18.725 PROBLEM SET 6

Due date: Friday, November 5 in lecture. Late work will be accepted only with a medical note or for another Institute-approved reason. You are strongly encouraged to work with others, but the final write-up should be entirely your own and based on your own understanding.

Read through all the problems. Write solutions to the "Required Problems", 1, 2, 3, and 4 and 1 additional problem of your choice to a total of 5 problems. I realize some optional problems have follow-up problems, which might seem at odds with writing up only 1 additional problem. You are encouraged to work through problems you don't write up. You are also allowed to write the solution to a problem without writing the solution to the problem it follows-up.

Required Problem 1: This is a "multilinear algebra problem" introducing the derivative and Hessian of a polynomial. The next problem relates the Hessian of a homogeneous polynomial on $\mathbb{P}^2$ to the *flex lines* of the associated plane curve.

For every finite-dimensional k-vector space V denote by $V^\vee$ the dual vector space $\mathrm{Hom}_k(V,k)$. Denote by $k[V^\vee]$ the ring of polynomial functions on V, i.e., the k-subalgebra of $\mathrm{Hom}_{\mathrm{Set}}(V,k)$ generated by $V^\vee$. There is a unique $\mathbb{Z}_{\geq 0}$-grading on $k[V^\vee]$ such that $k[V^\vee]_0 = k$ and $k[V^\vee]_1 = V^\vee$. For every integer $r \geq 0$, denote by $S^r(V^\vee)$ the k-vector space $k[V^\vee]_r$, called the r^{th} *symmetric power of* $V^\vee$. Denote by $(\mathbb{A}V, \mathcal{O}_{\mathbb{A}V})$ the unique affine variety whose underlying point-set is V and whose coordinate ring $\mathcal{O}_{\mathbb{A}V}(\mathbb{A}V)$ is $k[V^\vee]$. (Usually this variety is just denoted $(V, \mathcal{O}_V)$, but in this problem this notation distinguishes V as a k-vector space from V as an affine variety.)

(a) Denote by M the (left) $k[V^\vee]$-module $M = k[V^\vee] \otimes_k V^\vee$ where $f \cdot (g \otimes x) := (fg) \otimes x$ for every $f, g \in k[V^\vee]$ and $x \in V^\vee$. Prove there exists a unique k-derivation $d : k[V^\vee] \to M$ such that $d(x) = 1 \otimes x$ for every $x \in V^\vee = k[V^\vee]_1$. The induced homomorphism of $k[V^\vee]$-modules, $\Omega_{k[V^\vee]/k} \to M$, is an isomorphism (you need not prove this).

Solution, Uniqueness: Let $d_1, d_2 : k[V^\vee] \to M$ be k-derivations such that $d_1(x) = d_2(x) = 1 \otimes x$ for every $x \in V^\vee$. Then $\partial := d_1 - d_2 : k[V^\vee] \to M$ is a k-derivation such that $\partial(x) = 0$ for every $x \in V^\vee$. For every ring homomorphism $\phi : R \to S$ and every R-derivation $\partial : S \to M$, the kernel of ∂ is an R-subalgebra of S. The smallest k-subalgebra of $k[V^\vee]$ containing $V^\vee$ is all of $k[V^\vee]$. Therefore $\partial = 0$, i.e., $d_1 = d_2$.

Existence: Let $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ be an ordered basis for V and let $(x_1, \dots, x_n)$ be the dual ordered basis for $V^\vee$. A basis for $k[V^\vee]$ is the set of all monomials $\{\underline{x}^{\underline{i}} := x_1^{i_1} \cdots x_n^{i_n} | \underline{i} = (i_1, \dots, i_n) \in (\mathbb{Z}_{\geq 0})^n\}$. Define $d : k[V^\vee] \to M$ to be the unique k-linear map such that $d(1) = 0$, and for every $\underline{i} \neq (0, \dots, 0)$,

$$d(\underline{x}^{\underline{i}}) = \sum_{1 \leq m \leq n, i_m > 0} (i_m x_m^{i_m - 1} \cdot \prod_{l \neq m} x_l^{i_l}) \otimes x_m.$$

If we tensor this map with the inclusion of rings $k[V^\vee] \subset k[x_1, \ldots, x_n][1/x_1, \ldots, 1/x_n]$, then for every $\underline{i} \in (\mathbb{Z}_{\geq 0})^n$, there is a formula,

$$d(\underline{x}^{\underline{i}}) = \sum_{m=1}^{n} (i_m \underline{x}^{\underline{i}}/x_m) \otimes m_j.$$

The claim is that d is a k-derivation. Because d is k-linear, it suffices to check for every pair of monomials $\underline{x}^{\underline{i}}, \underline{x}^{\underline{j}}$,

$$d(\underline{x}^{\underline{i}} \cdot \underline{x}^{\underline{j}}) = x^{\underline{i}} \cdot d(\underline{x}^{\underline{j}}) + \underline{x}^{\underline{j}} \cdot d(\underline{x}^{\underline{i}}).$$

This equation can be checked after tensoring with the larger ring, $k[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$:

$$d(\underline{x}^{\underline{i}+\underline{j}}) = \sum_{m=1}^{n} ((i_m + j_m)\underline{x}^{\underline{i}+\underline{j}}/x_m) \otimes x_n = \\ \underline{x}^{\underline{j}} \cdot (\sum_{m=1}^{n} (i_m \underline{x}^{\underline{i}}/x_m) \otimes x_m) + \underline{x}^{\underline{i}} \cdot (\sum_{m=1}^{n} (j_m \underline{x}^{\underline{j}}/x_m) \otimes x_m).$$

Finally, for every $x = a_1x_1 + \cdots + a_nx_n \in V^\vee$, $d(x) = a_1 d(x_1) + \cdots + a_n d(x_n) = a_1 \cdot 1 \otimes x_1 + \cdots + a_n \cdot 1 \otimes x_n = 1 \otimes (a_1x_1) + \cdots + 1 \otimes (a_nx_n) = 1 \otimes x$.

(b) For every integer $r \geq 0$, denote by $d_r : S^r(V^\vee) \to S^{r-1}(V^\vee) \otimes V^\vee$ the restriction of d, and denote by $\widetilde{d_r} : S^r(V^\vee) \to \mathrm{Hom}_k(V, S^{r-1}(V^\vee))$ the composition of d_r with the canonical isomorphism $S^{r-1}(V^\vee) \otimes_k V^\vee \cong \mathrm{Hom}_k(V, S^{r-1}(V^\vee))$. Given $F \in S^r(V^\vee)$, denote the image under d_r by d_rF, and denote the induced linear map by $\widetilde{d_r}F : V \to S^{r-1}(V^\vee)$. Let $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ be an ordered basis for V and let $(x_1, \ldots, x_n)$ be the dual ordered basis for $V^\vee$. Prove for every $F \in S^r(V^\vee)$ and every $i = 1, \ldots, n$,

$$\widetilde{d_r}F(\mathbf{e}_i) = \frac{\partial F}{\partial x_i}.$$

Solution: By the construction in (a), for every monomial $\underline{x}^{\underline{i}}$,

$$\widetilde{d_r}\underline{x}^{\underline{i}}(\mathbf{e}_m) = i_m \underline{x}^{\underline{i}}/x_m = \frac{\partial \underline{x}^{\underline{i}}}{\partial x_m}.$$

Because both sides of the equation are k-linear, and because the monomials form a k-basis for $k[V^\vee]$, the equation holds for every polynomial F.

(c) For every integer $r \geq 0$, denote by $\mathrm{Hess}_r : S^r(V^\vee) \to \mathrm{Hom}_k(V, S^{r-2}(V^\vee) \otimes_k V^\vee)$ the unique linear map $F \mapsto \mathrm{Hess}_r(F)$ such that for every $v \in V$, $\mathrm{Hess}_r(F)(v) = d_{r-1}((\widetilde{d_r}F)(v))$. This is the *Hessian of* F. Let $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ be an ordered basis for V, and let $(x_1, \ldots, x_n)$ be the dual ordered basis for $V^\vee$. Prove that for every $F \in S^r(V^\vee)$ and every $1 \leq j \leq n$,

$$\mathrm{Hess}_r(F)(\mathbf{e}_j) = \sum_{i=1}^{n} \frac{\partial^2 F}{\partial x_i \partial x_j} \otimes x_i.$$

Considering the terms $\partial^2 F/\partial x_i \partial x_j$ to be "coefficients", $Hess_r(F)$ is an $n \times n$ matrix whose (i,j)-entry is the degree $r-2$ homogeneous polynomial $\partial^2 F/\partial x_i \partial x_j$. For every point $p \in \mathbb{A}V$, denote by $\mathrm{Hess}_r(F)(p) : V \to V^\vee$ the k-linear map obtained by evaluating these degree $r-2$ homogeneous polynomials at p.

Solution: By (b), for every F and every m, $\widetilde{d_r}F(e_m) = \partial F/\partial x_m$. Applying (b) again,

$$d_{r-1}(\widetilde{d_r}F(e_m)) = \sum_{l=1}^{n} \frac{\partial}{\partial x_l} \left(\frac{\partial F}{\partial x_m} \right) \otimes x_l.$$

Required Problem 2: This problem continues the previous problem. Let $\dim_k V = 3$ so that $\mathbb{A}V \cong \mathbb{A}^3_k$. Denote by $(\mathbb{P}V, \mathcal{O}_{\mathbb{P}V})$ the projective variety $(\mathbb{A}V - \{0\})/(v \sim \lambda v) \cong \mathbb{P}^2_k$. Let $r \geq 1$, let $F \in S^r(V^\vee)$ be an irreducible polynomial, and let $C = \mathbb{V}(F) \subset \mathbb{P}V$ be the associated plane curve.

(a) Let $p \in C$, and let $v \in V$. Prove that $(\widetilde{d_r}F(v))(p) = 0$ iff there exists a line $L \subset \mathbb{P}V$ tangent to C at p and such that the associated affine cone $\mathbb{A}L \subset \mathbb{A}V$ contains v. **Hint:** If $v \in \mathbb{A}\{p\}$ this is trivial, and if $v \notin \mathbb{A}\{p\}$, choose an ordered basis $(\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2)$ for V such that $p = [1,0,0]$ and $v = (0,1,0)$.

Solution: If v is proportional to p then by Euler identity, Problem 1 on Problem Set 4, $(\widetilde{d_r}F(v))(p)$ is a scalar multiple of $rF(p) = 0$. Also, for every tangent line L to C at p, $\mathbb{A}L$ contains the 1-dimensional subspace corresponding to p; in particular $v \in \mathbb{A}L$.

Thus assume v is not proportional to p. Let $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be an ordered basis for V such that $p = [\mathbf{e}_1]$ and $v = \mathbf{e}_2$. By the solution to Problem 2(b), Problem Set 4, $\mathrm{span}(p, v) = \mathbb{V}(x_2)$ is a tangent line to C at p iff $\partial F/\partial x_0(p) = \partial F/\partial x_1(p) = 0$. By the Euler identity, $\partial F/\partial x_0(p) = F(p) = 0$. Hence $\mathbb{V}(x_2)$ is a tangent line to C at p iff $\partial F/\partial x_1(p) = 0$. By Problem 1(b), $(\widetilde{d_r}F(v))(p) = \frac{\partial F}{\partial x_1}(p)$. Therefore $(\widetilde{d_r}F(v))(p) = 0$ iff there exists a line $L \subset \mathbb{P}V$ tangent to C at p such that $v \in \mathbb{A}L$.

(b) Assume $\mathrm{char}(k)$ does not divide $2(r-1)$. For every point $p \in C$ a tangent line L to C at p, $L \subset \mathbb{P}V$, is a *flex line to C at p* if the germ at p of the restriction to L of the dehomogenization of F is contained in $\mathfrak{m}_p^3\mathcal{O}_{L,p}$, i.e., the restriction of F to L vanishes to order ≥ 3 at p. Prove there is a flex line to C at p iff the 3×3 Hessian $\mathrm{Hess}_r(F)(p)$ is not an isomorphism, i.e., iff with respect to some (and hence any) basis, the determinant of the 3×3 Hessian matrix is 0. **Hint:** There are 2 cases depending on whether p is a smooth or singular point of C. In both cases, choose an ordered basis $(\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2)$ for V such that $p = [1,0,0]$ and such that tangent line under consideration is $\{[a,b,0] | a, b \in k\}$.

Solution: Let $(\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2)$ be an ordered basis such that $p = [1,0,0]$ and such that $L = \mathbf{V}(x_2)$ is a tangent line to C at p. Expand the polynomial F about $(1,0,0)$ as,

$$F(x_0, x_1, x_2) = x_0^r F(1, \tfrac{x_1}{x_0}, \tfrac{x_2}{x_0}) =$$
$$x_0^r \left[b_2 \tfrac{x_2}{x_1} + c_{1,1} \tfrac{x_1}{x_0}^2 + c_{1,2} \tfrac{x_1}{x_0} \tfrac{x_2}{x_0} + c_{2,2} \tfrac{x_2}{x_0}^2 + \dots \right].$$

In particular the restriction of F to L is $F(1,t,0) = c_{1,1}t^2 + \dots$. If $c_{1,1} = 0$, the restriction of F to L vanishes to order ≥ 3 at p. Suppose that $c_{1,1} \neq 0$. If $b_2 = 0$, then every line M containing p is a tangent line to C at p. Let $\lambda \in k$ be a solution of the quadratic equation $c_{1,1}t^2 + c_{1,2}t + c_{2,2}$. The restriction of F to the line $M = \mathbb{V}(x_1 - \lambda x_2)$ is $F(1, \lambda t, t) = (c_{1,1}\lambda^2 + c_{1,2}\lambda + c_{2,2})t^2 + \dots$, which vanishes to order ≥ 3 at $t = 0$. So if $c_{1,1} = 0$ or $b_2 = 0$, there exists a flext line to C at p.

Conversely, suppose that $c_{1,1}, b_2 \neq 0$. Then the unique tangent line to C at p is L, and the restriction of F to L is $F(1,t,0) = c_{1,1}t^2 + \dots$, which vanishes to order 2 at $t = 0$. Therefore there exists a flex line to C at p iff $b_2 = 0$ or $c_{1,1} = 0$.

Next consider the partial derivatives of F at p. By Problem 2(b) from Problem Set 4, $\partial F/\partial x_0(p) = \partial F/\partial x_1(p) = 0$. With the notation above, $\partial F/\partial x_2(p) = b_2$. By the Euler identity, $\partial^2 F/\partial x_0^2(p) = (r-1)\partial F/\partial x_0(p) = 0$, $\partial^2 F/\partial x_0 \partial x_1(p) =$

$(r-1)\partial F/\partial x_1(p) = 0$, and $\partial^2 F/\partial x_0 \partial x_2(p) = (r-1)b_2$. Because char$(k)$ does not divide $r-1$, this is 0 iff $b_2 = 0$. Finally, with the notation above, $c_{1,1} = \partial^2 F/\partial x_1^2(p)$.

Denote $a_{i,j} = \partial^2 F/\partial x_i \partial x_j(p)$ for $0 \le i, j \le 2$. Then, by (c),

$$\text{Hess}_r(F)(p) = \begin{pmatrix} 0 & 0 & a_{0,2} \\ 0 & a_{1,1} & a_{1,2} \\ a_{0,2} & a_{1,2} & a_{2,2} \end{pmatrix}.$$

The determinant is $-a_{0,2}^2 a_{1,1}$. Hence the determinant is 0 iff either $a_{0,2} = 0$ or $a_{1,1} = 0$, i.e., iff either $b_2 = 0$ or $c_{1,1} = 0$. Therefore there is a flex line to C at p iff the determinant of $\text{Hess}_r(F)(p) = 0$.

(c) Assume char(k) does not divide 6. Compute all the flex lines to the smooth cubic plane curve $\mathbb{V}(x_0^3 + x_1^3 + x_2^3) \subset \mathbb{P}_k^2$. **Hint:** There are 9 of them.

Solution: The Hessian matrix is,

$$\begin{pmatrix} 6x_0 & 0 & 0 \\ 0 & 6x_1 & 0 \\ 0 & 0 & 6x_2 \end{pmatrix}.$$

The determinant of the Hessian matrix is $6^3 x_0 x_1 x_2$. Because char(k) does not divide 6, the determinant is 0 at $p = [a_0, a_1, a_2]$ iff $a_0 a_1 a_2 = 0$. By (b), p is in C and C has a flex line to C at p iff, $a_0^3 + a_1^3 + a_2^3 = a_0 a_1 a_2 = 0$. If $a_0 = 0$, the first equation reduces to $a_1^3 + a_2^3 = 0$, which has 3 solutions $[0, 1, -1], [0, 1, -\omega], [0, 1, -\omega^2]$ where ω is a zero of $t^2 + t + 1$. The corresponding flex lines are $\mathbb{V}(x_1 + x_2), \mathbb{V}(x_1 + \omega^2 x_2), \mathbb{V}(x_1 + \omega x_2)$. Permuting the roles of x_0, x_1, and x_2, it follows the flex lines to C are precisely,

$$\begin{aligned} L_1 &= \mathbb{V}(x_1 + x_2), \\ L_2 &= \mathbb{V}(x_1 + \omega^2 x_2), \\ L_3 &= \mathbb{V}(x_1 + \omega x_2), \\ L_4 &= \mathbb{V}(x_0 + x_2), \\ L_5 &= \mathbb{V}(x_0 + \omega^2 x_2), \\ L_6 &= \mathbb{V}(x_0 + \omega x_2), \\ L_7 &= \mathbb{V}(x_0 + x_1), \\ L_8 &= \mathbb{V}(x_0 + \omega^2 x_2), \\ L_9 &= \mathbb{V}(x_0 + \omega x_2) \end{aligned}$$

Required Problem 3: This problem together with the next problem work through the construction of the *blowing up of an affine variety along an ideal.* You are encouraged to do these problems in whatever order makes most sense to you (the sketch below is only one of many approaches). You are also encouraged to read the section about blowing up in *The Geometry of Schemes* by Eisenbud and Harris.

Let $X \subset \mathbb{A}_k^n$ be an affine algebraic set and denote $A = k[X]$. Let $I \subset A$ be an ideal and let $f_1, \dots, f_r \in I$ be generators. Let $\mathbb{A}_k^r$ be an affine space with coordinates $y_1, \dots, y_r$. For every $f \in I$, denote by $F_f : D(f) \to \mathbb{A}_k^r$ the regular morphism,

$$p \mapsto (f_1(p)/f(p), \dots, f_r(p)/f(p)),$$

and denote by $Y_f \subset X \times \mathbb{A}_k^r$ the Zariski closure of the image of $\text{Id}_X \times F_f : D(f) \to X \times \mathbb{A}_k^r$. Denote by $B_I A$ the *blowup algebra of I in A*, i.e., the $\mathbb{Z}_{\le 0}$-graded A-algebra,

$$B_I A := A[I/t] \subset A[1/t], \text{ i.e., } A \oplus I \oplus \cdots \oplus I^k \oplus \dots$$

where $\deg(a) = 0$ for all $a \in A$ and $\deg(1/t) = -1$ (cf. also, Eisenbud's *Commutative Algebra*, p. 148; the reason for using $1/t$ as a variable instead of t is to make this compatible with the *Rees algebra*, p. 170).

(a) Denote by $\mathrm{pr}_{X,f} : Y_f \to X$ the restriction of $\mathrm{pr}_X : X \times \mathbb{A}^r_k \to X$ and denote by $G_f : D(f) \to Y_f$ the morphism induced by $\mathrm{Id}_X \times F_f$. Prove the composition $G_f^\# \circ \mathrm{pr}_X^\# : A \to A[1/f]$ is the usual k-algebra homomorphism $i : A \to A[1/f]$.

Solution: The composition $\mathrm{pr}_X \circ G_f = \mathrm{Id}_X|_{D(f)}$ by definition of G_f. Therefore $G_f^\# \circ \mathrm{pr}_X^\# = (\mathrm{pr}_X \circ G_f)^\#$ is the usual k-algebra homomorphism i of restriction to $D(f)$.

(b) Prove that $(\mathrm{Id}_X \times F_f)^\# : k[Y_f] \to A[1/f]$ is injective and the image is the subring of $A[1/f]$ generated by A and $\{g/f | g \in I\}$. Prove this is the smallest k-subalgebra $A_{I,f} \subset A[1/f]$ containing $i(A)$ and such that $i(I) \cdot A_{I,f}$ equals the principal ideal $\langle f \rangle A_f$.

Solution: Denote $\phi_f = (\mathrm{Id}_X \times F_f)^\# : A[x_1, \dots, x_r] \to A[1/f]$, the unique A-algebra homorphism such that $\phi_f(x_j) = f_j/f$ for every $j = 1, \dots, r$. The image of ϕ_f is the smallest subalgebra generated by $i(A)$ and $f_1/f, \dots, f_r/f$. Because every element of I is an A-linear combination of $f_1, \dots, f_r$, the image of ϕ_f is $A_{I,f}$.

By Problem 13(b) from Problem Set 1, $k[Y_f] = A[x_1, \dots, x_r]/\ker(\phi_f)$ so that $G_f^\# : k[Y_f] \to A_{I,f}$ is an isomorphism.

Since $i(I)A_{I,f}$ is generated by $i(f_1), \dots, i(f_r)$, and since every $i(f_j) = i(f) \cdot (f_j/f)$, $i(I)A_{I,f}$ is the principal ideal $\langle i(f) \rangle A_{I,f}$. Let $B \subset A[1/f]$ be any A-subalgebra such that $i(I)B$ is $\langle i(f) \rangle B$. Then for every f_j, $i(f_j) = i(f)b$ for some $b \in B$. In $A[1/f]$, $b = f_j/f$ is the unique element such that $i(f)b = i(f_j)$. Therefore $f_j/f \in B$ for $j = 1, \dots, r$, i.e., $A_{I,f} \subset B$.

(c) Denote by $\alpha_f : A_{I,f} \otimes_A B_I A \to A_{I,f}$ the unique homomorphism of $A_{I,f}$-algebras such that for every $g \in I$, $\alpha_f(1 \otimes (g/t)) = g/f$; in particular $\alpha_f(1 \otimes (f/t)) = 1$. Let $(H : T \to X, \beta)$ be a pair of a regular morphism of algebraic varieties $H : T \to X$ together with a homomorphism of $\mathcal{O}_T(T)$-algebras $\beta : \mathcal{O}_T(T) \otimes_A B_I A \to \mathcal{O}_T(T)$ such that $\beta(1 \otimes (f/t)) = 1$. Prove there is a unique regular morphism $\phi : T \to Y_f$ such that

(i) $H = \mathrm{pr}_{X,f} \circ \phi$, and
(ii) via the canonical isomorphisms $\mathcal{O}_T(T) \otimes_{A_{I,f}} (A_{I,f} \otimes_A B_I A) \cong \mathcal{O}_T(T) \otimes_A B_I A$ and $\mathcal{O}_T(T) \otimes_{A_{I,f}} A_{I,f} \cong \mathcal{O}_T(T)$, β equals $\mathrm{Id}_{\mathcal{O}_T(T)} \otimes \alpha_f : \mathcal{O}_T(T) \otimes_{A_{I,f}} (A_{I,f} \otimes_A B_I A) \to \mathcal{O}_T(T) \otimes_{A_{I,f}} A_{I,f}$.

Hint: Use the universal property of affine varieties together with Proposition 8.5.

Solution: First let's prove there exists an $A_{I,f}$-algebra homomorphism $\alpha_f : A_{I,f} \otimes_A B_I A \to A_{I,f}$ such that $\alpha_f(1 \otimes (g/t)) = g/f$ for every $g \in I$. This is equivalent to an A-algebra homomorphism, $\widetilde{\alpha}_f : B_I A \to A_{I,f}$ such that $\widetilde{\alpha}_f(g/t) = g/f$ for every $g \in I$. There is a unique A-algebra homomorphism $A[t, 1/t] \to A[1/f]$ such that $t \mapsto f$. The restriction to $B_I A \subset A[t, 1/t]$ is an A-algebra homomorphism whose image is contained in $A_{I,f} \subset A[1/f]$. Denote this by $\widetilde{\alpha}_f : B_I A \to A_{I,f}$. By construction, for every $g \in I$, $\widetilde{\alpha}_f(g/t) = g/f$. Because $B_I A$ is generated as an A-algebra by I/t, it is clear that $\widetilde{\alpha}_f$ is the unique A-algebra homomorphism with this property.

The next step is to prove $\widetilde{\alpha}_f$ is an isomorphism. One way to prove this is to relate $A_{I,f}$ to the graded localization $(B_IA[1/(f/t)])_0$. By construction, $\widetilde{\alpha}_f((f/t)-1) = 0$, i.e., $\widetilde{\alpha}_f$ factors through an A-algebra homomorphism, $B_IA/\langle (f/t)-1\rangle \to A_{I,f}$. By Proposition 8.5, $B_IA/\langle (f/t)-1\rangle \cong (B_IA[1/(f/t)])_0$. Denote the associated A-algebra homomorphism, $\widehat{\alpha}_f : (B_IA[1/(f/t)])_0 \to A_{I,f}$. The inclusion $B_IA \subset A[t,1/t]$ induces an A-algebra homomorphism $B_IA[1/(f/t)] \to A[t,1/t,1/f]$. The kernel is $\langle b \in B_IA | \exists r > 0, f^r b = 0\rangle$. As elements in $A[t,1/t]$, $f^r b = 0$ iff $(f/t)^r b = 0$ iff b is in the kernel of $B_IA \to B_IA[1/(f/t)]$. Therefore $B_IA[1/(f/t)] \to A[t,1/t,1/f]$ is injective. Clearly $(B_IA[1/(f/t)])_0 = A[1/f] \cap (B_IA[1/(f/t)])_0$ inside $A[t,1/t,1/f]$. In other words, the composition of $\widehat{\alpha}_f$ with the inclusion $A_{I,f} \subset A[1/f]$ is the injective A-algebra homomorphism $(B_IA[1/(f/t)])_0 \to A[1/f]$; thus $\widehat{\alpha}_f$, and hence $\widetilde{\alpha}_f$, is injective. The image of $\widetilde{\alpha}_f$ is an A-subalgebra of $A_{I,f}$ that contains I/f, thus it is all of $A_{I,f}$. So $\widetilde{\alpha}_f$ is also surjective, i.e., $\widetilde{\alpha}_f$ is an isomorphism of A-algebras.

Now let $(H : T \to X, \beta)$ be a pair as above. Consider the induced A-algebra homomorphism $\widetilde{\beta} : B_IA \to \mathcal{O}_T(T)$, $\widetilde{\beta}(b) = \beta(1 \otimes b)$. Because $\beta(1 \otimes (f/t)) = 1$, $\widetilde{\beta}$ factors through $B_IA/\langle (f/t)-1\rangle$. Denote by $\phi^{\#}$ the composition $\widetilde{\beta} \circ (\widetilde{\alpha}_f)^{-1} : A_{I,f} \to \mathcal{O}_T(T)$. This is clearly the unique A-algebra homomorphism such that $\mathrm{Id}_{\mathcal{O}_T(T)} \otimes_{A_{I,f}} \alpha_f : \mathcal{O}_T(T) \otimes_{A_{I,f}} (A_{I,f} \otimes_A B_IA) \to \mathcal{O}_T(T) \otimes_{A_{I,f}} A_{I,f}$ equals β via the obvious isomorphims. By the universal property of affine varieties, there exists a unique regular morphism $\phi : T \to Y_f$ whose associated pullback homomorphism is $\phi^{\#}$. Because $\phi^{\#}$ is a homomorphism of A-algebras, $\mathrm{pr}_{X,f} \circ \phi = F$. Thus $\phi : T \to Y_f$ is the unique regular morphism such that $\mathrm{pr}_{X,f} \circ \phi = F$ and such that the pullback of α_f by ϕ equals β.

Required Problem 4: This problem continues the previous problem. Again, you are encouraged to approach this problem in the way that makes most sense to you. You do not need to write out all details, but you should understand how your approach settles the details.

(a) For every ordered pair $(f,g) \in I \times I$, define $Y_{f,g} \subset Y_f$ to be $D(g/f)$. Denote by $\alpha'_{g,f} : k[Y_{f,g}] \otimes_A B_IA \to k[Y_{f,g}]$ the unique $k[Y_{f,g}]$-algebra homomorphism commuting with α_f and the $A_{I,f}$-module homomorphisms $A_{I,f} \to A_{I,f}[(g/f)^{-1}] = k[Y_{f,g}]$, $A_{I,f} \otimes_A B_IA \to k[Y_{f,g}] \otimes_A B_IA$. Denote by $\alpha''_{g,f}$ the unique $\mathbb{Z}_{\leq 0}$-graded $k[Y_{f,g}]$-algebra automorphisms,

$$\alpha''_{g,f} : k[Y_{f,g}] \otimes_A B_IA \to k[Y_{f,g}] \otimes_A B_IA,$$
$$a/t^n \mapsto (g/f)^{-n} \cdot a/t^n.$$

Denote $\alpha_{g,f} = \alpha'_{g,f} \circ \alpha''_{g,f}$. Prove the pair $(\mathrm{pr}_{X,f}|_{Y_{f,g}} : Y_{f,g} \to X, \alpha_{g,f})$ satisfies the condition in (c) of Problem 3 for $g \in I$. Deduce existence of a unique regular morphism $\phi_{g,f} : Y_{g,f} \to Y_g$ such that $\mathrm{pr}_{X,g} \circ \phi_{g,f} = \mathrm{pr}_{X,f}|_{Y_{f,g}}$ and $\phi^*_{g,f}\alpha_g = \alpha_{g,f}$.

Solution: It is not obvious that there exists a homomorphism $\alpha''_{g,f}$ as above. There is certainly a unique A-algebra homomorphism $\chi : A_{I,f} \otimes_A B_IA \to A[t,1/t,1/f,/g]$ such that $\chi(1 \otimes (a/t)) = f/g \cdot a/t$. It needs to be checked this factors through $(A_{I,f} \otimes_A B_IA)[1/(g/f)]$. The induced homomorphism $(A_{I,f} \otimes_A B_IA)[1/(g/f)] \to A[t,1/t,1/f,1/g]$ is injective by a similar argument to the one in Problem 3(c). The image $\chi(A_{I,f} \otimes \{1\})$ is contained in the image of $(A_{I,f} \otimes_A B_IA)[1/(g/f)]$. And for every $a \in I$, $\chi(1 \otimes (a/t)) = 1/(g/f) \cdot a/t$, which is in the image. Therefore the image of χ is in $(A_{I,f} \otimes_A B_IA)[1/(g/f)]$. Also denote by χ the induced A-algebra homomorphism $\chi : A_{I,f} \otimes_A B_I A \to (A_{I,f} \otimes_A B_I A)[1/(g/f)]$. Because $\chi(g/f \otimes 1) = g/f \otimes 1$ is invertible, there is an induced $A_{I,f}[1/(g/f)]$-algebra homomorphism $\alpha''_{g,f} : (A_{I,f} \otimes_A B_I A)[1/(g/f)] \to (A_{I,f} \otimes_A B_I A)[1/(g/f)]$.

The homomorphism $\alpha'_{g,f} \circ \alpha''_{g,f} : (A_{I,f}[1/(g/f)]) \otimes_A B_I A \to A_{I,f}[1/(g/f)]$ is an $A_{I,f}[1/(g/f)]$-algebra homomorphism such that for every $a \in I$, $\alpha'_{g,f} \circ \alpha''_{g,f}(1 \otimes (a/t)) = \alpha'_{g,f}(1/(g/f) \otimes (a/t)) = 1/(g/f) \cdot a/f$. In particular, $1 \otimes (g/t) \mapsto 1/(g/f) \cdot g/f = 1$. So $(\mathrm{pr}_{X,f} : Y_{f,g} \to X, \alpha_{g,f})$ satisfies the hypotheses of Problem 3(c) for $g \in I$. Therefore there is a unique regular morphism $\phi_{g,f} : Y_{f,g} \to Y_g$ such that $\mathrm{pr}_{X,g} \circ \phi_{g,f} = \mathrm{pr}_{X,f}$ and $\phi^*_{g,f}\alpha_g = \alpha_{g,f}$.

(b) Prove that $\phi_{g,f}(Y_{f,g}) \subset Y_{g,f}$ and prove that $\phi_{g,f}$ and $\phi_{f,g}$ are inverse isomorphisms $Y_{f,g} \cong Y_{g,f}$.

Solution: The pullback homomorphism $\phi^{\#}_{g,f} : A_{I,g} \to A_{I,f}[1/(g/f)]$ is an A-algebra homomorphism. In particular, $f = \phi^{\#}_{g,f}(g \cdot f/g) = g \cdot \phi^{\#}_{g,f}(f/g)$. Therefore $\phi^{\#}_{g,f}(f/g) = 1/(g/f)$, which is invertible. So the image of $\phi_{g,f}$ is contained in $D(f/g) = Y_{g,f}$. Both $\phi_{f,g} \circ \phi_{g,f} : Y_{f,g} \to Y_f$ and the inclusion morphism are regular morphisms that commute with projection to X and pullback α_f to $\alpha'_{g,f}$. So, by the universal property, they are equal. Therefore, for every $(f, g) \in I \times I$, $\phi_{f,g} \circ \phi_{g,f} : Y_{f,g} \to Y_{f,g}$ equals the identity morphism. In particular, this also holds for (g, f). Thus $\phi_{g,f} : Y_{f,g} \to Y_{g,f}$ and $\phi_{f,g} : Y_{g,f} \to Y_{f,g}$ are inverse regular morphisms.

(c) For every triple $(f, g, h) \in I \times I \times I$, prove $\phi^{-1}_{g,f}(Y_{g,h}) = Y_{f,g} \cap Y_{f,h}$ and prove $\phi_{h,g} \circ \phi_{g,f}|_{Y_{f,g} \cap Y_{f,h}} = \phi_{h,f}|_{Y_{f,g} \cap Y_{f,h}}$. Therefore the collection $((Y_f | f \in I), (Y_{f,g} | (f, g) \in I \times I), (\phi_{g,f} | (f, g) \in I \times I))$ satisfies the gluing lemma for objects. Denote the induced family of morphisms by $(\phi_f : Y_f \to Y)$. The variety Y is the *blowing up of X along the ideal I*.

Solution: This is very similar to the last part. To see that $\phi_{h,g} \circ \phi_{g,f} = \phi_{h,f}$, observe that both pullback α_h to the restriction of α_f to $Y_{f,g} \cap Y_{f,h}$. So by the universal property they are equal. This is the main step in proving that the collection satisfies the gluing lemma for objects.

(d) Prove the collection of morphisms $(\mathrm{pr}_{X,f} \circ \phi^{-1}_f : \phi_f(Y_f) \to X | f \in I)$ satisfies the gluing lemma for morphisms. The induced morphism $\mathrm{pr}_X : Y \to X$ is the *projection morphism*. The restriction $\mathrm{pr}_X : \mathrm{pr}^{-1}_X(X - \mathbb{V}(I)) \to (X - \mathbb{V}(I))$ is an isomorphism.

Solution: The morphisms $\phi_{g,f}$ commute with projection to X. Therefore the collection of morphisms $(\mathrm{pr}_{X,f} \circ \phi^{-1}_f : \phi_f(Y_f) \to X | f \in I)$ satisfies the gluing lemma for morphisms.

Next consider $\mathrm{pr}_X : \mathrm{pr}^{-1}_X(X - \mathbb{V}(I)) \to (X - \mathbb{V}(I))$. For every $f \in I$, $D(f)$ is an open subset of $X - \mathbb{V}(I)$. Consider $\mathrm{pr}_{X,f} : \mathrm{pr}^{-1}_{X,f}(D(f)) \to D(f)$. The induced map of k-algebras is $A[1/f] \to A_{I,f}[1/f] = A[I/f][1/f]$. This is clearly an isomorphism of k-algebras, therefore $\mathrm{pr}_{X,f} : \mathrm{pr}^{-1}_{X,f}(D(f)) \to D(f)$ is an isomorphism of varieties. Define $p^{-1}_f : D(f) \to \mathrm{pr}^{-1}_X(X - \mathbb{V}(I))$ to be the composition of $\mathrm{pr}^{-1}_{X,f} : D(f) \to \mathrm{pr}^{-1}_{X,f}(D(f))$ with ϕ_f. Because the morphisms $\phi_{g,f}$ commute with projection to X, the collection $(p^{-1}_f : D(f) \to \mathrm{pr}^{-1}_X(X - \mathbb{V}(I)) | f \in I)$ satisfies the gluing lemma for morphisms. Define $p^{-1} : (X - \mathbb{V}(I)) \to \mathrm{pr}_X^{-1}(X - \mathbb{V}(I))$ to be the induced morphism. By construction, $\mathrm{pr}_X \circ p^{-1}$ is the identity morphism on $X - \mathbb{V}(I)$. Moreover for every $f \in I$, the restriction of $p^{-1} \circ \mathrm{pr}_X$ to $\phi_f(\mathrm{pr}_{X,f}^{-1}(D(f)))$ is the inclusion morphism. So by the uniqueness part of the gluing lemma, $p^{-1} \circ \mathrm{pr}_X$ is the identity morphism on $\mathrm{pr}_X^{-1}(X - \mathbb{V}(I))$. Thus $p^{-1} = \mathrm{pr}_X^{-1}$ is an inverse of pr_X, i.e., $\mathrm{pr}_X : \mathrm{pr}_X^{-1}(X - \mathbb{V}(I)) \to (X - \mathbb{V}(I))$ is an isomorphism.

(e) For every $p \in Y$, prove the ideal $\mathrm{pr}_X^{\#} I \cdot \mathcal{O}_{Y,p}$ is a principal ideal in $\mathcal{O}_{Y,p}$.

Solution: For every $p \in Y$, there exists $f \in I$ and $q \in Y_f$ such that $p = \phi_f(q)$. So it suffices to prove for every $f \in I$ and every $q \in Y_f$, $\mathrm{pr}_{X,f}^{\#} I \cdot \mathcal{O}_{Y_f,q}$ is a principal ideal in $\mathcal{O}_{Y_f,q}$. In fact $\mathrm{pr}_{X,f}^{\#} I \cdot \mathcal{O}_{Y_f}(Y_f)$ is the principal ideal $\langle f\rangle \mathcal{O}_{Y_f}(Y_f)$ by Problem 3(b). So the image in the localization is also the principal ideal generated by f.

Problem 5: This problem continues the previous two problems. Let $g_1, \dots, g_s \in I$ be any other choice of generators, and define $(\mathrm{pr}'_{I,f} : Y'_f \to X | f \in I)$ and $\mathrm{pr}'_X : Y' \to X$ in the analogous manner as above with this choice of generators. Use the universal property of $(Y_f \to X, \alpha_f)$ to prove the variety Y'_f is canonically isomorphic to Y_f. Deduce that Y' is canonically isomorphic to Y, and the isomorphism commutes with pr_X and pr'_X.

Problem 6: Let $X = \mathbb{A}_k^{n+1}$, and let $I = \langle x_0, \dots, x_n\rangle \subset k[x_1, \dots, x_n]$. Let $E \subset \mathbb{A}_k^{n+1} \times \mathbb{P}_k^n$ be the tautological rank 1 subbundle from Problem 4(c) on Problem Set 5. Denote by $\pi_{\mathbb{A}_k^{n+1}} : E \to \mathbb{A}_k^{n+1}$ and $\pi_{\mathbb{P}_k^n} : E \to \mathbb{P}_k^n$ the projections. Let $\mathrm{pr}_X : Y \to \mathbb{A}_k^{n+1}$ be the blowing up of $\mathbb{A}_k^{n+1}$ along I.

(a) For every $f \in I_1$, define $E_f = \pi_{\mathbb{P}_k^n}^{-1}(D_+(f))$. Prove that $\pi_{\mathbb{A}_k^{n+1}}^{\#}(I) \cdot \mathcal{O}_E(E_f)$ is a principal ideal generated by $\pi_{\mathbb{A}_k^{n+1}}^{\#}(f)$. Use Problem 3(c) to deduce existence of a morphism $\eta_f : E_f \to Y_f$.

Solution: By the construction of $\mathbb{P}_k^n$, $D_+(f)$ is canonically isomorphic to the affine variety with coordinate ring $k[x_0/f, \dots, x_n/f] \subset k[x_0, \dots, x_n, 1/f]$. To distinguish the elements $x_i/f \in k[D_+(f)]$ from fractions that will occur, denote these elements by $[x_i/f]$. By definition, $E_f \subset \mathbb{A}_k^{n+1} \times D_+(f)$ is the closed subvariety $\mathbb{V}(x \cdot [y/f] - y \cdot [x/f] | x, y \in k[x_0, \dots, x_n]_1)$. In particular, the projection morphism $\mathrm{pr}_{\mathbb{A}_k^{n+1}} : E_f \to \mathbb{A}_k^{n+1}$ is a regular morphism of affine varieties with pullback morphism,

$$\mathrm{pr}_{\mathbb{A}_k^{n+1}}^{\#} : k[x_0, \dots, x_n] \to k[x_0, \dots, x_n, [x_0/f], \dots, [x_n/f]]/\langle x_i - f \cdot [x_i/f] | 0 \le i \le n\rangle.$$

Clearly there is a unique $k[x_0, \dots, x_n]$-algebra homomorphism $v_f : k[x_0, \dots, x_n][x_0/f, \dots, x_n/f] \to k[E_f]$ factoring $\mathrm{pr}_{\mathbb{A}_k^{n+1}}^{\#}$, and v_f is an isomorphism. So there is a unique isomorphism $\eta_f : E_f \to Y_f$ commuting with projection to $\mathbb{A}_k^{n+1}$.

(b) Prove the collection of morphisms $(\phi_f \circ \eta_f : E_f \to Y | f \in I)$ satisfies the hypotheses for the gluing lemma for morphisms. Denote by $\eta : E \to Y$ the induced morphism.

Solution: Let $f, g \in k[x_0, \dots, x_n]_1$ be nonzero elements. The intersection $E_f \cap E_g$ is an open affine subset of both E_f and E_g. The pullback map $k[E_f] \to k[E_f \cap E_g]$

factors through the localization,

$$k[x_0,\dots,x_n,[x_0/f],\dots,[x_n/f]]/\langle x_i - f\cdot[x_i/f]|0\le i\le n\rangle \to \\ (k[x_0,\dots,x_n,[x_0/f],\dots,[x_n/f]]/\langle x_i - f\cdot[x_i/f]|0\le i\le n\rangle)[1/[g/f]],$$

and the induced homomorphism $w_f : k[E_f][1/[g/f]] \to k[E_f\cap E_g]$ is an isomorphism. Similarly for $k[E_g]\to k[E_f\cap E_g]$. The induced isomorphism $w_{g,f} : k[E_g][1/[f/g]] \to k[E_f][1/[g/f]]$ maps $[h/g]$ to $[h/f]/[g/f]$.

The image $\eta_f(E_f\cap E_g)$ is precisely $Y_{f,g}$, and $\eta_g(E_f\cap E_g) = Y_{g,f}$. The induced homomorphism $\eta_f^\# : k[x_0,\dots,x_n][x_0/f,\dots,x_n/f][1/(g/f)] \to k[E_f][1/[g/f]]$ is an isomorphism, and $\eta_f^\#\circ\phi_{g,f}^\# = w_{g,f}\circ\eta_g^\#$. Hence $\phi_g\circ\phi_{g,f}\circ\eta_f|_{E_f\cap E_g} = \phi_f\circ\eta_f|_{E_f\cap E_g}$ equals $\phi_g\circ\eta_g|_{E_f\cap E_g}$. Therefore the collection satisfies the hypotheses for the gluing lemma for morphisms.

(c) For every $i=1,\dots,n$, prove that $\eta_{x_i} : E_{x_i}\to Y_{x_i}$ is an isomorphism. Deduce that $\eta : E\to Y$ is an isomorphism. In particular, deduce that $\mathrm{pr}_X : Y\to X$ is a projective morphism.

Solution: As proved in (a), every morphism η_f is an isomorphism. For every f, define $e_f^{-1} : Y_f\to E$ to be the composition of η_f^{-1} with the inclusion $E_f\subset E$. By the same argument as in (b), the collection of morphisms $(e_f^{-1} : Y_f\to E|f\in k[x_0,\dots,x_n]_1)$ satisfies the gluing lemma for morphisms, and so defines a morphism $e^{-1} : Y\to E$. Restricting to the open subsets $E_f\subset E$ and $Y_f\subset Y$, $e^{-1}\circ\eta = \mathrm{Id}_E$ and $\eta\circ e^{-1} = \mathrm{Id}_Y$. Therefore $\eta : E\to Y$ is an isomorphism.

Because $E\subset \mathbb{A}_k^{n+1}\subset\mathbb{P}_k^n$ is a closed subvariety, the projection $\mathrm{pr}_{\mathbb{A}_k^{n+1}} : E\to \mathbb{A}_k^{n+1}$ is (strongly) projective (by definition of *strongly projective*). Because η is an isomorphism that commutes with projection to $\mathbb{A}_k^{n+1}$, also $\mathrm{pr}_{\mathbb{A}_k^{n+1}} : Y\to\mathbb{A}_k^{n+1}$ is a projective morphism.

Problem 7: Let $F : X'\to X$ be a regular morphism of affine varieties, let $I\subset k[X]$ be an ideal, and denote $I' = F^\#(I)\cdot k[X']$. Let $\mathrm{pr}_X : Y\to X$ be the blowing up of X along I, and let $\mathrm{pr}_{X'} : Y'\to X'$ be the blowing up of X' along I'.

(a) For every $f\in I$, denote $f' = F^\#(f)\in I'$. Consider the composition $F\circ\mathrm{pr}_{X',f'} : Y'_{f'}\to X$. Prove that

$$(F\circ\mathrm{pr}_{X',f'})^\#(I)\cdot k[Y'_{f'}] = \mathrm{pr}^\#_{X',f'}(I')\cdot k[Y'_{f'}]$$

is a principal ideal generated by $(F\circ\mathrm{pr}_{X',f'})^\#(f)$. Use Problem 3(c) to deduce existence of a morphism $F_{I,f} : Y'_{f'}\to Y_f$ such that $\mathrm{pr}_{X,f}\circ F_f = F\circ\mathrm{pr}_{X',f'}$.

Solution: Denote $A = k[X]$ and $A' = k[X']$. The composition of the pullback homomorphism $F^\# : A\to A'$ with the inclusion $A'\to A'[1/f']$ is a homomorphism mapping f to an invertible element. So there is a unique homomorphism $F_f^\# : A[1/f]\to A'[1/f']$ factoring $F^\#$. Since $F^\#(A), F^\#(I/f)\subset A'_{I',f'}$ and since I/f generates $A_{I,f}$ as an A-algebra, $F^\#(A_{I,f})\subset A'_{I',f'}$, i.e., there is an induced A-algebra homomorphism $F^\#_{I,f} : A_{I,f}\to A'_{I',f'}$. By the universal property of affine varieties, there is a unique regular morphism $F_{I,f} : Y'_{f'}\to Y_f$ whose pullback homomorphism is $F^\#_{I,f}$. There is a canonical surjection of graded A'-algebras, $A'\otimes_A B_IA\to B_{I'}A'$ whose restriction to degree 1 graded pieces is the canonical surjection $A'\otimes_A I\to I' = I\cdot A'$. The regular morphism $F_{I,f}$ is the unique morphism from

Problem 3(c) such that $F_{I,f}^* \alpha_f$ is the composition of this canonical surjection with $\alpha'_{f'}$.

(b) Prove the collection of morphisms $(\phi_f \circ F_{I,f} : Y'_{f'} \to Y | f \in I)$ satisfies the hypotheses for the gluing lemma for morphisms. Denote by $F_I : Y' \to Y$ the induced morphism.

Solution: Let $f, g \in I$. Clearly $F_{I,f}^{-1}(Y_{f,g}) = Y'_{f',g'}$ and $F_{I,g}^{-1}(Y_{g,f}) = Y'_{g',f'}$. To prove that $F_{I,g} \circ \phi'_{g',f'} = \phi_{g,f} \circ F_{I,f}$, compare the pullbacks of α_g.

(c) Consider the morphism, $\mathrm{pr}_X \times F_I : Y' \to X' \times_X Y$. Prove this is a closed immersion whose image is the Zariski closure of the open subset $(X' - \mathbb{V}(I')) \times_{X - \mathbb{V}(I)} \mathrm{pr}_X^{-1}(X - \mathbb{V}(I)) \cong (X' - \mathbb{V}(I'))$. **Hint:** For every $f \in I$, consider $\mathrm{pr}_X \times F_I : F_I^{-1}(Y_f) \to X' \times_X Y_f$. Using that $(X' \times_X Y_f | f \in I)$ is an open covering of $X' \times_X Y$, deduce the result.

Solution: To prove a regular morphism $H : S \to T$ is a closed immersion, it suffices to prove for some open covering (T_i) of T, that every morphism $H : H^{-1}(T_i) \to T_i$ is a closed immersion. In this case, this reduces to proving that every morphism $\mathrm{pr}_X \times F_{I,f} : Y'_{f'} \to X' \times_X Y_f$ is a closed immersion. The pullback map on algebras is,

$$(\mathrm{pr}_X \times F_{I,f})^\# : A' \otimes_A A[I/f] \to A'[I'/f'].$$

It is clear this is surjective. So denoting the kernel by Q_f, $\mathrm{pr}_X \times F_{I,f}$ is an isomorphism to $\mathbb{V}(Q_f)$, i.e., $\mathrm{pr}_X \times F_{I,f}$ is a closed immersion.

(d) In particular, bringing the construction full circle, let $f_1, \dots, f_r \in I$ be generators, and define $F = (f_1, \dots, f_r) : X \to \mathbb{A}_k^r$. Show this morphism satisfies the hypotheses of the problem with X replaced by $\mathbb{A}_k^r$, with $F : X' \to X$ replaced by $F : X \to \mathbb{A}_k^r$, with I replaced by $\langle x_1, \dots, x_r \rangle$ and with I' replaced by I. Combining (c) with Problem 6(c), deduce that $\mathrm{pr}_X : Y \to X$ is a projective morphism.

Solution: By Problem 6(c), the blowing up of $\langle x_1, \dots, x_r \rangle$ in $\mathbb{A}_k^r$ is $\mathrm{pr}_{\mathbb{A}_k^r} : E \to \mathbb{A}_k^r$. As sketched, the choices above together with (c) determine a closed immersion of Y inside $X \times_{\mathbb{A}_k^r} E$. Because $\mathrm{pr}_{\mathbb{A}_k^r} : E \to \mathbb{A}_k^r$ is strongly projective, also $\mathrm{pr}_X : X \times_{\mathbb{A}_k^r} E \to X$ is strongly projective ("the base change of a closed immersion is a closed immersion"). Also, the composition of a closed immersion and a closed immersion is a closed immersion. Therefore the composition $Y \to X \times_{\mathbb{A}_k^r} E \to X$ is strongly projective, i.e., $\mathrm{pr}_X : Y \to X$ is strongly projective.

Problem 8: Let $X = \mathbb{A}_k^2$, and for every integer $n \geq 1$, let $I_n = \langle x^n, y^n \rangle \subset k[x, y]$. Let $\mathrm{pr}_{X,n} : Y_n \to X$ be the blowing up of X along I_n.

(a) If m divides n, prove there exists a unique regular morphism $F_{m,n} : Y_n \to Y_m$ such that $\mathrm{pr}_{X,m} \circ F_{m,n} = \mathrm{pr}_{X,n}$.

Solution: Using Problem 6 and Problem 7, for every integer $n \geq 1$, $Y_n = \mathbb{V}(x^n v_n - y^n u_n) \subset \mathbb{A}_k^2 \times \mathbb{P}_k^1$ where (u_n, v_n) are homogeneous coordinates on $\mathbb{P}_k^1$. If m divides n, say $n = lm$, define $F_{m,n} : Y_n \to Y_m$ by $((x, y), [u_n, v_n]) \mapsto ((x, y), [u_n^l, v_n^l])$. It is straightforward to check this is well-defined and is a regular morphism.

(b) If $m \neq n$, prove there does *not* exist an isomorphism $F_{m,n} : Y_n \to Y_m$ such that $\mathrm{pr}_{X,m} \circ F_{m,n} = \mathrm{pr}_{X,n}$. This shows that for ideals $I, J \subset k[X]$ with $\mathbb{V}(I) = \mathbb{V}(J)$, it is not necessarily true that the blowing up of X along I equals the blowing up of X along J.

Solution: Let p be the least common multiple of m and n. There exist unique morphisms $F_{m,p} : Y_p \to Y_m$ and $F_{n,p} : Y_p \to Y_n$ as in (a). These morphisms satisfy $F_{m,p}^{-1}(Y_{m,x^m}) = F_{n,p}^{-1}(Y_{n,x^n}) = Y_{p,x^p}$. The induced morphisms of affine varieties have pullback morphisms,

$$k[x,y,(y/x)^p] \subset k[x,y,(y/x)^m],$$
$$k[x,y,(y/x)^p] \subset k[x,y,(y/x)^n]$$

Considered as subrings of $k[x,y,1/x] \subset k(\mathbb{A}_k^2)$, $k[x,y,(y/x)^m] \neq k[x,y,(y/x)^n]$. Therefore there is no isomorphism of Y_{n,x^n} and Y_{p,x^p} commuting with projection to $\mathbb{A}_k^2$.

Problem 9, The twenty-seven lines on a cubic surface I: A classical result, that continues to inspire algebraic geometers, is that every smooth cubic hypersurface in $\mathbb{P}_k^3$ over an algebraically closed field k contains 27 lines. Off the Building 2 corridor on the first floor there is a display case containing models of surfaces, one of which is a real cubic surface containing 27 real lines (the maximum possible). This problem and the next give an example of a cubic surface for which you can compute the equations of the 27 lines.

Assume char(k) does not divide 6. Let $F(x_0,x_1,x_2,x_3) = x_0^3 + x_1^3 + x_2^3 + x_3^3$ and let $X = \mathbb{V}(F) \subset \mathbb{P}_k^3$. For each pairing $\{0,1,2,3\} = \{\{i,j\},\{k,l\}\}$, for each pair of elements, $[a_i,a_j],[a_k,a_l] \in \mathbb{P}_k^1$, consider the line $\mathbb{V}(a_ix_i + a_jx_j, a_kx_k + a_lx_l) \subset \mathbb{P}_k^3$. Among all such lines, which ones are contained in X? Prove X contains at least 27 lines.

Solution: The cubic surface in this example is called the *Fermat cubic surface* (more generally any hypersurface $\mathbb{V}(x_0^d + \cdots + x_n^d) \subset \mathbb{P}_k^n$ is called a *Fermat hypersurface* in analogy with the famous Fermat plane curves). Without loss of generality, assume $a_i = a_k = 1$. Then homogeneous coordinates on the line are x_j and x_l. With respect to these coordinates, the pullback of F is $x_j^3 + (-a_jx_j^3) + x_l^3 + (-a_lx_l^3) = (1-a_j^3)x_j^3 + (1-a_k^3)x_k^3$. This is the zero polynomial iff $a_j^3 = 0$ and $a_k^3 = 0$. Let ω be a nontrivial solution of t^2+t+1. The 27 lines of this form contained in X are displayed in Table 1 (on the last page).

There is an action of $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ on $\mathbb{P}_k^3$ (or better, of $\mu_3 \times \mu_3 \times \mu_3$ on $\mathbb{P}_k^3$), by $(e_1,e_2,e_3)\cdot[x_0,x_1,x_2,x_3] = [x_0, \omega^{e_1}x_1, \omega^{e_2}x_2, \omega^{e_3}x_3]$. The pullback of F is $x_0^3 + (\omega^{e_1}x_1)^3 + (\omega^{e_2}x_2)^3 + (\omega^{e_3}x_3)^3 = x_0^3 + x_1^3 + x_2^3 + x_3^3$. Therefore the action maps X to itself, i.e., there action restricts to an action on X. There is an induced action on the set of lines on X, and the 27 lines above are a (faithful) orbit of that action.

Problem 10, The twenty-seven lines on a cubic surface II: This problem continues the previous problem and proves the 27 lines from the previous problem are the only lines on X.

(a) Prove that every line $M \subset X$ intersects one of the 27 lines L identified in the previous problem. **Hint:** Find 3 lines L_1, L_2, L_3 from the previous problem and a hyperplane $H \subset \mathbb{P}_k^3$ such that $H \cap X = L_1 \cup L_2 \cup L_3$. Apply Problem 9(a) from Problem Set 5 to M and H.

Solution: For instance $L_1 \cup L_2 \cup L_3 = \mathbb{V}(x_0 + x_1) \cap X$. By Problem 9(a) from Problem Set 5, M intersects H. Because $M \subset X$, M intersects $H \cap X$. Therefore M intersects L_1, L_2 or L_3.

(b) Find a change-of-coordinates that preserves X and maps L to the line $\mathbb{V}(x_0 + x_1, x_2 + x_3)$. Prove there exists a hyperplane $H = \mathbb{V}(a(x_0 + x_1) + b(x_2 + x_3))$ containing L and M.

Solution: The change of variables is just the action of $(0, 0, e_3)$ on $\mathbb{P}^3_k$ for some choice of $e_3 \in \mathbb{Z}/3\mathbb{Z}$. If M equals L, we are done. Thus assume L and M are distinct.

After the coordinate change, consider the affine cones $\mathbb{A}L, \mathbb{A}M \subset \mathbb{A}^4_k$. These are 2-dimensional vector spaces. Because $L \cap M \neq \emptyset$, the intersection contains a 1-dimensional subspace. Because $L \neq M$, the intersection has dimension exactly 1. So the span has dimension 2. This is the affine cone over a hyperplane H containing $L \cup M$. Every hyperplane containing L is of the form $\mathbb{V}(a(x_0 + x_1) + b(x_2 + x_3))$ for some $[a, b] \in \mathbb{P}^1_k$.

(c) Without loss of generality, assume $b \neq 0$ and rewrite the defining equation of H as $(x_2 + x_3) = a(x_0 + x_1)$. For homogeneous coordinates on H, use $u = x_0 + x_1$, $v = x_0 - x_1$ and $w = x_2 - x_3$. Prove the restriction of F to H is of the form,

$$F|_H = uG(u, v, w) = u(c_u(a)u^2 + c_v(a)v^2 + c_w(a)w^2),$$

for particular polynomials $c_u(t), c_v(t), c_w(t) \in k[t]$. Therefore $H \cap X = L \cup \mathbb{V}(G(u, v, w))$.

Solution: Expanding F in u, v and w gives,

$$4F = (x_0 + x_1)((x_0 + x_1)^2 + 3(x_0 - x_1)^2) + (x_2 + x_3)((x_2 + x_3)^2 + 3(x_2 - x_3)^2) = \\ u(u^2 + 3v^2) + au(a^2u^2 + 3w^2) = u((1 + a^3)u^2 + 3v^2 + 3aw^2).$$

Thus $c_u(t) = 1 + t^3$, $c_v = 3$ and $c_w = 3t$.

(d) Use Problem 6 from Problem Set 1 to determine the values of a for which $\mathbb{V}(G(u, v, w)) \subset H$ contains a line M. Prove that all of these cases are already accounted for by the 27 lines from Problem 9.

Solution: By Problem 6 from Problem Set 1, $\mathbb{V}(G(u, v, w))$ contains a line iff $c_u(a) = 0$, $c_v(a) = 0$ or $c_w(a) = 0$. The third happens iff $a = 0$, the second never happens, and the first happens iff $a = -1, -\omega$ or $-\omega^2$.

In each of these 4 possibilities, we have,

$$\begin{array}{rcl} \mathbb{V}(x_2 + x_3) \cap X & = & L_1 \cup L_4 \cup L_7, \\ \mathbb{V}((x_0 + x_1) + (x_2 + x_3)) \cap X & = & L_1 \cup L_{10} \cup L_{19}, \\ \mathbb{V}(\omega(x_0 + x_1) + (x_2 + x_3)) \cap X & = & L_1 \cup L_{18} \cup L_{27}, \\ \mathbb{V}(\omega^2(x_0 + x_1) + (x_2 + x_3)) \cap X & = & L_1 \cup L_{14} \cup L_{23}. \end{array}$$

Since every line is accounted for by the 27 lines, and using that the 27 lines are an orbit of the action of $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, it follows that these are the only lines on X.

$$
\begin{array}{rcl}
L_1 & = & \mathbb{V}(x_0 + x_1, x_2 + x_3), \\
L_2 & = & \mathbb{V}(x_0 + x_1, x_2 + \omega x_3), \\
L_3 & = & \mathbb{V}(x_0 + x_1, x_2 + \omega^2 x_3), \\
L_4 & = & \mathbb{V}(x_0 + \omega x_1, x_2 + x_3), \\
L_5 & = & \mathbb{V}(x_0 + \omega x_1, x_2 + \omega x_3), \\
L_6 & = & \mathbb{V}(x_0 + \omega x_1, x_2 + \omega^2 x_3), \\
L_7 & = & \mathbb{V}(x_0 + \omega^2 x_1, x_2 + x_3), \\
L_8 & = & \mathbb{V}(x_0 + \omega^2 x_1, x_2 + \omega x_3), \\
L_9 & = & \mathbb{V}(x_0 + \omega^2 x_1, x_2 + \omega^2 x_3), \\
L_{10} & = & \mathbb{V}(x_0 + x_2, x_1 + x_3), \\
L_{11} & = & \mathbb{V}(x_0 + x_2, x_1 + \omega x_3), \\
L_{12} & = & \mathbb{V}(x_0 + x_2, x_1 + \omega^2 x_3), \\
L_{13} & = & \mathbb{V}(x_0 + \omega x_2, x_1 + x_3), \\
L_{14} & = & \mathbb{V}(x_0 + \omega x_2, x_1 + \omega x_3), \\
L_{15} & = & \mathbb{V}(x_0 + \omega x_2, x_1 + \omega^2 x_3), \\
L_{16} & = & \mathbb{V}(x_0 + \omega^2 x_2, x_1 + x_3), \\
L_{17} & = & \mathbb{V}(x_0 + \omega^2 x_2, x_1 + \omega x_3), \\
L_{18} & = & \mathbb{V}(x_0 + \omega^2 x_2, x_1 + \omega^2 x_3), \\
L_{19} & = & \mathbb{V}(x_0 + x_3, x_1 + x_2), \\
L_{20} & = & \mathbb{V}(x_0 + x_3, x_1 + \omega x_2), \\
L_{21} & = & \mathbb{V}(x_0 + x_3, x_1 + \omega^2 x_2), \\
L_{22} & = & \mathbb{V}(x_0 + \omega x_3, x_1 + x_2), \\
L_{23} & = & \mathbb{V}(x_0 + \omega x_3, x_1 + \omega x_2), \\
L_{24} & = & \mathbb{V}(x_0 + \omega x_3, x_1 + \omega^2 x_2), \\
L_{25} & = & \mathbb{V}(x_0 + \omega^2 x_3, x_1 + x_2), \\
L_{26} & = & \mathbb{V}(x_0 + \omega^2 x_3, x_1 + \omega x_2), \\
L_{27} & = & \mathbb{V}(x_0 + \omega^2 x_3, x_1 + \omega^2 x_2)
\end{array}
$$

Figure 1. The 27 lines on the Fermat cubic surface