# 18.725 PROBLEM SET 1

**Due date:** Friday, September 17 in lecture. Late work will be accepted only with a medical note or for another Institute-approved reason. You are strongly encouraged to work with others, but the final write-up should be entirely your own and based on your own understanding.

Read through all the problems. Write solutions to the "Required Problems", 1–4, together with 5 others of your choice to a total of 9 problems. Difficult problems are labelled "Difficult Problems".

**Required Problem 1** Do Exercise 1.11 from the notes for Lecture 1. Try to use the Nullstellensatz only when necessary.

**Required Problem 2 (a)** Prove that $\mathbb{A}^1_k$ with the Zariski topology is not Hausdorff.

**(b)** Prove that any bijection $F : \mathbb{A}^1_k \to \mathbb{A}^1_k$ is a homeomorphism with respect to the Zariski topology.

**Required Problem 3** Let $V \subset \mathbb{A}^m_k$ and $W \subset \mathbb{A}^n_k$ be affine algebraic sets with $\mathbb{I}(V) = I \subset k[x_1, \dots, x_m]$ and $\mathbb{I}(W) = J \subset k[y_1, \dots, y_n]$ respectively. Define $K \subset k[z_1, \dots, z_m, z_{m+1}, \dots, z_{m+n}]$ to be the ideal,

$$K = \langle f(z_1, \dots, z_m) | f(x_1, \dots, x_m) \in I \rangle + \langle g(z_{m+1}, \dots, z_{m+n}) | g(y_1, \dots, y_n) \in J \rangle.$$

**(a)** Prove the map

$$(\pi_1, \pi_2) : \mathbb{A}^{m+n}_k \to \mathbb{A}^m_k \times \mathbb{A}^n_k,\ (z_1, \dots, z_m, z_{m+1}, \dots, z_{m+n}) \mapsto ((z_1, \dots, z_m), (z_{m+1}, \dots, z_{m+n})),$$

restricts to a bijection from $\mathbb{V}(K)$ to $V \times W$.

**(b)** Prove the projections $\pi_1 : \mathbb{V}(K) \to V$, $\pi_2 : \mathbb{V}(K) \to W$ are regular morphisms.

**(c)** For every affine algebraic set $T$ prove the following set map is a bijection,

$$\text{Regular morphisms}(T, \mathbb{V}(K)) \to \text{Regular morphisms}(T, V) \times \text{Regular morphisms}(T, W),$$
$$(f : T \to \mathbb{V}(K)) \mapsto ((\pi_1 \circ f : T \to V), (\pi_2 \circ f : T \to W))$$

In other words, the pair of regular morphisms $(\pi_1, \pi_2)$ is a product of $V$ and $W$ in the category of affine algebraic sets.

**Required Problem 4(a)** Prove the induced topology on every subset of a Noetherian topological space is Noetherian.

**(b)** Prove every Noetherian topological space is quasi-compact. (**Hint:** Given an open covering $\mathcal{U}$ of $X$ by open subsets, consider the collection of closed subsets that are complements of unions of finite subcoverings.)

**Problem 5** Give an example of a Jacobson ring that is not a finitely-generated algebra over a field. Prove your example is a Jacobson ring.

**Problem 6** Denote $f(X,Y) = C_{2,0,0}X^2 + C_{1,1,0}XY + C_{0,2,0}Y^2 + C_{1,0,1}X + C_{0,1,1}Y + C_{0,0,2}$ for coefficients $C_{i,j,k} \in k$ satisfying $(C_{2,0,0}, C_{1,1,0}, C_{0,2,0}) \neq (0,0,0)$.

**(a)** Prove $\mathbb{V}(f) \subset \mathbb{A}^2_k$ is nonempty.

**(b)** If the following symmetric matrix $M$ is invertible, prove $f$ is irreducible (and thus $\mathbb{V}(f)$ is irreducible).

$$M = \begin{pmatrix} 2C_{2,0,0} & C_{1,1,0} & C_{1,0,1} \\ C_{1,1,0} & 2C_{0,2,0} & C_{0,1,1} \\ C_{1,0,1} & C_{0,1,1} & 2C_{0,0,2} \end{pmatrix}$$

**(c)** If $M$ has rank at least 2, prove $f$ is not the square of a linear polynomial (and thus $\mathbb{V}(f)$ is not a line).

**Problem 7** With notation from Problem 6 and assuming $\text{char}(k) \neq 2$, prove that $\mathbb{V}(f)$ is a line if $M$ has rank 1, and that $\mathbb{V}(f)$ is reducible if $M$ has rank 2. **Don't write up:** What if $\text{char}(k) = 2$?

**Difficult Problem 8** With notation as in Problem 3, prove that $K$ is a radical ideal. **Warning:** You will need to use that $k$ is algebraically closed; for $k$ not a perfect field there are examples where the ideals $I$ and $J$ are radical, but $K$ is not radical.

**Problem 9** Prove $V = \{(t, t^2, t^3) | t \in k\}$ is an affine algebraic subset of $\mathbb{A}^3_k$ and find $\mathbb{I}(V) \subset k[x_1, x_2, x_3]$.

**Difficult Problem 10** Prove the subset $V = \{(s^3, s^2t, st^2, t^3) | s, t \in k\}$ is an affine algebraic subset of $\mathbb{A}^4_k$ and find $\mathbb{I}(V) \subset k[x_0, x_1, x_2, x_3]$. **Don't write up:** If you do both Problem 9 and Problem 10, compare your answers.

**Problem 11** Let $g \geq 1$ be an integer, let $a_1, a_2, \dots, a_{2g-1} \in k - \{0, 1\}$ be distinct elements, and denote $f = y^2 - x(x-1)(x-a_1)\dots(x-a_{2g-1}) \in k[x, y]$.

**(a)** Prove $f$ is an irreducible polynomial. (**Hint:** Eisenstein's criterion.)

**(b)** Prove the ring $k[x, y]/\langle f \rangle$ is not a unique factorization domain.

**(c)** Conclude the affine algebraic set $\mathbb{V}(f) \subset \mathbb{A}^2_k$ is not isomorphic to $\mathbb{A}^1_k$. This affine algebraic set is the affine part of a *genus $g$ hyperelliptic curve*.

**Difficult Problem 12** With notation from Problem 11, prove there is no non-constant regular morphism $F : \mathbb{A}^1_k \to \mathbb{V}(f)$. (**Hint:** If there where such a morphism, what could you say about the irreducible factors of $F^*y$, $F^*x$, $F^*(x-1)$, etc.)

**Problem 13** Let $F : V \to W$ be a regular morphism of affine algebraic sets, and let $F^* : k[W] \to k[V]$ be the induced $k$-algebra homomorphism on coordinate rings.

**(a)** Prove $\text{Kernel}(F^*)$ is a radical ideal of $k[W]$.

**(b)** Describe the ideal $\mathbb{I}(F(V))$.

**(c)** Give a geometric interpretation to the condition that $F^*$ is injective.

**(d)** Give an example where $F^*$ is injective, but $F(V) \neq W$.

**Problem 14** Give an example of a homeomorphic regular morphism of affine algebraic sets that is *not* an isomorphism of affine algebraic sets. **Don't write up:** Try to find an example where the coordinate ring of the target is a unique factorization domain.

**Problem 15** For every choice of $a, b \in k$, find the irreducible components of the affine algebraic set $\mathbb{V}(xy - z, bx + ay - z - ab) \subset \mathbb{A}^3_k$.