18.725 PROBLEM SET 6

Due date: Friday, November 5 in lecture. Late work will be accepted only with a medical note or for another Institute-approved reason. You are strongly encouraged to work with others, but the final write-up should be entirely your own and based on your own understanding.

Read through all the problems. Write solutions to the "Required Problems", 1, 2, 3, and 4 and 1 additional problem of your choice to a total of 5 problems. I realize some optional problems have follow-up problems, which might seem at odds with writing up only 1 additional problem. You are encouraged to work through problems you don't write up. You are also allowed to write the solution to a problem without writing the solution to the problem it follows-up.

Required Problem 1: This is a "multilinear algebra problem" introducing the derivative and Hessian of a polynomial. The next problem relates the Hessian of a homogeneous polynomial on $\mathbb{P}^2$ to the *flex lines* of the associated plane curve.

For every finite-dimensional k-vector space V denote by $V^\vee$ the dual vector space $\mathrm{Hom}_k(V, k)$. Denote by $k[V^\vee]$ the ring of polynomial functions on V, i.e., the k-subalgebra of $\mathrm{Hom}_{\mathrm{Set}}(V, k)$ generated by $V^\vee$. There is a unique $\mathbb{Z}_{\geq 0}$-grading on $k[V^\vee]$ such that $k[V^\vee]_0 = k$ and $k[V^\vee]_1 = V^\vee$. For every integer $r \geq 0$, denote by $S^r(V^\vee)$ the k-vector space $k[V^\vee]_r$, called the r^{th} *symmetric power of* $V^\vee$. Denote by $(\mathbb{A}V, \mathcal{O}_{\mathbb{A}V})$ the unique affine variety whose underlying point-set is V and whose coordinate ring $\mathcal{O}_{\mathbb{A}V}(\mathbb{A}V)$ is $k[V^\vee]$. (Usually this variety is just denoted $(V, \mathcal{O}_V)$, but in this problem this notation distinguishes V as a k-vector space from V as an affine variety.)

(a) Denote by M the (left) $k[V^\vee]$-module $M = k[V^\vee] \otimes_k V^\vee$ where $f \cdot (g \otimes x) := (fg) \otimes x$ for every $f, g \in k[V^\vee]$ and $x \in V^\vee$. Prove there exists a unique k-derivation $d : k[V^\vee] \to M$ such that $d(x) = 1 \otimes x$ for every $x \in V^\vee = k[V^\vee]_1$. The induced homomorphism of $k[V^\vee]$-modules, $\Omega_{k[V^\vee]/k} \to M$, is an isomorphism (you need not prove this).

(b) For every integer $r \geq 0$, denote by $d_r : S^r(V^\vee) \to S^{r-1}(V^\vee) \otimes V^\vee$ the restriction of d, and denote by $\widetilde{d_r} : S^r(V^\vee) \to \mathrm{Hom}_k(V, S^{r-1}(V^\vee))$ the composition of d_r with the canonical isomorphism $S^{r-1}(V^\vee) \otimes_k V^\vee \cong \mathrm{Hom}_k(V, S^{r-1}(V^\vee))$. Given $F \in S^r(V^\vee)$, denote the image under d_r by $d_r F$, and denote the induced linear map by $\widetilde{d_r}F : V \to S^{r-1}(V^\vee)$. Let $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ be an ordered basis for V and let $(x_1, \ldots, x_n)$ be the dual ordered basis for $V^\vee$. Prove for every $F \in S^r(V^\vee)$ and every $i = 1, \ldots, n$,

$$\widetilde{d_r}F(\mathbf{e}_i) = \frac{\partial F}{\partial x_i}.$$

(c) For every integer $r \geq 0$, denote by $\mathrm{Hess}_r : S^r(V^\vee) \to \mathrm{Hom}_k(V, S^{r-2}(V^\vee) \otimes_k V^\vee)$ the unique linear map $F \mapsto \mathrm{Hess}_r(F)$ such that for every $v \in V$, $\mathrm{Hess}_r(F)(v) = d_{r-1}((\widetilde{d_r}F)(v))$. This is the *Hessian of* F. Let $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ be an ordered basis for V, and let $(x_1, \dots, x_n)$ be the dual ordered basis for $V^\vee$. Prove that for every $F \in S^r(V^\vee)$ and every $1 \leq j \leq n$,

$$\mathrm{Hess}_r(F)(\mathbf{e}_j) = \sum_{i=1}^{n} \frac{\partial^2 F}{\partial x_i \partial x_j} \otimes x_i.$$

Considering the terms $\partial^2 F/\partial x_i \partial x_j$ to be "coefficients", $Hess_r(F)$ is an $n \times n$ matrix whose (i,j)-entry is the degree $r-2$ homogeneous polynomial $\partial^2 F/\partial x_i \partial x_j$. For every point $p \in \mathbb{A}V$, denote by $\mathrm{Hess}_r(F)(p) : V \to V^\vee$ the k-linear map obtained by evaluating these degree $r-2$ homogeneous polynomials at p.

Required Problem 2: This problem continues the previous problem. Let $\dim_k V = 3$ so that $\mathbb{A}V \cong \mathbb{A}^3_k$. Denote by $(\mathbb{P}V, \mathcal{O}_{\mathbb{P}V})$ the projective variety $(\mathbb{A}V - \{0\})/(v \sim \lambda v) \cong \mathbb{P}^2_k$. Let $r \geq 1$, let $F \in S^r(V^\vee)$ be an irreducible polynomial, and let $C = \mathbb{V}(F) \subset \mathbb{P}V$ be the associated plane curve.

(a) Let $p \in C$, and let $v \in V$. Prove that $(\widetilde{d_r} F(v))(p) = 0$ iff there exists a line $L \subset \mathbb{P}V$ tangent to C at p and such that the associated affine cone $\mathbb{A}L \subset \mathbb{A}V$ contains v. **Hint:** If $v \in \mathbb{A}\{p\}$ this is trivial, and if $v \notin \mathbb{A}\{p\}$, choose an ordered basis $(\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2)$ for V such that $p = [1,0,0]$ and $v = (0,1,0)$.

(b) Assume $\mathrm{char}(k)$ does not divide $2(r-1)$. For every point $p \in C$ a tangent line L to C at p, $L \subset \mathbb{P}V$, is a *flex line to C at p* if the germ at p of the restriction to L of the dehomogenization of F is contained in $\mathfrak{m}_p^3 \mathcal{O}_{L,p}$, i.e., the restriction of F to L vanishes to order ≥ 3 at p. Prove there is a flex line to C at p iff the 3×3 Hessian $\mathrm{Hess}_r(F)(p)$ is not an isomorphism, i.e., iff with respect to some (and hence any) basis, the determinant of the 3×3 Hessian matrix is 0. **Hint:** There are 2 cases depending on whether p is a smooth or singular point of C. In both cases, choose an ordered basis $(\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2)$ for V such that $p = [1,0,0]$ and such that tangent line under consideration is $\{[a,b,0] | a, b \in k\}$.

(c) Assume $\mathrm{char}(k)$ does not divide 6. Compute all the flex lines to the smooth cubic plane curve $\mathbb{V}(x_0^3 + x_1^3 + x_2^3) \subset \mathbb{P}^2_k$. **Hint:** There are 9 of them.

Required Problem 3: This problem together with the next problem work through the construction of the *blowing up of an affine variety along an ideal.* You are encouraged to do these problems in whatever order makes most sense to you (the sketch below is only one of many approaches). You are also encouraged to read the section about blowing up in *The Geometry of Schemes* by Eisenbud and Harris.

Let $X \subset \mathbb{A}^n_k$ be an affine algebraic set and denote $A = k[X]$. Let $I \subset A$ be an ideal and let $f_1, \dots, f_r \in I$ be generators. Let $\mathbb{A}^r_k$ be an affine space with coordinates $y_1, \dots, y_r$. For every $f \in I$, denote by $F_f : D(f) \to \mathbb{A}^r_k$ the regular morphism,

$$p \mapsto (f_1(p)/f(p), \dots, f_r(p)/f(p)),$$

and denote by $Y_f \subset X \times \mathbb{A}^r_k$ the Zariski closure of the image of $\mathrm{Id}_X \times F_f : D(f) \to X \times \mathbb{A}^r_k$. Denote by $B_I A$ the *blowup algebra of I in A*, i.e., the $\mathbb{Z}_{\leq 0}$-graded A-algebra,

$$B_I A := A[I/t] \subset A[1/t], \text{ i.e., } A \oplus I \oplus \dots \oplus I^k \oplus \dots$$

where $\deg(a) = 0$ for all $a \in A$ and $\deg(1/t) = -1$ (cf. also, Eisenbud's *Commutative Algebra*, p. 148; the reason for using $1/t$ as a variable instead of t is to make this compatible with the *Rees algebra*, p. 170).

(a) Denote by $\mathrm{pr}_{X,f} : Y_f \to X$ the restriction of $\mathrm{pr}_X : X \times \mathbb{A}^r_k \to X$ and denote by $G_f : D(f) \to Y_f$ the morphism induced by $\mathrm{Id}_X \times F_f$. Prove the composition $G_f^\# \circ \mathrm{pr}_X^\# : A \to A[1/f]$ is the usual k-algebra homomorphism $i : A \to A[1/f]$.

(b) Prove that $(\mathrm{Id}_X \times F_f)^\# : k[Y_f] \to A[1/f]$ is injective and the image is the subring of $A[1/f]$ generated by A and $\{g/f | g \in I\}$. Prove this is the smallest k-subalgebra $A_{I,f} \subset A[1/f]$ containing $i(A)$ and such that $i(I) \cdot A_{I,f}$ equals the principal ideal $\langle f \rangle A_f$.

(c) Denote by $\alpha_f : A_{I,f} \otimes_A B_I A \to A_{I,f}$ the unique homomorphism of $A_{I,f}$-algebras such that for every $g \in I$, $\alpha_f(1 \otimes (g/t)) = g/f$; in particular $\alpha_f(1 \otimes (f/t)) = 1$. Let $(H : T \to X, \beta)$ be a pair of a regular morphism of algebraic varieties $H : T \to X$ together with a homomorphism of $\mathcal{O}_T(T)$-algebras $\beta : \mathcal{O}_T(T) \otimes_A B_I A \to \mathcal{O}_T(T)$ such that $\beta(1 \otimes (f/t)) = 1$. Prove there is a unique regular morphism $\phi : T \to Y_f$ such that

(i) $H = \mathrm{pr}_{X,f} \circ \phi$, and

(ii) via the canonical isomorphisms $\mathcal{O}_T(T) \otimes_{A_{I,f}} (A_{I,f} \otimes_A B_I A) \cong \mathcal{O}_T(T) \otimes_A B_I A$ and $\mathcal{O}_T(T) \otimes_{A_{I,f}} A_{I,f} \cong \mathcal{O}_T(T)$, β equals $\mathrm{Id}_{\mathcal{O}_T(T)} \otimes \alpha_f : \mathcal{O}_T(T) \otimes_{A_{I,f}} (A_{I,f} \otimes_A B_I A) \to \mathcal{O}_T(T) \otimes_{A_{I,f}} A_{I,f}$.

Hint: Use the universal property of affine varieties together with Proposition 8.5.

Required Problem 4: This problem continues the previous problem. Again, you are encouraged to approach this problem in the way that makes most sense to you. You do not need to write out all details, but you should understand how your approach settles the details.

(a) For every ordered pair $(f, g) \in I \times I$, define $Y_{f,g} \subset Y_f$ to be $D(g/f)$. Denote by $\alpha'_{g,f} : k[Y_{f,g}] \otimes_A B_I A \to k[Y_{f,g}]$ the unique $k[Y_{f,g}]$-algebra homomorphism commuting with α_f and the $A_{I,f}$-module homomorphisms $A_{I,f} \to A_{I,f}[(g/f)^{-1}] = k[Y_{f,g}]$, $A_{I,f} \otimes_A B_I A \to k[Y_{f,g}] \otimes_A B_I A$. Denote by $\alpha''_{g,f}$ the unique $\mathbb{Z}_{\leq 0}$-graded $k[Y_{f,g}]$-algebra automorphisms,

$$\alpha''_{g,f} : k[Y_{f,g}] \otimes_A B_I A \to k[Y_{f,g}] \otimes_A B_I A,$$
$$a/t^n \mapsto (g/f)^{-n} \cdot a/t^n.$$

Denote $\alpha_{g,f} = \alpha'_{g,f} \circ \alpha''_{g,f}$. Prove the pair $(\mathrm{pr}_{X,f}|_{Y_{f,g}} : Y_{f,g} \to X, \alpha_{g,f})$ satisfies the condition in (c) of Problem 3 for $g \in I$. Deduce existence of a unique regular morphism $\phi_{g,f} : Y_{g,f} \to Y_g$ such that $\mathrm{pr}_{X,g} \circ \phi_{g,f} = \mathrm{pr}_{X,f}|_{Y_{f,g}}$ and $\phi^*_{g,f}\alpha_g = \alpha_{g,f}$.

(b) Prove that $\phi_{g,f}(Y_{f,g}) \subset Y_{g,f}$ and prove that $\phi_{g,f}$ and $\phi_{f,g}$ are inverse isomorphisms $Y_{f,g} \cong Y_{g,f}$.

(c) For every triple $(f, g, h) \in I \times I \times I$, prove $\phi^{-1}_{g,f}(Y_{g,h}) = Y_{f,g} \cap Y_{f,h}$ and prove $\phi_{h,g} \cap \phi_{g,f}|_{Y_{f,g} \cap Y_{f,h}} = \phi_{h,f}|_{Y_{f,g} \cap Y_{f,h}}$. Therefore the collection $((Y_f | f \in I), (Y_{f,g} | (f, g) \in I \times I), (\phi_{g,f} | (f, g) \in I \times I))$ satisfies the gluing lemma for objects. Denote the induced family of morphisms by $(\phi_f : Y_f \to Y)$. The variety Y is the *blowing up of X along the ideal I*.

(d) Prove the collection of morphisms $(\mathrm{pr}_{X,f} \circ \phi_f^{-1} : \phi_f(Y_f) \to X | f \in I)$ satisfies the gluing lemma for morphisms. The induced morphism $\mathrm{pr}_X : Y \to X$ is the *projection morphism*. The restriction $\mathrm{pr}_X : \mathrm{pr}_X^{-1}(X - \mathbb{V}(I)) \to (X - \mathbb{V}(I))$ is an isomorphism.

(e) For every $p \in Y$, prove the ideal $\mathrm{pr}_X^\# I \cdot \mathcal{O}_{Y,p}$ is a principal ideal in $\mathcal{O}_{Y,p}$.

Problem 5: This problem continues the previous two problems. Let $g_1, \dots, g_s \in I$ be any other choice of generators, and define $(\mathrm{pr}'_{I,f} : Y'_f \to X | f \in I)$ and $\mathrm{pr}'_X : Y' \to X$ in the analogous manner as above with this choice of generators. Use the universal property of $(Y_f \to X, \alpha_f)$ to prove the variety Y'_f is canonically isomorphic to Y_f. Deduce that Y' is canonically isomorphic to Y, and the isomorphism commutes with pr_X and pr'_X.

Problem 6: Let $X = \mathbb{A}_k^{n+1}$, and let $I = \langle x_0, \dots, x_n \rangle \subset k[x_1, \dots, x_n]$. Let $E \subset \mathbb{A}_k^{n+1} \times \mathbb{P}_k^n$ be the tautological rank 1 subbundle from Problem 4(c) on Problem Set 5. Denote by $\pi_{\mathbb{A}_k^{n+1}} : E \to \mathbb{A}_k^{n+1}$ and $\pi_{\mathbb{P}_k^n} : E \to \mathbb{P}_k^n$ the projections. Let $\mathrm{pr}_X : Y \to \mathbb{A}_k^{n+1}$ be the blowing up of $\mathbb{A}_k^{n+1}$ along I.

(a) For every $f \in I$, define $E_f = \pi_{\mathbb{P}_k^n}^{-1}(D_+(f))$. Prove that $\pi^\#_{\mathbb{A}_k^{n+1}}(I) \cdot \mathcal{O}_E(E_f)$ is a principal ideal generated by $\pi^\#_{\mathbb{A}_k^{n+1}}(f)$. Use Problem 3(c) to deduce existence of a morphism $\eta_f : E_f \to Y_f$.

(b) Prove the collection of morphisms $(\phi_f \circ \eta_f : E_f \to Y | f \in I)$ satisfies the hypotheses for the gluing lemma for morphisms. Denote by $\eta : E \to Y$ the induced morphism.

(c) For every $i = 1, \dots, n$, prove that $\eta_{x_i} : E_{x_i} \to Y_{x_i}$ is an isomorphism. Deduce that $\eta : E \to Y$ is an isomorphism. In particular, deduce that $\mathrm{pr}_X : Y \to X$ is a projective morphism.

Problem 7: Let $F : X' \to X$ be a regular morphism of affine varieties, let $I \subset k[X]$ be an ideal, and denote $I' = F^\#(I) \cdot k[X']$. Let $\mathrm{pr}_X : Y \to X$ be the blowing up of X along I, and let $\mathrm{pr}_{X'} : Y' \to X'$ be the blowing up of X' along I'.

(a) For every $f \in I$, denote $f' = F^\#(f) \in I'$. Consider the composition $F \circ \mathrm{pr}_{X',f'} : Y'_{f'} \to X$. Prove that

$$(F \circ \mathrm{pr}_{X',f'})^\#(I) \cdot k[Y'_{f'}] = \mathrm{pr}^\#_{X',f'}(I') \cdot k[Y'_{f'}]$$

is a principal ideal generated by $(F \circ \mathrm{pr}_{X',f'})^\#(f)$. Use Problem 3(c) to deduce existence of a morphism $F_{I,f} : Y'_{f'} \to Y_f$ such that $\mathrm{pr}_{X,f} \circ F_f = F \circ \mathrm{pr}_{X',f'}$.

(b) Prove the collection of morphisms $(\phi_f \circ F_{I,f} : Y'_{f'} \to Y | f \in I)$ satisfies the hypotheses for the gluing lemma for morphisms. Denote by $F_I : Y' \to Y$ the induced morphism.

(c) Consider the morphism, $\mathrm{pr}_X \times F_I : Y' \to X' \times_X Y$. Prove this is a closed immersion whose image is the Zariski closure of the open subset $(X' - \mathbb{V}(I')) \times_{X - \mathbb{V}(I)} \mathrm{pr}_X^{-1}(X - \mathbb{V}(I)) \cong (X' - \mathbb{V}(I'))$. **Hint:** For every $f \in I$, consider $\mathrm{pr}_X \times F_I : F_I^{-1}(Y_f) \to X' \times_X Y_f$. Using that $(X' \times_X Y_f | f \in I)$ is an open covering of $X' \times_X Y$, deduce the result.

(d) In particular, bringing the construction full circle, let $f_1, \dots, f_r \in I$ be generators, and define $F = (f_1, \dots, f_r) : X \to \mathbb{A}_k^r$. Show this morphism satisfies the hypotheses of the problem with X replaced by $\mathbb{A}_k^r$, with $F : X' \to X$ replaced by $F : X \to \mathbb{A}_k^r$, with I replaced by $\langle x_1, \dots, x_r \rangle$ and with I' replaced by I. Combining (c) with Problem 6(c), deduce that $\mathrm{pr}_X : Y \to X$ is a projective morphism.

Problem 8: Let $X = \mathbb{A}_k^2$, and for every integer $n \geq 1$, let $I_n = \langle x^n, y^n \rangle \subset k[x, y]$. Let $\mathrm{pr}_{X,n} : Y_n \to X$ be the blowing up of X along I_n.

(a) If m divides n, prove there exists a unique regular morphism $F_{m,n} : Y_n \to Y_m$ such that $\mathrm{pr}_{X,m} \circ F_{m,n} = \mathrm{pr}_{X,n}$.

(b) If $m \neq n$, prove there does *not* exist an isomorphism $F_{m,n} : Y_n \to Y_m$ such that $\mathrm{pr}_{X,m} \circ F_{m,n} = \mathrm{pr}_{X,n}$. This shows that for ideals $I, J \subset k[X]$ with $\mathbb{V}(I) = \mathbb{V}(J)$, it is not necessarily true that the blowing up of X along I equals the blowing up of X along J.

Problem 9, The twenty-seven lines on a cubic surface I: A classical result, that continues to inspire algebraic geometers, is that every smooth cubic hypersurface in $\mathbb{P}^3_k$ over an algebraically closed field k contains 27 lines. Off the Building 2 corridor on the first floor there is a display case containing models of surfaces, one of which is a real cubic surface containing 27 real lines (the maximum possible). This problem and the next give an example of a cubic surface for which you can compute the equations of the 27 lines.

Assume $\mathrm{char}(k)$ does not divide 6. Let $F(x_0, x_1, x_2, x_3) = x_0^3 + x_1^3 + x_2^3 + x_3^3$ and let $X = \mathbb{V}(F) \subset \mathbb{P}^3_k$. For each pairing $\{0, 1, 2, 3\} = \{\{i, j\}, \{k, l\}\}$, for each pair of elements, $[a_i, a_j], [a_k, a_l] \in \mathbb{P}^1_k$, consider the line $\mathbb{V}(a_i x_i + a_j x_j, a_k x_k + a_l x_l) \subset \mathbb{P}^3_k$. Among all such lines, which ones are contained in X? Prove X contains at least 27 lines.

Problem 10, The twenty-seven lines on a cubic surface II: This problem continues the previous problem and proves the 27 lines from the previous problem are the only lines on X.

(a) Prove that every line $M \subset X$ intersects one of the 27 lines L identified in the previous problem. **Hint:** Find 3 lines L_1, L_2, L_3 from the previous problem and a hyperplane $H \subset \mathbb{P}^3_k$ such that $H \cap X = L_1 \cup L_2 \cup L_3$. Apply Problem 9(a) from Problem Set 5 to M and H.

(b) Find a change-of-coordinates that preserves X and maps L to the line $\mathbb{V}(x_0 + x_1, x_2 + x_3)$. Prove there exists a hyperplane $H = \mathbb{V}(a(x_0 + x_1) + b(x_2 + x_3))$ containing L and M.

(c) Without loss of generality, assume $b \neq 0$ and rewrite the defining equation of H as $(x_2 + x_3) = a(x_0 + x_1)$. For homogeneous coordinates on H, use $u = x_0 + x_1$, $v = x_0 - x_1$ and $w = x_2 - x_3$. Prove the restriction of F to H is of the form,

$$F|_H = uG(u, v, w) = u(c_u(a)u^2 + c_v(a)v^2 + c_w(a)w^2),$$

for particular polynomials $c_u(t), c_v(t), c_w(t) \in k[t]$. Therefore $H \cap X = L \cup \mathbb{V}(G(u, v, w))$.

(d) Use Problem 6 from Problem Set 1 to determine the values of a for which $\mathbb{V}(G(u, v, w)) \subset H$ contains a line M. Prove that all of these cases are already accounted for by the 27 lines from Problem 9.