18.725 PROBLEM SET 4

Due date: Friday, October 15 in lecture. Late work will be accepted only with a medical note or for another Institute-approved reason. You are strongly encouraged to work with others, but the final write-up should be entirely your own and based on your own understanding.

Read through all the problems. Write solutions to the "Required Problems", 1, 2, 3, and 4, together with 2 others of your choice to a total of 6 problems. The last 5 problems on this problem set are taken from Problem Set 2 (the solutions to these problems were not given). You can use them for the non-required problems only if you did not use them for Problem Set 2.

Required Problem 1: Let F be an element of $k[X_0,\dots,X_n]_e$. Prove the *Euler identity*,

$$e\cdot F(X_0,\dots,X_n) = X_0\frac{\partial F}{\partial X_0} + \cdots + X_n\frac{\partial F}{\partial X_n}.$$

Remark: This isn't a proof, but to see where this identity comes from, differentiate with respect to t both sides of the identity,

$$t^e F(X) = F(tX).$$

Solution: By linearity, it suffices to prove the case when F is a monomial $X^e = X_0^{e_0}\cdot\dots\cdot X_n^{e_n}$. For every $i=0,\dots,n$, $\partial F/\partial X_i = e_i F/X_i$ so that $X_i(\partial F/\partial X_i) = e_i F$. Therefore $X_0(\partial F/\partial X_0)+\cdots+X_n(\partial F/\partial X_n) = e_0F+\cdots+e_nF = (e_0+\cdots+e_n)F = eF$.

Required Problem 2: Let X_0, X_1, X_2 be homogeneous coordinates on $\mathbb{P}^2_k$. Let $(\mathbb{P}^2_k)^\vee$ be a copy of $\mathbb{P}^2_k$ with homogeneous coordinates Y_0, Y_1, Y_2. Denote by $(\mathbb{P}^2_k \times (\mathbb{P}^2_k)^\vee, \pi_1, \pi_2)$ a product of $(\mathbb{P}^2_k, (\mathbb{P}^2_k)^\vee)$. Define $\Lambda \subset \mathbb{P}^2_k \times (\mathbb{P}^2_k)^\vee$ to be,

$$\{([a_0,a_1,a_2],[b_0,b_1,b_2]) | a_0b_0 + a_1b_1 + a_2b_2 = 0\}.$$

A *projective line* in $\mathbb{P}^2_k$ is $\mathbb{V}(s)$ for any nonzero $s \in k[X_0,X_1,X_2]_1$.

(a) Prove there is a bijection between $(\mathbb{P}^2_k)^\vee$ and the set of lines in $\mathbb{P}^2_k$ defined by $q \in (\mathbb{P}^2_k)^\vee \mapsto \pi_1(\Lambda\cap\pi_2^{-1}(q))$.

Solution: Every nonzero element $s \in k[X_0,X_1,X_2]_1$ is of the form $b_0X_0+b_1X_1+b_2X_2$, thus $\mathbb{V}(s) = \pi_1(\Lambda\cap\pi_2^{-1}(q))$ for $q=[b_0,b_1,b_2]$. If $q=[b_0,b_1,b_2]$ and $r=[c_0,c_1,c_2]$ are such that $\pi_1(\Lambda\cap\pi_2^{-1}(q)) = \pi_1(\Lambda\cap\pi_2^{-1}(r))$, then by the projective ideal variety correspondence $\langle b_0X_0+b_1X_1+b_2X_2\rangle = \langle c_0X_0+c_1X_1+c_2X_2\rangle$, from which it easily follows that $[b_0,b_1,b_2]=[c_0,c_1,c_2]$ as elements of $(\mathbb{P}^2_k)^\vee$.

(b) Let $F \in k[X_0,X_1,X_2]_e$ be an irreducible polynomial. Denote $C = \mathbb{V}(F) \subset \mathbb{P}^2_k$. Let $p = [a_0,a_1,a_2]$ be an element of C. A line $L \subset \mathbb{P}^2_k$ is *tangent to C at p* if $p \in L$ and the restriction of F to L has a repeated root at p. Assuming char(k) does not divide e, prove the line L associated to $[b_0, b_1, b_2] \in (\mathbb{P}^2_k)^\vee$ is tangent to C at $[a_0, a_1, a_2]$ iff the following matrix has rank 1,

$$\begin{pmatrix} (\partial F)/(\partial X_0)(a_0,a_1,a_2) & (\partial F)/(\partial X_1)(a_0,a_1,a_2) & (\partial F)/(\partial X_2)(a_0,a_1,a_2) \\ b_0 & b_1 & b_2 \end{pmatrix}.$$

(**Hint:** After a change of coordinates, arrange that $(a_0, a_1, a_2) = (1, 0, 0)$ and $(b_0, b_1, b_2) = (0, 0, 1)$. Combine this with the Euler identity from Problem 1.)

Solution: There is an action of $\mathbf{GL}_3$ on $\mathbb{P}^2_k$ and $(\mathbb{P}^2_k)^\vee$. For every $g \in \mathbf{GL}_3$, clearly L is tangent to C at p iff $g \cdot L$ is tangent to $g \cdot C$ at $g \cdot p$. Moreover, for $q = [b_0, b_1, b_2]$, the matrix M^g above for $g \cdot F$ and $g \cdot q$ is simply $M \cdot g^\dagger$. Thus M^g has rank 1 iff M has rank 1. So it suffices to prove the result after applying an element of $\mathbf{GL}_3$. It is easy to prove that $\mathbf{GL}_3$ acts transitively on Λ, so assume $p = [a_0, a_1, a_2] = [1, 0, 0]$ and $q = [b_0, b_1, b_2] = [0, 0, 1]$. Then L is tangent to C at p iff $F(x_0, x_1, 0)$ has a repeated root at $(1, 0, 0)$, i.e., iff $f(t) = F(1, t, 0)$ has a repeated root at $t = 0$. This is true iff $F(1, 0, 0) = 0$ and $(\partial F)/(\partial X_1)(1, 0, 0) = 0$. Because $\mathrm{char}(k)$ does not divide e, $F(1, 0, 0) = 0$ iff $(\partial F)/(\partial X_0)(1, 0, 0) = 0$. Therefore L is tangent to C at p iff $(\partial F)/(\partial X_0)(1, 0, 0) = (\partial F)/(\partial X_1)(1, 0, 0) = 0$. This is precisely the condition that the following matrix has rank 1,

$$\begin{pmatrix} \frac{\partial F}{\partial X_0}(1,0,0) & \frac{\partial F}{\partial X_1}(1,0,0) & \frac{\partial F}{\partial X_2}(1,0,0) \\ 0 & 0 & 1 \end{pmatrix}.$$

(c) A line $L \subset \mathbb{P}^2_k$ is *tangent to* C if there exists $p \in L$ such that L is tangent to C at p. Using (b) and the universal closedness of $\mathbb{P}^2_k$, prove the following subset of $(\mathbb{P}^2_k)^\vee$ is Zariski closed,

$$\left\{ q | \pi_1(\Lambda \cap \pi_2^{-1}(q)) \text{ is tangent to } C \right\}.$$

Solution: The 2×2-minors of the matrix from (b) are bihomogeneous in X and Y of bidegree $(e - 1, 1)$. The vanishing locus is a Zariski closed subset of $(\mathbb{P}^2_k) \times (\mathbb{P}^2_k)^\vee$. So the intersection with Λ is a Zariski closed subset. By universal closedness, the image of this closed set under π_2 is a closed subset of $(\mathbb{P}^2_k)^\vee$. The set above is precisely this set.

Remark: Even if $\mathrm{char}(k)$ does divide e, this set is closed. In (b), the condition on the matrix is not enough to guarantee that L is tangent to C at p. But the condition on the matrix together with the condition $F(a_0, a_1, a_2) = 0$ is equivalent to the condition that L is tangent to C at p, with no hypothesis on $\mathrm{char}(k)$. These conditions define a Zariski closed subset of Λ, whose image under π_2 is a Zariski closed subset of $(\mathbb{P}^2_k)^\vee$.

Required Problem 3: Let k be an algebraically closed field and let R be a finitely generated, reduced k-algebra. Define the *max spectrum of* R, $\mathrm{Spec}_{\max}(R)$, to be the set of k-algebra homomorphisms $\phi : R \to k$. For every element $r \in R$, there is a mapping $\tilde{r} : \mathrm{Spec}_{\max}(R) \to \mathbb{A}^1_k = k$ by $\tilde{r}(\phi) = \phi(r)$. Define the *Zariski topology on Spec* $_{max}(R)$ to be the weakest topology such that $\tilde{r}$ is continuous (with respect to the Zariski topology on $\mathbb{A}^1_k$) for every $r \in R$. Denote by $\mathcal{F}$ the sheaf on $\mathrm{Spec}_{\max}(R)$ of all continuous maps from open subsets to $\mathbb{A}^1_k$. Define the *structure sheaf of Spec* $_{max}(R)$, $\mathcal{O}$, to be the smallest subsheaf of $\mathcal{F}$ such that,

(i) for every nonempty open subset $U \subset \mathrm{Spec}_{\max}(R)$, the constant mappings are in $\mathcal{O}(U)$,

(ii) for every open subset $U \subset \text{Spec}_{\max}(R)$ and every $g \in \mathcal{O}(U)$ that is everywhere nonzero, also $1/g \in \mathcal{O}(U)$, and
(iii) for every $r \in R$, $\tilde{r} \in \mathcal{O}(\text{Spec}_{\max}(R))$.

(a) Prove that a basis for the topology on $\text{Spec}_{\max}(R)$ is given by the *basic open affines*, $D(r) := \{\phi : R \to k | \phi(r) \neq 0\}$.

Solution: The weakest topology such that all of the maps $\tilde{r}$ is continuous is the topology with basis $\tilde{r}_1^{-1}(U_1) \cap \cdots \cap \tilde{r}_n^{-1}(U_n)$ for $r_1, \dots, r_n \in R$ and $U_1, \dots, U_n \subset \mathbb{A}_k^1$ Zariski open sets. The Zariski open subsets $U \subset \mathbb{A}_k^1$ are the sets $D(f)$ for $f \in k[x]$. So a basis for the topology consists of

$$\tilde{r}_1^{-1}(D(f_1)) \cap \cdots \cap \tilde{r}_n^{-1}(D(f_n)) = D(f_1 \circ r_1) \cap \cdots \cap D(f_n \circ r_n) = D((f_1 \circ r_1) \cdot \cdots \cdot (f_n \circ r_n)).$$

(b) Prove that for every open U, every continuous map $g : U \to \mathbb{A}_k^1$ and every point $\phi \in U$, there exists a neighborhood $\phi \in V \subset U$ such that $g|_V$ is in $\mathcal{O}(V)$ iff there exist $h, s \in R$ such that $\phi \in D(s) \subset U$ and $g|_{D(s)} = \tilde{h}/\tilde{r}$. Using Theorem 4.5, prove that for every $s \in R$, $\mathcal{O}(D(s)) \cong R[1/s]$.

Solution: By definition of $\mathcal{O}$, $\tilde{h}/\tilde{s} \in \mathcal{O}(D(s))$. It remains to prove that if $g|_V \in \mathcal{O}_V$, then there exist $h, s \in R$ such that $g|_{D(s)} = \tilde{h}/\tilde{r}$. Denote by $\mathcal{F}$ the sub-presheaf of the sheaf of continuous maps to $\mathbb{A}_k^1$, where

$$\mathcal{F}(U) = \{\tilde{h}/\tilde{s} | h, s \in R, \ U \subset D(s)\}.$$

By definition, $\mathcal{O}$ is the sheafification of $\mathcal{F}$. As discussed in lecture, the stalk of $\mathcal{O}$ at ϕ equals the stalk of $\mathcal{F}$ at ϕ, as subsets of the stalk at ϕ of the sheaf of all continous functions. So there exist $f, q \in R$ such that $\phi(s) \neq 0$ and the stalk $(g)_\phi$ equals $(\tilde{f}/\tilde{q})_\phi$. By the definition of the stalk, there exists $\phi \in V \subset U \cap D(q)$ such that $g|_V = (\tilde{f}/\tilde{q})_V$. By (a), there exists $r \in R$ such that $\phi \in D(r) \subset V$. Define $s = rq$ and $h = rf$. Then $\phi \in D(s) \subset V$ and $g|_{D(s)} = \tilde{h}/\tilde{s}$.

By the sheaf axiom, a continuous function $g : U \to \mathbb{A}_k^1$ is in $\mathcal{O}(U)$ iff for every element $\phi \in U$ there exists $\phi \in V \subset U$ such that $g|_V \in \mathcal{O}(V)$. By the last paragraph, $g \in \mathcal{O}(U)$ iff for every $\phi \in U$, there exists $h, s \in R$ such that $\phi(s) \neq 0$ and $g|_{D(s)} = \tilde{h}/\tilde{s}$. This is precisely the same as the definition of regularity of functions on a quasi-affine algebraic set. Therefore, by exactly the same argument as in the case of affine algebraic sets, the k-algebra of regular functions on $D(s)$ is $R[1/s]$.

(c) Prove that $(\text{Spec}_{\max}(R), \mathcal{O})$ is an affine variety. **Not to be written up:** What is the universal property of this affine variety?

Solution: Let $r_1, \dots, r_n \in R$ be a finite set of generators. Define $\psi : k[x_1, \dots, x_n] \to R$ to be the k-algebra homomorphism $f(x_1, \dots, x_n) \mapsto f(r_1, \dots, r_n)$. Define $I \subset k[x_1, \dots, x_n]$ to be the kernel. Define $X = \mathbb{V}(I) \subset \mathbb{A}_k^n$ and define $(X, \mathcal{O}_X)$ to be the associated algebraic variety. Because R is reduced, I is radical ideal, so $k[X] = k[x_1, \dots, x_n]/I$. There is an induced k-algebra homomorphism $\psi : k[X] \to R$. By the universal property of affine varieties, there exists a regular morphism $F : \text{Spec}_{\max}(R) \to X$ such that $\psi = F^*$. Of course, ψ is an isomorphism and $F(\phi) = (\phi(r_1), \dots, \phi(r_n))$. By the Weak Nullstellensatz, F is a bijection. By (a), F identifies the standard basis for the Zariski topology on $\text{Spec}_{\max}(R)$ with the standard basis for the Zariski topology on X, i.e., F is a homeomorphism. Denote by $G : X \to \text{Spec}_{\max}(R)$ the inverse homeomorphism. By (b), for every open

$U \subset \text{Spec}_{\max}(R)$, for every $\phi \in U$ and every continuous $g : U \to \mathbb{A}^1_k$, g is regular at ϕ iff $G^*(g)$ is regular at $F(\phi)$ as a function on a quasi-affine algebraic set. Hence, for every $g \in \mathcal{O}(U)$, $G^*(g) \in \mathcal{O}_X(G^{-1}(U))$, i.e., G is a regular morphism. Since F and G are inverse regular morphisms, $\text{Spec}_{\max}(R)$ is isomorphic to X, i.e., $\text{Spec}_{\max}(R)$ is an affine variety.

Using the isomorphisms F and ψ, for every SWF $(T, \mathcal{O}_T)$, the following set map is a bijection:

$$\text{Hom}_{SWF}((T, \mathcal{O}_T), (\text{Spec}_{\max}(R), \mathcal{O})) \to \text{Hom}_{k-\text{alg}}(R, \mathcal{O}_T(T)),$$
$$(F : T \to \text{Spec}_{\max}(R)) \mapsto (F^* : R \to \mathcal{O}_T(T)).$$

Required Problem 4: Let $F : X \to Y$ be a regular morphism of affine algebraic sets.

(a) For every element $y \in Y$, denote by $\mathfrak{m}_y \subset k[Y]$ the corresponding maximal ideal. Prove there is a bijection between the elements of $F^{-1}(\{y\})$ and the maximal ideals of $k[X]/F^*(\mathfrak{m}_y)k[X]$.

Solution: There is a bijection between maximal ideals of $k[X]/F^*(\mathfrak{m}_y)k[X]$ and maximal ideals of $k[X]$ containing $F^*(\mathfrak{m}_y)k[X]$. By the Nullstellensatz, there is a bijection between the maximal ideals of $k[X]$ containing $F^*(\mathfrak{m}_y)k[X]$ and the elements of X contained in $\mathbb{V}(F^*(\mathfrak{m}_y))$. Of course $\mathbb{V}(F^*(\mathfrak{m}_y)) = F^{-1}\mathbb{V}(\mathfrak{m}_y)$, i.e., $F^{-1}(\{y\})$.

(b) If F is a finite morphism, and if $F^{-1}(\{y\})$ is empty, prove there exists $g \in k[Y]$ such that $g(y) \neq 0$ and $F^*(g) = 0$, i.e., $F^*(g) \cdot k[X] = \{0\}$. (**Hint**: Apply Nakayama's lemma to the finitely-generated $k[Y]$-module $k[X]$.)

Solution: Denote by $I \subset k[Y]$ the ideal $\mathfrak{m}_y$. Denote by M the finitely-generated $k[Y]$-module, $M = k[X]$. By hypothesis, $M/IM = \{0\}$. By Nakayama's lemma, there exists $g \in k[Y]$ such that $g \cong 1 \mod I$ and $g \cdot M = \{0\}$. Therefore, $g(y) = 1$ and $F^*(g) = \{0\}$.

(c) If F is a finite morphism, conclude that $F(X) \subset Y$ is a closed subset: if $y \in Y - F(X)$, then there exists $g \in k[Y]$ such that $y \in D(g) \subset Y - F(X)$. **Not to be written up:** Combined with Corollary 14.19, conclude that finite morphisms of algebraic varieties are universally closed.

Solution: The subset $F(X) \subset Y$ is closed iff the complement $Y - F(X)$ is open. Let $y \in Y - F(X)$. Because $F^{-1}(\{y\})$ is empty, by (b) there exists $g \in k[Y]$ such that $g(y) = 1$ and $F^*(g) = \{0\}$, i.e., $y \in D(g)$ and $F^{-1}(D(g)) = \emptyset$. Therefore $y \in D(g) \subset Y - F(X)$, proving $Y - F(X)$ is open.

Problem 5 (a): Assume $\text{char}(k) \neq 2$. Prove the subset of $(\mathbb{P}^2_k)^\vee$ parametrizing lines tangent to $C = \mathbb{V}(X_0^2 + X_1^2 + X_2^2)$ is $\mathbb{V}(Y_0^2 + Y_1^2 + Y_2^2)$. For a "general" element $p \in \mathbb{P}^2_k$, how many tangent lines to C contain p?

(b) Let $F \in k[X_0, X_1, X_2]_e$ be an irreducible polynomial. Define $U = \mathbb{V}(F) - \mathbb{V}(\partial F/\partial X_0, \partial F/\partial X_1, \partial F/\partial X_2)$. Prove the following mapping $U \to (\mathbb{P}^2_k)^\vee$ is a regular morphism whose image is contained in the set of lines tangent to $\mathbb{V}(F)$ (this mapping is the *Gauss map*):

$$[p] \in U \mapsto [(\partial F/\partial X_0)(p), (\partial F/\partial X_1)(p), (\partial F/\partial X_2)(p)].$$

Problem 6: Let R be a finitely-generated k-algebra that is not necessarily reduced. Repeat the definition of $\text{Spec}_{\max}(R)$ and $\mathcal{O}^{\text{red}}$ as in Problem 3 (except that, for reasons that will become clear, the sheaf is denoted $\mathcal{O}^{\text{red}}$ instead of $\mathcal{O}$). Prove $(\text{Spec}_{\max}(R), \mathcal{O}^{\text{red}})$ is an affine variety, and identify the k-algebra $\mathcal{O}^{\text{red}}(\text{Spec}_{\max}(R))$.

Solution: Denote by R^{red} the reduced k-algebra of R, i.e., the quotient of R by the nilradical. Denote by $p : R \to R^{\text{red}}$ the canonical surjection. There is an induced set map $F : \text{Spec}_{\max}(R^{\text{red}}) \to \text{Spec}_{\max}(R)$, by $F(\phi) = \phi \circ p$. This is a bijection since the nilradical is contained in every maximal ideal of R. For every element $r \in R$, the function $\widetilde{p(r)} \circ F$ equals $\widetilde{r}$. First this implies that $F(D(r)) = D(p(r))$ and $F^{-1}(D(p(r))) = D(r)$, i.e., p^* is a homeomorphism. Second the image of R in the k-algebra of continuous functions $\text{Spec}_{\max}(R) \to \mathbb{A}^1_k$ equals $F^*(R^{\text{red}})$. It follows that $F_*\mathcal{O}^{\text{red}}$ equals the $\mathcal{O}$ as subsheaves of the sheaf of continuous functions $\text{Spec}_{\max}(R^{\text{red}}) \to \mathbb{A}^1_k$. Therefore $F : (\text{Spec}_{\max}(R), \mathcal{O}^{\text{red}}) \to (\text{Spec}_{\max}(R^{\text{red}}), \mathcal{O})$ is an isomorphism of SWFs.

Problem 7: Another proof of existence of sheafification Let X be a topological space and let $\mathcal{F}$ be a presheaf of sets. Define the *éspace étalè* as a *set* in Definition 10.8, $p : |\mathcal{F}| \to X$.

(a) Let $U \subset X$ be an open set, $p \in U$ an element and $f, g \in \mathcal{F}(U)$ elements whose images are equal in the stalk $\mathcal{F}_p$. Prove there exists an open neighborhood $p \in V \subset U$ such that $f|_V = g|_V$.

Solution: This is part of the definition of direct limits.

(b) For every open set $U \subset X$ and every $f \in \mathcal{F}(U)$, define $D(U, f) \subset |\mathcal{F}|$ to be the set of pairs (p, f_p) of an element $p \in U$ and the image f_p of f in $\mathcal{F}_p$. Prove these sets form the basis for a topology on $|\mathcal{F}|$, called the *natural topology*.

Solution: This is not technically correct, because the empty set should be added. The other axioms for a basis are satisfied. First of all, for every $p \in X$ and every $f_p \in \mathcal{F}_p$, there exists an open set $p \in U \subset X$ and $f \in \mathcal{F}(U)$ such that f_p is the germ of f at p. So $(p, f_p) \in D(U, f)$. Next, let (p, h_p) be an element of $D(U, f) \cap D(V, g)$. Then by (a), there exists $p \in W \subset U \cap V$ such that $f|_W = g|_W$. So $(p, h_p) \in D(W, f|_W)$.

(c) For every open set $U \subset X$ and every $f \in \mathcal{F}(U)$, prove the induced set map $\tilde{f} : U \to |\mathcal{F}|$ is continuous with respect to the natural topology on $|\mathcal{F}|$.

Solution: It suffices to prove that for every pair (V, g), $\tilde{f}^{-1}(D(V, g))$ is open. For every $p \in \tilde{f}^{-1}(D(V, g))$, $f_p = g_p$. By (a), there exists $p \in W \subset U \cap V$ such that $f|_W = g|_W$. Therefore $W \subset \tilde{f}^{-1}(D(V, g))$, proving $\tilde{f}^{-1}(D(V, g))$ is open.

(d) Denote by $\mathcal{F}^+$ the sheaf of sections of the continuous mapping $p : |\mathcal{F}| \to X$ as in Example 10.4(ii). By (c) there is a presheaf homomorphism $\phi : \mathcal{F} \to \mathcal{F}^+$. Prove this is a sheafification of $\mathcal{F}$.

Solution: First of all, it is easy to prove $\mathcal{F}^+$ is a sheaf because continuous maps satisfy the gluing lemma. To prove $\phi : \mathcal{F} \to \mathcal{F}^+$ is a sheafification, it suffices to prove for every $p \in X$ that the induced map of stalks is a bijection, $\phi_p : \mathcal{F}_p \to \mathcal{F}^+_p$.

Injectivity: Let $U \subset X$ be an open set, $p \in U$ an element and $f, g \in \mathcal{F}(U)$ elements such that $\phi_p(f_p) = \phi_p(g_p)$, i.e., $\tilde{f}_p = \tilde{g}_p$. By (a), there exists $p \in V \subset U$ such that $\tilde{f}|_V = \tilde{g}|_V$. In particular, $f_p = \tilde{f}|_V(p) = \tilde{g}|_V(p) = g_p$.

Surjectivity: Let $p \in X$, let $p \in U \subset X$ be an open neighborhood, and let $f \in \mathcal{F}^+(U)$. There exists an open subset $p \in V \subset U$ and $g \in \mathcal{F}(V)$ such that $f(p) = (p, g_p)$, i.e., $p \in f^{-1}(D(V, g))$. Because f is continuous, the subset $W := f^{-1}(D(V, g))$ is open. By definition, $f|_W = \tilde{g}|_W$. So $f_p = \phi_p(g_p)$.

Problem 8 Let $\mathcal{A}$ and $\mathcal{B}$ be categories. An *adjoint pair of functors* is a pair of functors (L, R), $L : \mathcal{A} \to \mathcal{B}$, $R : \mathcal{G} \to \mathcal{A}$, together with a rule associating to every object A of $\mathcal{A}$ and every object B of $\mathcal{B}$ a bijection,

$$\eta_{A,B} : \mathrm{Hom}_{\mathcal{B}}(L(A), B) \to \mathrm{Hom}_{\mathcal{A}}(A, R(B)),$$

which is a *natural bijection* in the sense that for every object A of $\mathcal{A}$, resp. every object B of $\mathcal{B}$, the induced transformation of functors $\mathcal{B} \to \mathrm{Sets}$,

$$\eta_{A,*} : \mathrm{Hom}_{\mathcal{B}}(L(A), *) \Rightarrow \mathrm{Hom}_{\mathcal{A}}(A, R(*)),$$

is a natural transformation, resp. the induced transformation of contravariant functors $\mathcal{A} \to \mathrm{Sets}$,

$$\eta(*, B) : \mathrm{Hom}_{\mathcal{B}}(L(*), B) \Rightarrow \mathrm{Hom}_{\mathcal{A}}(*, R(B)),$$

is a natural transformation.

(a) Let $\mathcal{A} = \mathrm{Sets}$ and let $\mathcal{B} = \mathrm{Groups}$, Rings, or $R-$ modules. Define $R : \mathcal{B} \to \mathcal{A}$ to be the functor that sends each object to its underlying set of elements. Prove there is a functor $L : \mathcal{A} \to \mathcal{B}$ and a natural bijection η so that (L, R) is an adjoint pair. **Hint:** For each $\mathcal{B}$, there is a notion of a *free object.*

Solution: For $\mathcal{B} = \mathrm{Groups}$, for every set S define F_S together with the set map $i : S \to F_S$ to be the *free group on* S, i.e., the group whose elements are all finite words $w = x_1 x_2 \dots x_n$ where every x_i is either an element of S or the formal inverse of an element of S, and product is defined by concatenating words and contracting inverses. The free group has the universal property that for every group G, the following set map is a bijection,

$$\mathrm{Hom}_{\mathrm{Groups}}(F_S, G) \to \mathrm{Hom}_{\mathrm{Sets}}(S, G), \quad (\phi : F_S \to G) \mapsto (\phi \circ i : S \to G).$$

This is precisely the condition for an adjoint pair. The construction for rings and for R-modules is similar.

(b) In each case above, prove that (L, R) has the additional property that a morphism $f : B \to B'$ in $\mathcal{B}$ is an isomorphism iff $R(f)$ is an isomorphism (this is not an axiom for an adjoint pair).

Solution: The point is that a homomorphism of groups, rings or R-modules is invertible iff the underlying set map is a bijection. This is because the inverse set map automatically preserves the group product, resp. addition and multiplication, resp. addition and scaling by elements in R.

Problem 9: Let $\mathcal{A} = \mathrm{Sets}$, let $\mathcal{B}$ be a category, and let (L, R, η) be an adjoint pair such that for every morphism $f : B \to B'$ in $\mathcal{B}$, f is an isomorphism iff $R(f)$ is an isomorphism. Let X be a topological space, and let $\mathcal{F}$ be a presheaf of objects in $\mathcal{B}$ on X.

(a) Prove that $\mathcal{F}$ is a sheaf iff the presheaf of sets $R(\mathcal{F})$ on X is a sheaf.

Correction: The assertion is false. A corrected version of this exercise appears on the next problem set.

(b) Prove that $\mathcal{F}$ satisfies Axiom (A) from Definition 10.1 iff $\mathcal{F}$ satisfies Axiom (A') from Remark 10.2.

Correction: Same as above.

Difficult Problem 10: Let $F : \mathbb{P}^1_k \to \mathbb{P}^3_k$ be the regular morphism $[a_0, a_1] \mapsto [a_0^3, a_0^2 a_1, a_0 a_1^2, a_1^3]$. Denote by $C \subset \mathbb{P}^3_k$ the image of F (which is a projective subvariety by Problem 10 from PS# 2). For every element $p = [b_0, b_1, b_2, b_3] \in \mathbb{P}^3_k - F(\mathbb{P}^1_k)$, define a morphism $G_p : C \to \mathbb{P}^5_k$ by

$$[c_0, c_1, c_2, c_3] \mapsto [b_1c_0 - b_0c_1, b_2c_0 - b_0c_2, b_3c_0 - b_0c_3, b_2c_1 - b_1c_2, b_3c_1 - b_1c_3, b_3c_2 - b_2c_3].$$

(a) Prove there exists a linear embedding $H : \mathbb{P}^2_k \subset \mathbb{P}^5_k$ whose image contains the image of G_p.

Solution: Choose homogeneous coordinates on $\mathbb{P}^5_k$, $(Z_{(i,j)} | 0 \leq i < j \leq 3)$. Then, up to relabelling coordinates, G_p is the restriction of a regular morphism $g_p : \mathbb{P}^1_k - \{p\} \to \mathbb{P}^5_k$ determined by $g_p^* Z_{i,j} = b_j X_i - b_i X_j$. Denote $Z_{(j,i)} := -Z_{(i,j)}$. There exists $0 \leq i \leq 3$ such that $b_i \neq 0$. For every $0 \leq j < k \leq 3$ with $j, k \neq i$,

$$g_p^* Z_{j,k} = -(b_k/b_i) g_p^* Z_{(i,j)} + (b_j/b_i) g_p^* Z_{(i,k)}.$$

Choose homogeneous coordinates on $\mathbb{P}^2_k$, $(Y_j | 0 \leq j \leq 3, j \neq i)$. Define $H : \mathbb{P}^2_k \to \mathbb{P}^5_k$ to be the regular morphism determined by $H^* Z_{(i,j)} = Y_j$ for $j \neq i$, and $H^* Z_{(j,k)} = -(b_k/b_i) Y_j + (b_j/b_i) Y_k$ for $j, k \neq i$. The image of g_p is contained in the image of H.

(b) With respect to your linear embedding, find the equation of the plane curve $C_p = H^{-1}(G_p(C))$ for $p = [1, 0, 0, 1]$. Write down all the elements $q \in C_p$ where there is *not* a unique tangent line to C_p at q.

Solution: Choose $i = 0$ in (a) above. There is a unique regular morphism $i_p : \mathbb{P}^3_k - \{p\} \to \mathbb{P}^2_k$ such that $H \circ i_p = g_p$, namely,

$$i_p^* Y_1 = -X_1, \quad i_p^* Y_2 = -X_2, \quad i_p^* Y_3 = X_0 - X_3.$$

The composition $i_p \circ F$ is $[a_0, a_1] = [-a_0^2 a_1, -a_0 a_1^2, a_0^3 - a_1^3]$. The equation of the image is $Y_1^3 - Y_2^3 + Y_1 Y_2 Y_3$. For every point q except $[Y_1, Y_2, Y_3] = [0, 0, 1]$ there is a unique tangent line, namely,

$$\mathbb{V}(a_1(2a_0^3 + a_1^3) Y_1 - a_0(a_0^3 + 2a_1^3) Y_2 + a_0^2 a_1^2 Y_3).$$

For the point $q = [0, 0, 1]$, every line containing q is a tangent line to C_p at q.

(c) A *secant line to* C is a projective line in $\mathbb{P}^3$ that intersects C in at least 2 distinct points. How many secant lines to C contain p? **Not to be written up:** What if p is another (general) element of $\mathbb{P}^3_k$? How many secant lines to C contain p? Pay special attention if you go to Alexei Oblomkov's PUMA-GRASS lecture.

Solution: The lines in $\mathbb{P}^3_k$ containing p are in bijective correspondence with the elements of $\mathbb{P}^2_k$ via $q \mapsto \overline{i_p^{-1}(\{q\})}$. Thus the secant lines to C containing p correspond to pairs of distinct points $r, s \in C$ such that $i_p(r) = i_p(s)$. For such a pair, the corresponding point $q = i_p(r) = i_p(s)$ is a point of C_p for which there is not a unique tangent line. Since there is precisely one such point on C_p, there is one secant line to C containing p, namely $\mathbb{V}(X_1, X_2) \subset \mathbb{P}^3_k$ which contains $p = [1, 0, 0, 1]$, contains $[1, 0, 0, 0] = F([1, 0])$ and contains $[0, 0, 0, 1] = F([0, 1])$.

It is true that there is a unique secant line to C containing p for every point $p \in \mathbb{P}^3_k - \mathbb{V}(Q)$, where

$$Q = 4(X_0X_3 - X_1^2)(X_1X_3 - X_2^2) - (X_0X_3 - X_1X_2)^2.$$

Moreover, for every point $p \in \mathbb{V}(Q) - C$, there is a unique tangent line to C containing p. This implies a peculiar property of C: every pair of distinct tangent lines to C in $\mathbb{P}^3_k$ are disjoint (for any non-planar curve, 2 *general* tangent lines are disjoint, but typically every tangent line intersects finitely many other tangent lines).

Problem 11: For every integer $n \in \mathbb{Z}$, define X_n to be a copy of the affine variety $\mathbb{V}(xy) \in \mathbb{A}^2$, define $X_{n,n+1} \subset X_n$ to be $D(x)$ and $X_{n,n-1} \subset X_n$ to be $D(y)$. Define $\phi_{n,n+1} : X_{n,n+1} \to X_{n+1,n}$ to be the regular morphism $(a, 0) \mapsto (0, 1/a)$. If $|m - n| > 1$, define $X_{m,n} = \emptyset$ and define $\phi_{m,n}$ to be the empty mapping.

(a) Prove that the collection $(\{X_n\}, \{X_{m,n}\}, \{\phi_{m,n}\})$ satisfy the axioms for Lemma 12.11, the Gluing Lemma for spaces with functions. Denote by X the associated space with functions.

Solution: This comes to the fact that $X_{n,n-1} \cap X_{n,n+1} = \emptyset$.

(b) Prove that X is a connected algebraic variety that is *not* quasi-compact.

Solution: The collection $(\phi_n(X_n)|n \in \mathbb{Z})$ is an open covering of X that has no finite subcovering.