18.725 PROBLEM SET 9

Due date: Wednesday, December 8 in lecture. Late work will be accepted only with a medical note or for another Institute-approved reason. You are strongly encouraged to work with others, but the final write-up should be entirely your own and based on your own understanding.

Read through all the problems. Write solutions to the "Required Problems", 1, 2, 3, and 4 together with 1 more problem to a total of 5.

Required Problem 1, Intersection Multiplicity: This problem is essentially (Hartshorne, Exer. I.5.4). Let $F, G \in k[X_0, X_1, X_2]$ be non-constant, irreducible, homogeneous polynomials, and denote $C = \mathbb{V}(F), D = \mathbb{V}(G)$ in $\mathbb{P}^2_k$. Let $p \in C \cap D$ be an element such that $\dim(C \cap D, p) = 0$, i.e., p is an isolated point of $C \cap D$. The *intersection multiplicity of C and D at p*, $i(C, D; p)$, is defined to be,

$$i(C, D; p) = \dim_k(\mathcal{O}_{\mathbb{P}^2,p}/\langle F_p, G_p\rangle),$$

where $F_p, G_p \in \mathcal{O}_{\mathbb{P}^2,p}$ are germs of dehomogenizations of F and G at p.

Let $P \subset k[X_0, X_1, X_2]$ be the homogeneous ideal corresponding to p. Form the graded $k[X_0, X_1, X_2]$-module, $M = \text{Image}(\phi_p)$, where ϕ_p is the homomorphism of graded modules,

$$\phi_p : k[X_0, X_1, X_2]/\langle F, G\rangle \to (k[X_0, X_1, X_2]/\langle F, G\rangle)_P.$$

(a) Prove that the Hilbert polynomial of M equals $i(C, D; p)$, i.e., for all $l \gg 0$, $\dim_k M_l = i(C, D; p)$. **Hint:** You may assume existence of a *Jordan-Hölder filtration of M*: a filtration of M by graded submodules, $M = M^0 \supset M^1 \supset \cdots \supset M^r = \{0\}$, such that for every $i = 1, \dots, r$, $M^{i-1}/M^i \cong (k[X_0, X_1, X_2]/P)(d_i)$ for some integer d_i. For every $X \in k[X_0, X_1, X_2]_1 - P$, the dehomogenization of M with respect to X equals $\mathcal{O}_{\mathbb{P}^2,p}/\langle F_p, G_p\rangle$ and has an induced Jordan-Hölder filtration whose associated graded pieces are the dehomogenizations of the graded modules M^{i-1}/M^i. Relate the length of the dehomogenization of M, the Hilbert polynomial of M and the integer r.

(b) This problem is rather difficult. Attempt it, but you don't have to solve it. Denote by $e(C; p)$, resp. $e(D; p)$, the Hilbert-Samuel multiplicity of C at p, resp. of D at p. Prove that $i(C, D; p) \geq e(C; p)e(D; p)$. **Hint:** Work in affine coordinates for which $p = (0, 0)$. First consider the case that $C = \mathbb{V}(f), D = \mathbb{V}(g)$ where f and g are relatively prime homogeneous polynomials in x, y. Next deduce the case where f and g are not necessarily homogeneous, but the tangent cones of C and D at p have no common irreducible component. The general case can be deduced from this one by an "semicontinuity" argument.

(c) Let X be a plane curve and $p \in X$ an element. Prove that for all but finitely many lines L in $\mathbb{P}^2$ containing p, $i(X, L; p) = e(X; p)$.

Required Problem 2, Bézout's Theorem in the Plane: This problem continues the previous problem. Let $d = \deg(F)$ and let $e = \deg(G)$. Assume $C \cap D = \{p_1, \dots, p_m\}$, i.e., $C \cap D$ has no irreducible component of dimension 1.

Define $M = k[X_0, X_1, X_2]/\langle F, G\rangle$ as a graded module. For every $i = 1, \dots, m$, define $M_i = \text{Image}(\phi_{P_i})$ where P_i is the homogeneous ideal of p_i and where $\phi_{P_i} : k[X_0, X_1, X_2]/\langle F, G\rangle \to (k[X_0, X_1, X_2]/\langle F, G\rangle)_{P_i}$ is the localization homomorphism.

For the following homomorphism of graded modules, prove both the kernel and cokernel have finite length:

$$\phi : M \to \oplus_{i=1}^m M_i.$$

Hint: This requires more about the Jordan-Hölder filtration and associated primes. For a graded module M, there exists a filtration of M, $M = M^0 \supset \cdots \supset M^r = \{0\}$, such that for every $j = 1, \dots, r$, $M^{j-1}/M^j \cong (k[X_0, X_1, X_2]/Q_j)(d_j)$ where Q_j is an associated prime of M. If Q is a minimal associated prime, then $(M^{j-1}/M^j)_P$ is nonzero iff $P_j = P$. So the graded pieces in the filtration of M_i are the associated graded pieces in the filtration of M such that $Q_j = P_i$.

Remark: It follows that the Hilbert polynomial of M equals the sum over i of the Hilbert polynomial of M_i. On the one hand, there is an exact sequence of graded modules,

$$0 \to k[X_0, X_1, X_2](-d-e) \xrightarrow{(G,-F)^\dagger} k[X_0, X_1, X_2](-d) \oplus k[X_0, X_1, X_2](-e)$$
$$\xrightarrow{(F,G)} k[X_0, X_1, X_2] \xrightarrow{k} [X_0, X_1, X_2]/\langle F, G\rangle \to 0,$$

from which it easily follows the Hilbert polynomial of M is de. On the other hand, by Problem 1, the Hilbert polynomial of each M_i is the intersection multiplicity $i(C, D; p_i)$. This gives *Bézout's theorem in the plane*,

$$\deg(C) \cdot \deg(D) = \sum_{p_i \in C \cap D} i(C, D; p_i).$$

Required Problem 3: This is essentially (Hartshorne, Exer. I.7.5). Let $C \subset \mathbb{P}^2_k$ be a plane curve of degree $d \geq 1$.

(a) If there exists $p \in C$ such that $e(C; p) = d$, prove C is a union of lines containing p.

(b) If C is irreducible, and $p \in C$ is a point such that $e(C; p) = d - 1$, prove the projection from p is birational: $\pi_p : (C - \{p\}) \to \mathbb{P}^1_k$.

Required Problem 4: Find an example of a weakly projective morphism $F : X \to Y$ that is not strongly projective. If you are ambitious, find an example where X and Y are quasi-compact and separated (one was given in lecture . . .).

Problem 5: Assume char(k) does not divide 6. Combine Problem 2 with Problem 2 from Problem Set 6 to deduce that every smooth plane curve C of degree $d \geq 3$ has at most $3d(d-2)$ flex lines.

Problem 6: If char(k) = 3, give an example of a smooth plane curve C of degree $d \geq 3$ having infinitely many flex lines. If you get stuck, look up (Hartshorne, Exer. IV.2.4).

Problem 7: Find two homogeneous polynomials $F_2 \in k[X_0, X_1, X_2, X_3]_2, F_3 \in k[X_0, X_1, X_2, X_3]_3$ such that $\mathbb{V}(F_2, F_3)$ is the rational normal curve $C = \{[s_0^3, s_0^2 s_1, s_0 s_1^2, s_1^3] \in \mathbb{P}^3_k | [s_0, s_1] \in \mathbb{P}^1_k\}$ Note that F_2, F_3 do *not* generate the homogeneous ideal $\mathbb{I}(C)$.

Problem 8: For every integer $n \geq 3$, find $n-1$ homogeneous polynomials $F_i \in k[X_0, \ldots, X_n]_i$, $i = 2, \ldots, n$, such that $\mathbb{V}(F_2, \ldots, F_n)$ is the rational normal curve $C = \{[s_0^n, s_0^{n-1}s_1, \ldots, s_0 s_1^{n-1}, s_1^n] \in \mathbb{P}_k^n | [s_0, s_1] \in \mathbb{P}_k^1\}$.

Problem 9: Let $C \subset \mathbb{P}_k^n$ be an irreducible curve contained in no hyperplane. Let $p \in C$ be any point, and let $\pi_p : C - \{p\} \to \mathbb{P}_k^{n-1}$ be projection from p. Denote by D the closure of the image of C. Prove that D is contained in no hyperplane and $\deg(D) \leq \deg(C) - 1$.

Problem 10: This problem continues Problem 9. Prove that the only irreducible curve $C \subset \mathbb{P}_k^n$ of degree 1 is a line and use this to prove that $\deg(C) \geq n$ if C is an irreducible curve contained in no hyperplane.