone}(u)) \to h^p(A'').$$

Also, associated to the morphism of complexes $w(u) : \mathrm{Cone}(u) \to A'[+1]$, there is a morphism,

$$h^p(w(u)) : h^p(\mathrm{Cone}(u)) \to h^p(A'[+1]) = h^{p+1}(A').$$

Therefore, there is a unique morphism,

$$\delta^p_\Sigma : h^p(A'') \to h^{p+1}(A')$$

such that $\delta_\Sigma \circ h^p(\widetilde{u''}) = h^p(w(u))$.

Because the cone construction is functorial in u and the association $t \mapsto \widetilde{t}$ is functorial in t, δ^p_Σ satisfies Axiom (ii) of a δ-functor. It remains to check Axiom (i), i.e, it remains to prove exactness of the complex,

$$\dots \longrightarrow h^p(A') \xrightarrow{h^p(u)} h^p(A) \xrightarrow{h^p(v)} h^p(A'') \xrightarrow{\delta^p_\Sigma} h^{p+1}(A') \longrightarrow \dots$$

Because h^p is half-exact and Σ is a short exact sequence, the long exact sequence is exact at $h^p(A)$. Again because h^p is half-exact, exactness of the following exact sequence gives exactness at $h^p(A'')$,

$$0 \longrightarrow A \xrightarrow{v(u)} \mathrm{Cone}(u) \xrightarrow{w(u)} A'[+1] \longrightarrow 0.$$

Finally, using the quasi-isomorphism $\widetilde{w(u)} : \mathrm{Cone}(v(u)) \to A'[+1]$, exactness at $h^{p+1}(A')$ follows from exactness of the following sequence

$$0 \longrightarrow \mathrm{Cone}(u) \xrightarrow{v(v(u))} \mathrm{Cone}(v(u)) \xrightarrow{w(v(u))} A[+1] \longrightarrow 0.$$

For every integer m, the restriction of this δ-functor to the full subcategory $\mathrm{Ch}^{\geq m}(\mathcal{A})$ is a universal cohomological δ-functor concentrated in degrees $p \geq m$. Similarly, for every integer n, the restriction of this δ-functor the full subcategory $\mathrm{Ch}^{\leq n}(\mathcal{A})$ is a universal homological δ-functor concentrated in degrees $p \leq n$.

Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. There are the obvious exact, fully faithful functors $\mathcal{A} \to \mathrm{Ch}^\bullet(A)$ and $\mathcal{B} \to \mathrm{Ch}^\bullet(A)$ sending an object M to $M[0] = ((C^p), (d^p))$ where $C^p = 0$ except for $C^0 = M$, and all d^p are zero. These functors are called the *inclusion functors*. There is a unique additive functor $\mathrm{Ch}^\bullet(F) : \mathrm{Ch}^\bullet(\mathcal{A}) \to \mathrm{Ch}^\bullet(\mathcal{B})$ compatible with the inclusion functors and F and also compatible with the translation functors. Note that the failure of F to preserve short exact sequences is precisely the same as the failure of $\mathrm{Ch}^\bullet(F)$ to preserve quasi-isomorphisms. However, since F preserves split exact sequences, $\mathrm{Ch}^\bullet(F)$ preserves exact sequences that are split term-by-term. Also $\mathrm{Ch}^\bullet(F)$ preserves identity morphisms and homotopies. Therefore $\mathrm{Ch}^\bullet(F)$ preserves homotopy equivalences. In particular, it sends homotopy equivalences to quasi-isomorphisms.

Denote by $\mathrm{Ch}^{\geq 0}(F)$ the restriction of $\mathrm{Ch}^\bullet(F)$ to the full subcategories $\mathrm{Ch}^{\geq 0}(\mathcal{A})$ and $\mathrm{Ch}^{\geq 0}(\mathcal{B})$. For every object A of $\mathcal{A}$, let $A[0] \to I$ be an injective resolution. By standard lemmas of homological algebra, an injective resolution is unique up to homotopy equivalence, and the homotopy equivalence is unique up to a null homotopy. Thus the object $F(I)$ in $\mathrm{Ch}^{\geq 0}(\mathcal{B})$ is well-defined up to homotopy equivalence and null homotopy. Therefore the cohomology groups $h^p(F(I))$ are well-defined in M, up to unique isomorphism, i.e., for any other injective resolution I', there are canonical isomorphisms $h^p(F(I)) \cong h^p(F(I'))$. These objects, essentially well-defined, are called the *right derived functors* of F,

$$R^pF(M) := h^p(F(I)).$$

Given a short exact sequence of objects of $\mathcal{A}$,

$$\Sigma : 0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

and given injective resolutions $M'[0] \to I'$ and $M''[0] \to I''$, the horseshoe lemma of homological algebra asserts there exists a commutative diagram of short exact sequences of complexes,

$$\begin{array}{ccccccccc}
\Sigma : 0 & \longrightarrow & M'[0] & \longrightarrow & M[0] & \longrightarrow & M''[0] & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\Sigma_I : 0 & \longrightarrow & I' & \longrightarrow & I & \longrightarrow & I'' & \longrightarrow & 0
\end{array}$$

It follows that the lower exact sequence is split term-by-term and that $M[0] \to I$ is an injective resolution. Therefore, the complex of complexes,

$$F(\Sigma_I) : 0 \longrightarrow F(I') \longrightarrow F(I) \longrightarrow F(I'') \longrightarrow 0$$

is an exact sequence split term-by-term. In particular, it is exact. Therefore, since $h = ((h^p), (\delta^p))$ is a δ-functor, the following is also a δ-functor from $\mathcal{A}$ to $\mathcal{B}$ concentrated in degrees ≥ 0,

$$R^\bullet F = ((R^p F)_{p\in\mathbb{Z}}, (\delta^p_F)_{p\in M}), \quad R^p F(M) := h^p(F(I)), \quad \delta^p_{F,\Sigma} := \delta^p_{h,F(\Sigma_I)}.$$

Moreover, by essentially the same proof as for h, RF is a universal cohomological δ-functor concentrated in degrees $p \geq 0$. The terms of this δ-functor are called the *right derived functors of* F. In particular $R^0 F$ equals the original functor F.

In general, i.e., without the hypothesis that $\mathcal{A}$ has enough injective objects, if there exists a universal cohomological δ-functor $((F^p), (\delta^p))$ concentrated in degrees $p \geq 0$ and with $F^0 = F$, then it is unique up to unique isomorphism. When it exists, it is called a *right satellite* of F. Thus, when $\mathcal{A}$ has enough injective objects, the construction of the right derived functors proves that F has a right satellite, any of which is canonically isomorphic to the right derived functors.

There is an exactly similar construction of left derived functors. The main examples of derived functors are as follows.

(i) For every Abelian category $\mathcal{A}$ and every object M of $\mathcal{A}$, if they exist, the right satellites of the functor,

$$F : \mathcal{A} \to \text{Ab}, \quad F(N) = \text{Hom}_{\mathcal{A}}(M, N),$$

are called the *Ext groups*, denoted $\text{Ext}^p_I(M, N)$. Even when $\mathcal{A}$ does not have enough injective objects, the Ext groups often exist (one can use Yoneda Ext, for instance).

(ii) Similarly, for every object N of $\mathcal{A}$, the right satellites of the contravariant additive functor,

$$G : \mathcal{A} \to \text{Ab}, \quad G(M) = \text{Hom}_{\mathcal{A}}(M, N),$$

are also called the Ext groups, $\text{Ext}^p_{II}(M, N)$. One of the basic results from homological algebra is that in almost every case, these two definitions of Ext groups agree and together make $\text{Ext}^p(M, N)$ into a bi-δ-functor. In particular, this holds if $\mathcal{A}$ has enough projective and injective objects (but also under much weaker hypotheses).

(ii) Let R be a (not necessarily commutative) ring and let $R-\text{mod}$ be the category of left R-modules. Let M be a right R-module and let $M\otimes_R : R-\text{mod} \to \text{Ab}$ be the right exact functor $N \mapsto M \otimes_R N$. The category $R-\text{mod}$ has enough projective objects (e.g., the free modules). Therefore the left derived functors of F exist. They are called *Tor groups*, and are denote by $\text{Tor}_n^{I,R}(M,N) = L^{-n}F(N)$.

(iii) As with Ext, there are also left derived functors $\text{Tor}_n^{II,R}(M,N)$ of the right exact functor $M \mapsto M\otimes_R N$. As with Ext, there are canonical isomorphism $\text{Tor}_n^{I,R}(M,N) \cong \text{Tor}_n^{II,R}(M,N)$. Together these make $\text{Tor}_n^R(M,N)$ into a bi-δ-functor.

(iv) In this class, the most important example is the following. Let X be a topological space, and let $\mathcal{O}_X$ be a sheaf of rings on X (but $(X,\mathcal{O}_X)$ is not necessarily a locally ringed space). Denote by $\mathcal{O}_X - \text{mod}$ the category of sheaves of left $\mathcal{O}_X$-modules. There is a left-exact functor,

$$\Gamma(X,-) : \mathcal{O}_X - \text{mod} \to \Gamma(X,\mathcal{O}_X) - \text{mod}, \quad \mathcal{F} \mapsto \Gamma(X,\mathcal{F}).$$

As we will see, this category has enough injective objects. The right derived functors are the *sheaf cohomologies*,

$$H^p(X,\mathcal{F}) := R^p\Gamma(X,-)(\mathcal{F}).$$

Of course $\mathcal{O}_X - \text{mod}$ is an additive subcategory of Abelian sheaves and $\Gamma(X,\mathcal{F}) - \text{mod}$ is an additive subcategory of Ab. Therefore, it is a priori possible that the sheaf cohomologies change when we change the sheaf of rings. We will see this does not happen: the underlying Abelian group of the $\Gamma(X,\mathcal{O}_X)$-module $H^p(X,\mathcal{F})$ is canonically isomorphic to the sheaf cohomology of $\mathcal{F}$ considered as an Abelian sheaf. This is why $\mathcal{O}_X$ is not part of the notation for the sheaf cohomology.

Lecture 16. April 10, 2006

Homework. Problem Set 8

Stated criterion: A δ-functor is universal if the higher terms are "effaceable"/"eraseable". Stated definition of an F-*acyclic* object. Combined with effaceable criterion to show that derived functor cohomology may be computed using F-acyclic resolutions. Did an example with Tor (the left derived functors of tensor product) to deduce that for a flat A-algebra B, for every A-module M and every B-module N, $\text{Tor}_p^B(B\otimes_A M, N)$ is canonically isomorphic to $\text{Tor}_p^A(M,N)$ as A-modules.

Explained that left adjoint functors preserve right exactness, colimits and projective objects. Also right adjoint functors preserve left exactness, limits (i.e., inverse limits) and injective objects. Used this to show that the category of $\mathcal{O}_X$-modules has enough injectives. Proved that injective $\mathcal{O}_X$-modules are flasque. Used this to prove flasque sheaves are acyclic for higher sheaf cohomology. Thus sheaf cohomology may be computed using flasque resolutions. Also it follows that sheaf cohomology defined as derived functors on the category of $\mathcal{O}_X$-modules agrees with sheaf cohomology defined as derived functors on the category of Abelian sheaves.

Skipped Grothendieck's vanishing theorem (reading the proof is a homework problem).

Lecture 17. April 12, 2006

Homework. Problem Set 9

Showed that Proposition II.5.6 implies that for every quasi-coherent sheaf $\mathcal{F}$ on an affine scheme, $H^1(X, \mathcal{F}) = 0$. Used this to prove Serre's criterion for affineness: a quasi-compact scheme X is affine if and only if for every quasi-coherent ideal sheaf $\mathcal{I}$, $H^1(X, \mathcal{I}) = 0$.

Stated the stronger result that for every quasi-coherent sheaf $\mathcal{F}$ on an affine scheme, $H^p(X, \mathcal{F}) = 0$ for every $p > 0$. Began the proof under the additional hypothesis that X is Noetherian: showed that it suffices to prove injective quasi-coherent sheaves are flasque.

Showed that for every Noetherian ring A, for every $a \in A$, and for every injective A-module I, the kernel of $I \to I[1/a]$ is an injective A-module (using the Krull Intersection Theorem). Also proved that every injective module over a Noetherian ring is *divisible*, i.e., $I \to I[1/a]$ is surjective.

Lecture 18. April 19, 2006

Homework. Problem Set 10

Finished the proof that an injective quasi-coherent sheaf on an affine Noetherian scheme is flasque. Deduced the result that every injective quasi-coherent sheaf on any Noetherian scheme (not necessarily affine) is flasque. Thus the derived functors of global sections on the category of quasi-coherent sheaves on a Noetherian scheme agree with the derived functors on the category of Abelian sheaves, i.e., agree with sheaf cohomology.

For a topological space X, defined the category of open coverings of X, associated to every open covering the associated simplicial object in the category of open coverings (roughly the nerve), and used this to define the *set of Čech cochains* associated to a sheaf, resp. *object of Čech cochains* for a more general sheaf. Observed this is a cosimplicial set, resp. cosimplicial object. For a sheaf of objects in an Abelian category (e.g., for an Abelian sheaf), defined the *Čech complex* to be the associated cochain complex of this cosimplicial object.

If the indexing set is well-ordered, defined the *reduced Čech complex* and stated the fact that the natural map from the Čech complex to the reduced Čech complex is a quasi-isomorphism. Thus the smaller reduced complex may be used for computations.

Defined the Čech cohomology of a sheaf relative to a covering. Observed the Čech complex, and thus the Čech cohomology, is functorial for refinements of coverings. Used this to define the (absolute) Čech cohomology of a sheaf to be the limit over all refinements of the Čech cohomology relative to the covering.

Computed the example of the Čech cohomology of $\mathbb{Z}_X$ on the circle X with respect to a standard covering. Observed it agrees with the singular cohomology of the circle.

Stated results to be proved next time.

(i) If X is an open subset of the covering, there is an explicit homotopy proving the Čech complex is acyclic.

(ii) Using this, the *complex of Čech sheaves* is a resolution of a sheaf.

(iii) The Čech complex of sheaves of a flasque sheaf is a flasque resolution. Therefore, for a flasque sheaf, the higher Čech cohomology is zero.

(iv) Using this, there is a well-defined, natural transformation from Čech cohomology to sheaf cohomology. But it is not always true that Čech cohomology gives a δ-functor.

(v) For a sheaf whose higher sheaf cohomology vanishes on all opens of the nerve of the covering, the map from Čech cohomology to usual cohomology is an isomorphism.

(vi) A slight variation of this argument will allow us to prove that for every affine scheme X (not necessarily Noetherian) the higher sheaf cohomology of every quasi-coherent sheaf is zero.

Lecture 19. April 24, 2006

Homework. Problem Set 10

Proved that if X is one of the open sets in the covering, the associated Čech complex is acyclic. Therefore the complex of Čech sheaves is a resolution of the original sheaf. Every resolution admits a morphism to any injective resolution, unique up to homotopy. Applying global sections and taking cohomology, one obtains natural transformations from Čech cohomology to sheaf cohomology. Also, for a flasque sheaf, the complex of Čech sheaves is a flasque resolution, hence computes sheaf cohomology, i.e., the natural tansformations are bijections for a flasque sheaf. Since the higher sheaf cohomology of a flasque sheaf vanishes, also the higher Čech cohomology of a flasque sheaf vanishes (for any open covering).

Taking a monomorphism of a given sheaf into a flasque sheaf and examining terms, we concluded that for every sheaf $\mathcal{F}$ the following map is a bijection,

$$\check{H}^1(X,\mathcal{F}) \to H^1(X,\mathcal{F}).$$

Also, if $H^1(U_\sigma,\mathcal{F})$ vanishes for every $\sigma = (\sigma_0,\dots,\sigma_p)$, then for every short exact sequence,

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0,$$

there is a long exact sequence of Čech cohomology, compatible with the long exact sequence of sheaf cohomology via the natural transformations above. In particular, using the case that $\mathcal{G}$ is flasque and using low degree terms, $\check{H}^1(\mathfrak{U},\mathcal{F}) \to H^1(X,\mathcal{F})$ is a bijection.

There are two further conclusions. First, if $H^i(U_\sigma,\mathcal{F}) = 0$ for all $0 < i < p$, then $H^1(U_\sigma,\mathcal{H}) = 0$ for all $0 < i < p-1$. Also $\check{H}^j(\mathfrak{U},\mathcal{H}) = \check{H}^{j+1}(\mathfrak{U},\mathcal{F})$ and also $H^j(X,\mathcal{H}) = H^{j+1}(X,\mathcal{F})$. Thus, working by induction on p, it follows that $H^i(\mathfrak{U},\mathcal{F}) \to H^i(X,\mathcal{F})$ is a bijection for every $i < p$.

One can do better if the condition on cohomology vanishing holds not just for a single covering $\mathfrak{U}$, but for a "cofinal collection" of coverings, i.e., every covering has a refinement by one in the collection. In this case, the compatibilities give isomorphisms $\check{H}^j(X,\mathcal{H}) \to \check{H}^{j+1}(X,\mathcal{F})$. But we know that $\check{H}^1(X,\mathcal{H}) = H^1(X,\mathcal{H})$. Therefore also $\check{H}^2(X,\mathcal{F}) = H^2(X,\mathcal{F})$. Note the only hypothesis was that $H^1(U_\sigma,\mathcal{F}) = 0$ for all σ and for all coverings $\mathfrak{U}$ in a cofinal collection. Suppose for the moment that all higher Čech cohomology of quasi-coherent sheaves on an affine scheme are known to be zero. Then this proves that both H^1 and H^2 of a quasi-coherent sheaf on an affine scheme are zero. But then $H^1(U_\sigma,\mathcal{H})$ is zero for every U_σ in an affine covering. Thus, for the same reason, $H^3(X,\mathcal{F}) = H^2(X,\mathcal{H}) = \check{H}^2(X,\mathcal{H}) = \check{H}^3(X,\mathcal{F})$. This is zero by assumption. Therefore

$H^1(U_\sigma, \mathcal{H}) = H^2(U_\sigma, \mathcal{H})$ both vanish, and one can continue in this manner to conclude that all higher sheaf cohomology of $\mathcal{F}$ vanish.

Thus, finally, to prove vanishing of all higher sheaf cohomology of a quasi-coherent sheaf on an affine scheme, it suffices to construct a cofinal collection of coverings such that all higher Čech cohomology vanishes for each covering. Let $X = \text{Spec } A$, let the sheaf be $\widetilde{M}$ for an A-module M. The cofinal coverings will be finite coverings by distinguished open affines $(D(a_1), \dots, D(a_r))$.

Since cohomology of the Čech complex equals the cohomology of the reduced Čech complex, we can compute this complex, which is

$$\textstyle\prod_{i_0} M[1/a_{i_0}] \longrightarrow \prod_{i_0<i_1} M[1/a_{i_0}a_{i_1}] \longrightarrow \dots$$

Observe this is just the tensor product of M with the complex

$$\textstyle\prod_{i_0} A[1/a_{i_0}] \longrightarrow \prod_{i_0<i_1} A[1/a_{i_0}a_{i_1}] \longrightarrow \dots$$

This is a complex of *flat* A-modules. Therefore, if this complex is acyclic, the same is true after tensoring with M. Thus, to prove the theorem, we are reduced to proving the complex is acyclic for $M = A$.

Because the localization $A[1/a]$ is the filtering direct limit (= colimit) of free A-modules $A\langle 1/a^N\rangle$ with basis $1/a^N$, the Čech complex is a filtering colimit of the following complexes,

$$\textstyle\prod_{i_0} A\langle 1/a_{i_0}^N\rangle \longrightarrow \prod_{i_0<i_1} A\langle 1/a_{i_0}^N a_{i_1}^N\rangle \longrightarrow \dots.$$

This is precisely the Koszul complex associated to the sequence $(a_1^N, \dots, a_r^N)$, except that the final term in the Koszul complex has been truncated. The claim is that the Koszul complex is exact, and thus the complex above is acyclic.

If the Koszul complex is exact at all places, then the ideal $\langle a_1^N, \dots, a_r^N\rangle A = A$. Of course this holds in our case because $D(a_1^N), \dots, D(a_r^N)$ cover Spec A. When $(a_1^N, \dots, a_r^N)$ generate the unit ideal, the general theory of Koszul complex gives that the Koszul complex of $(a_1^N, \dots, a_r^N)$ is exact if and only if the Koszul complex of $(b_1, \dots, b_s)$ is exact for any (and hence every) sequence $(b_1, \dots, b_s)$ generating the unit ideal, cf., Corollary 17.10 of Eisenbud's "Commutative algebra". Consider the sequence $(b_1, \dots, b_s) = (1)$. The associated Koszul complex is

$$\dots \longrightarrow 0 \longrightarrow A \xrightarrow{\text{Id}_A} A \longrightarrow 0 \longrightarrow \dots$$

which is clearly exact. Therefore the Koszul complex for $(a_1^N, \dots, a_r^N)$ is exact. Taking limits, the Čech complex is acyclic. This completes the proof of vanishing of the higher sheaf cohomology of any quasi-coherent sheaf on any affine scheme.

Lecture 20. April 26, 2006

Homework. Problem Set 11

Let A be a commutative ring, let P be a projective A-module of constant, finite rank r, and let $\phi : P \to A$ be an A-module homomorphism. Let $\bigwedge^\bullet P$ denote the (A-multilinear) exterior algebra of P,

$$\bigwedge^{\bullet} P = \oplus_{k=0}^r \bigwedge^k P.$$

There is a unique A-module homomorphism $d : \bigwedge^\bullet P \to \bigwedge^\bullet P$ satisfying,

(i) for every $p \in P = \bigwedge^1 P$, $dp = \phi(p)$ inside $A = \bigwedge^0 P$,

(ii) for every pair of homogeneous elements $\alpha, \beta \in \bigwedge^\bullet P$ of degrees a and b respectively, $d(\alpha\wedge\beta) = (d\alpha) \wedge \beta + (-1)^a \alpha \wedge (d\beta)$.

In fact, the map d is defined on "pure tensors" by

$$d(p_0 \wedge \cdots \wedge p_k) = \sum_{l=0}^{k} (-1)^l \phi(p_l) p_0 \wedge \cdots \wedge p_{l-1} \wedge p_{l+1} \wedge \cdots \wedge p_k.$$

The map d makes $\bigwedge^\bullet P$ into a cochain complex, where the cohomological degree of $\bigwedge^k P$ is $-k$. This complex is the (homological) *Koszul complex*. It is functorial in ϕ: given $\phi : P \to A$ and $\psi : P' \to P$, there is an induced morphism $K(\psi) : K(\phi \circ \psi) \to K(\phi)$ of cochain complexes of A-modules.

It is always the case that $h^0(K(\phi)) = A/\text{Image}(\phi)$. Thus, $K(\phi)$ is exact only if ϕ is surjective. In fact, if ϕ is surjective, then $K(\phi)$ is exact. Let $q \in P$ be an element such that $\phi(q) = 1$. There is a unique A-module homomorphism $s : \bigwedge^\bullet A \to \bigwedge^\bullet A$ such that

(i) $s(1) = q$,

(ii) for every pair of homogeneous elements $\alpha, \beta \in \bigwedge^\bullet P$ of degrees a and b respectively, $s(\alpha\wedge\beta) = (s\alpha) \wedge \beta + (-1)^a \alpha \wedge (s\beta)$.

In fact, the map s is defined on "pure tensors" by

$$s(p_0 \wedge \cdots \wedge s_k) = \sum_{l=0}^{k} (-1)^l p_0 \wedge \cdots \wedge p_{l-1} \wedge q \wedge p_l \wedge \cdots \wedge p_k.$$

The claim is that $ds + sd = \text{Id}$. Because both d and s satisfy the graded Leibniz rule, it suffices to check on elements of degree 0 and 1. In each of these cases, it is straightforward to verify.

One of the most important results about Koszul complexes is the following. Let $P = A^{\oplus r}$ and let $\phi : P \to A$ be the map $(c_1, \ldots, c_r) \mapsto c_1 a_1 + \cdots + c_r a_r$. The Koszul complex $K(\phi)$ is acyclic if and only if the sequence $(a_1, \ldots, a_r)$ is regular. Recall, a complex C is *acyclic* if the only nonvanishing cohomology is $h^0(C)$ – in this case $h^0(K(\phi)) = A/\langle a_1, \ldots, a_r\rangle$. And a sequence $(a_1, \ldots, a_r)$ is *regular* if a_1 is a nonzerodivisor, and for every $i = 2, \ldots, r$, the image of a_i in $A/\langle a_1, \ldots, a_{i-1}\rangle$ is a nonzerodivisor. Using this, it follows that the property of being regular is invariant under permutation, and also invariant under replacing $(a_1, \ldots, a_r)$ by powers $(a_1^{e_1}, \ldots, a_r^{e_r})$ with $e_1, \ldots, e_r$ positive.

Let A be a commutative ring, let P be a free A-module of finite rank r, and denote $S = \text{Sym}_A^\bullet P$. Let $\mathbb{P} = \text{Proj} S$. Denote by $\pi : \mathbb{P} \to \text{Spec } A$ the obvious morphism. There is a surjection $\pi^* \widetilde{P} \to \mathcal{O}(1)$, and this gives the universal property of $\mathbb{P}$. Using this, for every nonnegative integer d, there is a map $\mathrm{Sym}^d_A P \to H^0(\mathbb{P}, \mathcal{O}(d))$. One of the basic facts about cohomology of coherent sheaves on projective space is that this map is an isomorphism.

To compute the cohomology of $\mathcal{O}(d)$ on $\mathbb{P}$ for every integer d, first form the affine scheme $\mathbb{A} = \mathrm{Spec} S$. Denote by U the open subscheme whose complement has ideal S_+, the irrelevant ideal. Denote by $\rho : U \to \mathbb{P}$ the usual $\mathbb{G}_m$-torsor. Recall that $\rho_* \mathcal{O}_U = \oplus_{d \in \mathbb{Z}} \mathcal{O}(d)$. Because ρ is affine, $H^q(\mathbb{P}, \rho_* \mathcal{O}_U)$ is canonically isomorphic to $H^q(U, \mathcal{O}_U)$ for every integer q. Using a Čech covering, it will be clear that $H^q(\mathbb{P}, \oplus_d \mathcal{O}(d))$ is canonically isomorphic to $\oplus_d H^q(\mathbb{P}, \mathcal{O}(d))$. Therefore, to compute the cohomology of all invertible sheaves on $\mathbb{P}$, it suffices to compute the cohomology $H^q(U, \mathcal{O}_U)$, remembering the grading induced by the $\mathbb{G}_m$-action.

Let $x_0, \dots, x_{r-1}$ be an ordered basis for P. Denote by $\mathfrak{U} = (D(x_0), \dots, D(x_{r-1}))$ the associated open covering of U. This is an open affine covering. Denote by $\mathfrak{U}' = (D_+(x_0), \dots, D_+(x_{r-1}))$ the associated open covering of $\mathbb{P}$. This is also an open affine covering. Therefore the reduced Čech cohomology of any quasicoherent sheaf with respect to each of these coverings equals the sheaf cohomology. The reduced Čech complex for $\mathcal{O}_U$, respectively $\rho_* \mathcal{O}_U$, is

$$\check{C}(\mathfrak{U}, \mathcal{O}_U)_k = \prod_{0 \le i_0 < \dots < i_k \le r-1} S[1/(x_{i_0} \dots x_{i_k})]$$

with differential d the usual map. The grading of the terms in this complex associated to the $\mathbb{G}_m$-action is the usual grading. In particular, because the indexing set of the product is finite, it equals a direct sum,

$$\check{C}(\mathfrak{U}, \mathcal{O}_U)_k = \oplus_{0 \le i_0 < \dots < i_k \le r-1} S[1/(x_{i_0} \dots x_{i_k})].$$

From this it follows that the Čech cohomology of $\oplus_d \mathcal{O}(d)$ is the direct sum of the Čech cohomologies of the summands $\mathcal{O}(d)$.

One can compute the Čech cohomology above directly. But one can also realize this complex as a limit of complexes

$$\check{C}(\mathfrak{U}, \mathcal{O}_U) = \varprojlim_N \check{C}_N,$$

$$\check{C}_{N,k} = \oplus_{0 \le i_0 < \dots < i_k \le r-1} S\langle 1/(x_{i_0} \cdots x_{i_k})^N \rangle.$$

Here $S\langle e \rangle$ is just shorthand for the free S-module of rank 1 with basis element being the symbol e. The basis element of $\check{C}_{N,k}$ maps to the element $1/(x_{i_0} \cdots x_{i_k})^N$ inside $\check{C}(\mathfrak{U}, \mathcal{O}_U)_k$. Also, the differential map d on $\check{C}_{N,k}$ is the obvious differential. And the grading is the usual grading where the symbol $1(x_{i_0} \cdots x_{i_k})^N$ has degree $-N(k+1)$.

The point is that $\check{C}_N$ is closely related to a Koszul complex. Denote by P_n the graded, free S-module generated by the symbols $x_0^N, \dots, x_{r-1}^N$, and let $\phi_N : P_N \to S$ be the map sending the symbol x_i^N to the element x_i^N of S. The complex $\check{C}_N$ is related to $K(\phi_N)$ by,

$$\check{C}_{N,k} = \mathrm{Hom}_S(K(\phi_N)_{k+1}, S),$$

compatibly with the differentials and the natural gradings. On the other hand, Koszul complexes are self-dual,

$$K(\phi_N) \cong \mathrm{Hom}_S(K(\phi_N), \bigwedge^r P_N)[r],$$

compatibly with the gradings. Using this, $\check{C}_N$ is isomorphic to the *brutal truncation* of the shifted Koszul complex,

$$\check{C}_{N,k} \cong (K(\phi_N)[-r+1] \otimes_S \mathrm{Hom}_S(\bigwedge^r P_N, S))_{\geq 0}.$$

Because $(x_0, \ldots, x_{r-1})$ is a regular sequence in S, also $(x_0^N, \ldots, x_{r-1}^N)$ is a regular sequence. Therefore $K(\phi_N)$ is acyclic. For the moment assume that $r > 1$. Then it follows that $\check{C}_N$ has precisely two nonvanishing cohomologies,

$$h^0(\check{C}_{N,k}) = S\langle 1 \rangle,$$

$$h^{r-1}(\check{C}_{N,k}) = h^0(K(\phi_N)) \otimes_S \mathrm{Hom}_S(\bigwedge^r P_N, S) = S/\langle x_0^N, \ldots, x_{r-1}^N \rangle \otimes_S \mathrm{Hom}_S(\bigwedge^r P_N, S).$$

As a graded S-module the second cohomology is canonically isomorphic to

$$h^{r-1}(\check{C}_{N,k}) \cong \mathrm{Hom}_A(S/\langle x_0^N, \ldots, x_{r-1}^N \rangle, \mathrm{Hom}_A(\bigwedge^r P, A)),$$

where $\bigwedge^r P$ has degree r and $\mathrm{Hom}_A(\bigwedge^r P, A)$ has degree $-r$. In particular, for every integer d the d^{th} graded piece stabilizes for $N \geq d$. The final answer is that, for $r > 1$, for every integer d,

$$\left\{ \begin{array}{ccccll} H^0(\mathbb{P}, \mathcal{O}(d)) & = & \check{H}^0(\mathfrak{U}', \mathcal{O}(d)) & = & S_d, & \\ H^{r-1}(\mathbb{P}, \mathcal{O}(d)) & = & \check{H}^{r-1}(\mathfrak{U}', \mathcal{O}(d)) & = & \mathrm{Hom}_A(S_{-d-r}, \mathrm{Hom}_A(\bigwedge^r P, A)), & \\ H^q(\mathbb{P}, \mathcal{O}(d)) & = & \check{H}^q(\mathfrak{U}', \mathcal{O}(d)) & = & (0), & q \neq 0, r-1. \end{array} \right.$$

Something special happens when $r = 1$. In this case, the reduced Čech complex has only one term $S[1/x_0]$. Therefore, for every integer d, there is one nonvanishing cohomology,

$$\left\{ \begin{array}{cccl} H^0(\mathbb{P}, \mathcal{O}(d)) & = & P^{\otimes d}, & \\ H^q(\mathbb{P}, \mathcal{O}(d)) & = & (0) & q \neq 0. \end{array} \right.$$

For $d < 0$, $P^{\otimes d}$ is defined to be $\mathrm{Hom}_A(P^{-d}, A)$.

Of course when $r = 0$, $\mathbb{P}$ is the empty scheme. Thus all cohomologies are identically zero.

Lecture 21. May 1, 2006

Homework. Problem Set 11

Last time the Čech complex of $\oplus_d \mathcal{O}(d)$ on projective space $\mathbb{P} \cong \mathbb{P}_A^{r-1}$ was shown to be (a truncation of a shift of) a limit of Koszul complexes $K(\phi_N)$. The first Koszul complex that arises, $K(\phi_1)$, is very natural and can be used to give a reformulation of the theorem.

Let $\psi : \pi^* \widetilde{P} \to \mathcal{O}(1)$ be the universal invertible quotient. Twisting by $\mathcal{O}(-1)$ gives a surjective morphism of coherent sheaves $\phi : \pi^* \widetilde{P} \otimes \mathcal{O}(-1) \to \mathcal{O}_{\mathbb{P}}$. Just as with a morphism from a projective module to A, there is a Koszul complex associated to this map, $K(\phi)$. Moreover, $\rho^* K(\phi)$ is the restriction to U of the complex $\widetilde{K(\phi_1)}$ of coherent sheaves on $\mathbb{A}$ associated to the Koszul complex

$K(\phi_1)$ of S-modules. Thus, $K(\phi)$ is nothing more than the sheafified version of $K(\phi_1)$, taking the gradings into account. The terms in the complex are,

$$K(\phi)_k = \pi^* \widetilde{\bigwedge^k P} \otimes \mathcal{O}(-k)$$

for $k = 0, \dots, r$, and the differentials are,

$$d_k : K(\phi)_k \to K(\phi)_{k-1}, \quad d((p_0 \wedge \dots \wedge p_k) \otimes g) = \sum_{l=0}^{k} (-1)^l (p_0 \wedge \dots \wedge p_{l-1} \wedge p_{l+1} \wedge \dots \wedge p_k) \otimes (\psi(p_l) g),$$

where $p_0, \dots, p_k$ are elements of P, and g is a local section of $\mathcal{O}(-k)$.

Because ϕ is surjective, $K(\phi)$ is exact. For every k, denote

$$\Omega_\pi^k := \mathrm{Ker}(d_{k-1}) = \mathrm{Image}(d_k).$$

Because of the graded Leibniz rule, there is a map

$$\bigwedge^k \Omega_\pi^1 \to \Omega_\pi^k.$$

Because the complex is exact, this map is an isomorphism. Two of the terms Ω_π^k are clear: because d_{-1} is the zero map, $\Omega_\pi^0 = \mathcal{O}_{\mathbb{P}}$. Because d_{r-1} is the zero map, $\Omega_\pi^{r-1} = K(\phi)_r$. This second identity gives an interesting isomorphism,

$$\bigwedge^{r-1} \Omega_\pi^1 \cong K(\phi)_r = \pi^* \widetilde{\bigwedge^r P} \otimes \mathcal{O}(-r).$$

This sheaf comes up so often it is given a special name, the *dualizing sheaf*,

$$\omega_\pi := \Omega_\pi^{r-1} = \pi^* \widetilde{\bigwedge^r P} \otimes \mathcal{O}(-r).$$

For every k there is a short exact sequence,

$$\Sigma_k : 0 \longrightarrow \Omega_\pi^k \longrightarrow K(\phi)_k \longrightarrow \Omega_\pi^{k-1} \longrightarrow 0.$$

Thus there are connecting maps in cohomology, $\delta : H^{q-1}(\mathbb{P}, \Omega_\pi^{k-1}) \to H^q(\mathbb{P}, \Omega_\pi^k)$. Composing these connecting maps gives a morphism,

$$H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}) = H^0(\mathbb{P}, \Omega_\pi^0) \xrightarrow{\delta} H^1(\mathbb{P}, \Omega_\pi^1) \xrightarrow{\delta} \dots \xrightarrow{\delta} H^{r-1}(\mathbb{P}, \Omega_\pi^{r-1}) = H^{r-1}(\mathbb{P}, \omega_\pi).$$

Of course there is an A-module homomorphism $A \to H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}})$. Thus, putting all these homomorphisms together, there is an A-module homomorphism,

$$t' : A \to H^{r-1}(\mathbb{P}, \omega_\pi).$$

Although this is our main interest in $K(\phi)$, there is another aspect of $K(\phi)$ that is also very important. The sheaf Ω^1_π that is defined to be $\mathrm{Ker}(d_0)$ is, in fact, canonically isomorphic to the sheaf of relative differentials of π. This has not been formally defined yet. The short exact sequence Σ_1 is the dual sequence of the *Euler sequence* typically used to present the tangent sheaf of projective space.

For every coherent sheaf $\mathcal{F}$ on $\mathbb{P}$, there is a map

$$\mu_{\mathcal{F}} : \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{F}, \omega_\pi) \times H^{r-1}(\mathbb{P}, \mathcal{F}) \to H^{r-1}(\mathbb{P}, \omega_\pi).$$

This map is defined using the fact that $H^{r-1}(\mathbb{P}, -)$ is functorial. This map is A-bilinear and thus defines an A-module homomorphism,

$$\mu_{\mathcal{F}} : \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{F}, \omega_\pi) \otimes_A H^{r-1}(\mathbb{P}, \mathcal{F}) \to H^{r-1}(\mathbb{P}, \omega_\pi),$$

or equivalently,

$$\nu_{\mathcal{F}} : \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{F}, \omega_\pi) \to \mathrm{Hom}_A(H^{r-1}(\mathbb{P}, \mathcal{F}), H^{r-1}(\mathbb{P}, \omega_\pi)).$$

This leads to a reformulation of the theorem from last time.

(i) The A-module homomorphism $t' : A \to H^{r-1}(\mathbb{P}, \omega_\pi)$ is an isomorphism. Denote its inverse by $t : H^{r-1}(\mathbb{P}, \omega_\pi) \to A$.

(ii) If $r > 1$, then for every integer d the natural A-module homomorphism $S_d \to H^0(\mathbb{P}, \mathcal{O}(d))$ is an isomorphism. If $r = 1$, then for every integer d the natural A-module homomorphism $P^{\otimes d} \to H^0(\mathbb{P}, \mathcal{O}(d))$ is an isomorphism.

(iii) For every integer d, the map $\mu_{\mathcal{O}(d)}$ is a perfect pairing, i.e., $\nu_{\mathcal{O}(d)}$ is an isomorphism.

(iv) For every integer d and every $q \neq 0, r-1$, $H^q(\mathbb{P}, \mathcal{O}(d)) = (0)$.

The isomorphism t in (i) is called the *trace map*. Actually, (iii) implies that $\nu_{\mathcal{F}}$ is an isomorphism for *every* coherent sheaf $\mathcal{F}$ (at least when A is Noetherian). The point is that we can find a presentation,

$$\mathcal{O}(-e)^{\oplus M} \longrightarrow \mathcal{O}(-d)^{\oplus N} \longrightarrow \mathcal{F} \longrightarrow 0$$

This gives a commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{F}, \omega_\pi) & \longrightarrow & \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}(-d), \omega_\pi)^{\oplus N} & \longrightarrow & \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}(-e), \omega_\pi)^{\oplus M} \\
& \downarrow{\scriptstyle \nu_{\mathcal{F}}} & & \downarrow & & \downarrow \\
0 \longrightarrow & \mathrm{Hom}_A(H^{r-1}(\mathbb{P}, \mathcal{F}), A) & \longrightarrow & \mathrm{Hom}_A(H^{r-1}(\mathbb{P}, \mathcal{O}(-d)), A)^{\oplus N} & \longrightarrow & \mathrm{Hom}_A(H^{r-1}(\mathbb{P}, \mathcal{O}(-e)), A)^{\oplus M}
\end{array}$$

By (iii), the second and third vertical arrows are isomorphisms. Therefore $\nu_{\mathcal{F}}$ is also an isomorphism.

This is the most basic example of a *dualizing pair*. To avoid technical issues, we formulate this only when A is a field K. In this case, for a purely s-dimensional, proper K-scheme X, a *dualizing pair* is a pair (ω_X°, t) of a coherent sheaf ω_X° and a map of vector spaces $t : H^s(X, \omega_X^\circ) \to k$

satisfying the following universal property. For every coherent sheaf $\mathcal{F}$ on X, the following map is an isomorphism,

$$\nu_{\mathcal{F}} : \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X^\circ) \to \mathrm{Hom}_k(H^{r-1}(X,\mathcal{F}), H^{r-1}(X,\omega_X^\circ)) \xrightarrow{t\circ -} \mathrm{Hom}_k(H^{r-1}(X,\mathcal{F}), k).$$

In other words, (ω_X°, t) represents the contravariant functor,

$$\mathrm{Hom}_k(H^{r-1}(X,-),k) : \mathrm{Coh}_X \to k\text{-vector spaces}.$$

Therefore, if it exists, a dualizing pair is unique up to unique isomorphism. The main part of the reformulation above is that $\mathbb{P}_k^{r-1}$ does have a dualizing pair, namely (ω_π, t).

There are two other consequences of the theorem from last time. First of all, by a downward induction argument, we proved the following theorem of Serre. Assume A is Noetherian.

(i) For every coherent sheaf $\mathcal{F}$ on $\mathbb{P}$, for every $q > r-1$, $H^q(\mathbb{P},\mathcal{F}) = (0)$.

(ii) For every coherent sheaf $\mathcal{F}$ on $\mathbb{P}$, for every q, $H^q(\mathbb{P},\mathcal{F})$ is a finitely generated A-module.

(iii) For every coherent sheaf $\mathcal{F}$ on $\mathbb{P}$, there exists an $n_0 = n_0(\mathcal{F})$ such that for every $n \geq n_0$ and every $q > 0$, $H^q(\mathbb{P}, \mathcal{F}\otimes\mathcal{O}(n)) = (0)$.

(iv) For every coherent sheaf $\mathcal{F}$ on $\mathbb{P}$, the S-module $\Gamma_*(\mathcal{F}) := \oplus_{n\geq 0} H^0(\mathbb{P}, \mathcal{F}\otimes\mathcal{O}(n))$ is finitely generated.

Using this, we also proved another theorem of Serre characterizing ample invertible sheaves on proper schemes. Let X be a proper scheme over a Noetherian ring A. Let $\mathcal{L}$ be an invertible sheaf on X. Then the following are equivalent.

(i) The sheaf $\mathcal{L}$ is ample.

(ii) For every coherent sheaf $\mathcal{F}$ on X, there exists $n_0 = n_0(\mathcal{F})$ such that for every $n \geq n_0$ and every $q > 0$, $H^q(X, \mathcal{F}\otimes\mathcal{L}^{\otimes n}) = (0)$.

Finally, we defined the Ext modules, $\mathrm{Ext}^q_{\mathcal{O}_X}(\mathcal{F},-)$ and *Ext* sheaves, $\mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{F},-)$, to be the right derived functors of $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},-)$, respectively $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},-)$.

Lecture 22. May 3, 2006

Homework. Problem Set 12

Discussed basic facts about *Ext* and *Ext*. Proved that for a projective scheme X over a Noetherian ring A, for every pair of coherent sheaves $\mathcal{F}, \mathcal{G}$, there exists an integer l_0 such that for every $l \geq l_0$ and every integer q, the natural map,

$$\mathrm{Ext}^q_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}(l)) \to \Gamma(X, \mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}(l)))$$

is an isomorphism.

Let X be an n-dimensional projective scheme over a field k. Recalled the definition of a dualizing pair (ω_X°, t) for X. Assuming a dualizing pair exists, constructed a natural transformation of δ-functors,

$$\theta^q : \mathrm{Ext}^q(\mathcal{F}, \omega_X^\circ) \to H^{n-q}(X, \mathcal{F})^\vee.$$

The first δ-functor is universal. For $X = \mathbb{P}_k^n$, used the computation of cohomology of invertible sheaves to prove the second δ-functor is also coeffaceable, thus universal. Therefore, for projective space, every natural transformation θ^q is a natural isomorphism.

Lecture 23. May 8, 2006

Homework. Problem Set 12

Began by proving one version of the Riemann-Roch theorem for curves. Let X be a proper, reduced, connected curve over an algebraically closed field. Recall that for every coherent sheaf on X, there are two associated integers, the rank and the degree. The *arithmetic genus* of X is defined to be $p_a(X) = h^1(X, \mathcal{O}_X)$. The Riemann-Roch theorem for curves states that for every coherent sheaf $\mathcal{F}$ on X,

$$h^0(X, \mathcal{F}) - h^1(X, \mathcal{F}) = \deg(\mathcal{F}) + \mathrm{rank}(\mathcal{F})(1 - p_a(X)).$$

This is proved by proving it first for torsion sheaves, next for the structure sheaf, and then for every free $\mathcal{O}_X$-module of finite rank. For every coherent sheaf $\mathcal{F}$ with rank r, there exists a pair of short exact sequences,

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{F} \longrightarrow \mathcal{T} \longrightarrow 0,$$
$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{O}_X^{\oplus r} \longrightarrow \mathcal{T}' \longrightarrow 0,$$

Using the long exact sequence of cohomology,

$$\chi(X, \mathcal{F}) = \chi(X, \mathcal{K}) + \chi(X, \mathcal{T}) = \chi(X, \mathcal{O}_X^{\oplus r}) - \chi(X, \mathcal{T}') + \chi(X, \mathcal{T}).$$

Using the Riemann-Roch theorem for $\mathcal{O}_X^{\oplus r}$, $\mathcal{T}$ and $\mathcal{T}'$, this gives,

$$\chi(X, \mathcal{F}) = r(1 - p_a(X)) + \deg(\mathcal{T}) - \deg(\mathcal{T}') = r(1 - p_a(X)) + \deg(\mathcal{F}).$$

Alternatively, one could *define* the degree of a coherent sheaf $\mathcal{F}$ to be,

$$\deg(\mathcal{F}) := \chi(X, \mathcal{F}) + r(p_a(X) - 1),$$

and then use the argument above to prove it satisfies the axioms of Exercise II.6.12. Once we have finished proving Serre duality for projective morphisms, there will be another formulation of Riemann-Roch for locally free sheaves E on X,

$$h^0(X, E) - h^0(X, E^\vee \otimes \omega_X) = \deg(E) + \mathrm{rank}(E)(1 - p_a(X)).$$

Recalled the definition of a dualizing pair (ω_X°, t_X) and the associated map of contravariant δ-functors,

$$\theta^i : \mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \omega_X^\circ) \to H^{d-i}(X, \mathcal{F})^\vee,$$

on a d-dimensional projective scheme X. Recalled the statement of the duality theorem for $X = \mathbb{P}^n$: there exists a dualizing pair, and the associated map of δ-functors is an isomorphism. Proved a version of "relative duality" for closed immersions. Let $\iota : X \to \mathbb{P}^n$ is a closed immersion of codimension c. Define ω_X° to be a coherent sheaf on X satisfying,

$$\iota_* \omega_X^\circ = \mathcal{E}xt_{\mathcal{O}_{\mathbb{P}^n}}(\iota_* \mathcal{O}_X, \omega_{\mathbb{P}^n}).$$

There exists an element,

$$t_\iota \in \mathrm{Ext}^c_{\mathcal{O}_{\mathbb{P}^n}}(\iota_* \omega_X^\circ, \omega_{\mathbb{P}^n})$$

such that the pair $(\omega_X^\circ, t_\iota)$ represents the functor,

$$\mathrm{Coh}_X \to k - \text{Vector Spaces}, \quad \mathcal{F} \mapsto \mathrm{Ext}^c_{\mathcal{O}_{\mathbb{P}^n}}(\iota_* \mathcal{F}, \omega_{\mathbb{P}^n}).$$

From duality for $\mathbb{P}^n$, t_ι is equivalent to an element t_X in

$$H^d(X, \omega_X^\circ)^\vee \cong H^{n-c}(\mathbb{P}^n, \iota_* \omega_X^\circ)^\vee = \mathrm{Ext}^c_{\mathcal{O}_{\mathbb{P}^n}}(\iota_* \omega_X^\circ, \omega_{\mathbb{P}^n}).$$

The claim is that the pair (ω_X°, t_X) is a dualizing pair for X. Indeed, for every coherent $\mathcal{F}$ on X, by relative duality,

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X^\circ) \cong \mathrm{Ext}^c_{\mathcal{O}_{\mathbb{P}^n}}(\iota_* \mathcal{F}, \omega_{\mathbb{P}^n}).$$

And by duality for $\mathbb{P}^n$,

$$\mathrm{Ext}^c_{\mathcal{O}_{\mathbb{P}^n}}(\iota_* \mathcal{F}, \omega_{\mathbb{P}^n}) \cong H^{n-c}(\mathbb{P}^n, \iota_* \mathcal{F})^\vee \cong H^d(X, \mathcal{F})^\vee.$$

Therefore every projective scheme has a dualizing pair.

Asked the question when is the associated transformation $(\theta^i)_i$ of δ functors an isomorphism. Observed this is equivalent to coeffaceability of the δ-functor $H^{d-i}(X, \mathcal{F})^\vee$ (since the other δ-functor is coeffaceable). Stated and proved the Serre duality theorem: the natural transformation $(\theta^i)_i$ is an isomorphism if and only if the second δ-functor is coeffaceable if and only if X is Cohen-Macaulay and equidimensional.

Stated the fact that if X is Gorenstein and equidimensional, then ω_X° is an invertible sheaf. Stated the result, to be proved next time, that for a reduced, local complete intersection scheme X, the dualizing sheaf is isomorphic to the determinant of the sheaf of relative differentials $\Omega_{X/k}$ (which is also to be defined).

Lecture 24. May 10, 2006

Homework. Problem Set 13

Philosophical discussion of connectedness, comparison of the Zariski and "classical" topologies on the set of closed points of a finite type scheme over $\mathbb{C}$, and Hartog's phenomenon. Stated and sketched the proof of Hartshorne's connectedness theorem. Stated and proved the Enriques-Severi-Zariski lemma. Stated the theorem that an ample divisor on a projective, $R_0 + S_2$ scheme over a field is connected. Stated the weak Lefschetz theorem for the classical topology. Explained why the previous theorem is the π_0-case. Stated the versions of the π_1-case of the Lefschetz theorem from

SGA2. Proved the surjectivity part of this version. Defined relative differentials and explained the universal property (existence of a universal derivation).

Lecture 25. May 15, 2006

Homework. Problem Set 13

Stated the basic properties of the sheaf of relative differentials. Computed the sheaf of relative differentials for the vector bundle and the projective bundle associated to a locally free sheaf. Combined this with the earlier Koszul complex to deduce that the dualizing sheaf of a projective bundle equals the canonical sheaf. Analyzed the Koszul complex associated to a regular embedding to prove that for every reduced, pure-dimensional, local complete intersection, projective scheme over a field, the dualizing sheaf is isomorphic to the canonical sheaf.

Defined the Hodge groups. Stated the Hodge theorem. Observed that Serre duality for the Hodge groups is compatible with Poincare duality via the Hodge decomposition.

Stated the theorem on formal functions. Explained one consequence of Zariski's main theorem: a rational function on a normal variety extends to a regular morphism if and only if it extends to a continuous function (for the Zariski or classical topology, if the variety is over $\mathbb{C}$).

Lecture 26. May 17, 2006

Recalled the statement of the theorem on formal functions. Deduced the following corollary: For a scheme X with a projective morphism of relative dimension d to an affine scheme, for every coherent sheaf $\mathcal{F}$, $H^i(X, \mathcal{F}) = 0$ for $i > d$. Defined the Stein factorization of a proper morphism $f : X \to Y$. Proved that if $f_*\mathcal{O}_X = \mathcal{O}_Y$, then f has connected fibers. Deduced the connectedness formulation of Zariski's Main Theorem: a proper, birational morphism to a normal (Noetherian) scheme has connected fibers. Discussed the five formulations of Zariski's Main Theorem given in Mumford's The red book of varieties and schemes. Proved the theorem on formal functions.