Name: ______________________________ Problem 1: ____________ /20

Problem 1(20 points) Define a *natural transformation* between two covariant functors. Then, for the category of vector spaces over a field F, prove there exists a natural transformation θ from the identity functor to the double dual functor,

$$V \mapsto (V^*)^* = \mathrm{Hom}_F(\mathrm{Hom}_F(V, F), F),$$

$$[T : V \to W] \mapsto [(T^\dagger)^\dagger : (V^*)^* \to (W^*)^*],$$

such that for the F-vector space F, θ_F is an isomorphism.
Extra credit. (3 points) For your choice of isomorphism θ_F, is θ the only natural transformation? Prove your answer.

Given functors $\mathcal{F}, \mathcal{G} : \mathcal{C} \to \mathcal{D}$, a natural transformation $\theta : \mathcal{F} \Rightarrow \mathcal{G}$ is a rule associating to every object c of $\mathcal{C}$ a morphism in $\mathcal{D}$, $\theta_c : \mathcal{F}c \to \mathcal{G}c$ such that for every morphism in $\mathcal{C}$, $u : c \to c'$, the following diagram commutes

$$\begin{tikzcd} \mathcal{F}c \arrow[r, "\mathcal{F}u"] \arrow[d, "\theta_c"'] & \mathcal{F}c' \arrow[d, "\theta_{c'}"] \\ \mathcal{G}c \arrow[r, "\mathcal{G}u"'] & \mathcal{G}c' \end{tikzcd}$$

.

Let $\mathcal{C} = \mathcal{D}$ be the category of F-vector spaces. Let λ be an element of F (later $\lambda \neq 0$). Let $\mathcal{F}$ be the identity functor. Let $\mathcal{G}$ be the double dual functor. Define θ^λ as follows. For every F-vector space V, $\theta^\lambda_V : V \to \mathrm{Hom}_F(\mathrm{Hom}_F(V, F), F)$ is the map sending each $\vec{v}$ in V to the F-linear map $\theta^\lambda_V(\vec{v}) : \mathrm{Hom}_F(V, F) \to F$, $(\chi : V \to F) \mapsto \lambda \chi(\vec{v}) \in F$.

Let $T : W \to V$ be an F-linear transformation of F-vector spaces. For every F-linear map $\mathcal{S} : \mathrm{Hom}_F(W, F) \to F$, the map $(T^\dagger)^\dagger(\mathcal{S}) : \mathrm{Hom}_F(V, F) \to F$ sends each $(\chi : V \to F)$ to $\mathcal{S}(\chi \circ T)$. Thus $(T^\dagger)^\dagger(\theta^\lambda_W(\vec{w}))$ equals

$$(\chi : V \to F) \mapsto \theta^\lambda_W(\vec{w})(\chi \circ T) = \lambda(\chi \circ T)(\vec{w})$$

$$\lambda \chi(T\vec{w}) = \theta^\lambda_V(T\vec{w})$$

$(\theta^\lambda_V \circ T)(\vec{w})$. Thus $(T^\dagger)^\dagger \circ \theta^\lambda_W$ equals $\theta^\lambda_V \circ T$ as required for a natural transformation. (over $\downarrow$)

Notice that $(\theta^{\lambda}_F(1))(Id_F : F \to F)$ equals λ. So if $\lambda \neq 0$, then θ^{λ}_F is an isomorphism. For any λ, suppose $\eta : \mathcal{F} \Rightarrow \mathcal{G}$ is a natural transformation with $(\eta_F(1))(Id_F : F \to F)$ equals λ. For every F-vector space V and for every vector $\vec{v}$ in V, define $T_{\vec{v}} : F \to V$ by $T_{\vec{v}}(\mu) := \mu \vec{v}$. Since η is a natural transformation, $(T_{\vec{v}}^{\,t})^t \circ \eta_F$ equals $\eta_V \circ T_{\vec{v}}$.
In particular, $\eta_V(\vec{v}) = (\eta_V \circ T_{\vec{v}})(1) = (T_{\vec{v}}^{\,t})^t \circ \eta_F(1)$. As described above, for $(\chi : V \to F)$, $(\,(T_{\vec{v}}^{\,t})^t(\eta_F(1))\,)(\chi)$ equals $\eta_F(1)(\chi \circ T_{\vec{v}})$. And $\chi \circ T_{\vec{v}} : F \to F$ equals $\chi(\vec{v}) \cdot Id_F$.
So, since $\eta_F(1)$ is F-linear, $\eta_F(1)(\chi(\vec{v}) \cdot Id_F) = \chi(\vec{v})\, \eta_F(1)(Id_F)$, i.e. $\chi(\vec{v}) \cdot \lambda$. So $\eta_V(\vec{v})$ equals $\theta^{\lambda}_V(\vec{v})$ for every V and for every $\vec{v}$ in V. Thus η equals θ^{λ}.
So θ^{λ} is the unique natural transformation $\mathcal{F} \Rightarrow \mathcal{G}$ with $\theta^{\lambda}_F(1)(Id_F)$ equal to λ.
(This is a special case of the general proof that a natural transformation from a representable functor, $\theta : h^c \Rightarrow \mathcal{G}$, is uniquely determined by its value $\theta_c(Id_c) \in \mathcal{G}c$. In this case the identity functor $\mathcal{F} = Id_{F\text{-vect.}}$ is representable: $\mathcal{F} = h^F$.)

Name: ______________________ Problem 2: __________ /30

Problem 2(30 points) Let R be a commutative ring with 1. For every $\mathbb{Z}$-module Q give $\mathrm{Hom}_{\mathbb{Z}}(R,Q)$ the structure of R-module as in the homework exercises, i.e., for $\phi : R \to Q$ in $\mathrm{Hom}_{\mathbb{Z}}(R,Q)$ and for r in R, $(r * \phi)(s) := \phi(sr)$. Prove that if Q is an injective $\mathbb{Z}$-module, then $\mathrm{Hom}_{\mathbb{Z}}(R,Q)$ is an injective R-module.

Let $f: M \to \mathrm{Hom}_{\mathbb{Z}}(R,Q)$ be an R-module homomorphism, $m \mapsto (f_m : R \to Q)$. Let $i: M \to N$ be an injective R-module homomorphism. Define $\tilde{f}: M \to Q$ by $\tilde{f}(m) := f_m(1)$. Since f is a group homomorphism, also $\tilde{f}$ is a group homomorphism: $\tilde{f}(m_1+m_2) = f_{m_1+m_2}(1) = f_{m_1}(1) + f_{m_2}(1) = \tilde{f}(m_1) + \tilde{f}(m_2)$.

Since i is an injective group homomorphism, and since Q is an injective $\mathbb{Z}$-module, there exists a group homomorphism $\tilde{g}: N \to Q$ such that $\tilde{g} \circ i$ equals $\tilde{f}$.

Define $g: N \to \mathrm{Hom}_{\mathbb{Z}}(R,Q)$, $n \mapsto (g_n : R \to Q)$, by $g_n(r) = \tilde{g}(rn)$. Then $(r * g_n)(s) = g_n(sr) = \tilde{g}((sr)n) = \tilde{g}(s(rn)) = g_{rn}(s)$. So g is an R-module homomorphism (it is obviously a group homomorphism since $\tilde{g}$ is).

And $(g \circ i)(m) = g_{i(m)}$ sending r to $g_{i(m)}(r) = \tilde{g}(r\, i(m))$. Since i is an R-module homomorphism, $r\, i(m)$ equals $i(rm)$. So $\tilde{g}(r\, i(m)) = \tilde{g}(i(rm)) = \tilde{f}(rm) = f_{rm}(1)$. And since f is an R-module homomorphism, $f_{rm}(1) = (r * f_m)(1) = f_m(r)$. So $(g \circ i)(m)$ equals $f_m = f(m)$. Thus $g \circ i$ equals f.

Name: ______________________ Problem 3: ____________ /30

Problem 3(30 points) In the following commutative diagram, the rows are exact.

$$
\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & A \arrow[r, "p_A"] \arrow[d, "u"'] & A'' \arrow[r] \arrow[d, "u''"] & 0 \\
0 \arrow[r] & B' \arrow[r, "q_B"] \arrow[d, "v'"'] & B \arrow[r, "p_B"] \arrow[d, "v"'] & B'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C' \arrow[r, "q_C"] & C \arrow[r] & 0 &
\end{tikzcd}
$$

Assuming that u'' is surjective and v' is injective, carefully prove that $\mathrm{Ker}(v)$ equals $\mathrm{Image}(u)$. You may use the Snake Lemma or the Long Exact Sequence of Homology. But if so, you must precisely and correctly state the result you are using. And you must justify your application of the result.

Added during exam: vou equals zero so that all columns are complexes.

This diagram is a short exact sequence of complexes, $0 \to M'_\bullet \xrightarrow{q_\bullet} M_\bullet \xrightarrow{p_\bullet} M''_\bullet \to 0$ where $M'_\bullet$ is the first column, etc.

The Long Exact Sequence of Homology gives an exact sequence

$$0 \to 0 \to \mathrm{Ker}(u) \xrightarrow{p_A} \mathrm{Ker}(u'') \xrightarrow{\delta} \mathrm{Ker}(v') \xrightarrow{q_B} \frac{\mathrm{Ker}(v)}{\mathrm{Im}(u)} \xrightarrow{p_B} \mathrm{Coker}(u'') \xrightarrow{\delta} \mathrm{Coker}(v') \xrightarrow{q_C} \mathrm{Coker}(v).$$

By hypothesis, $\mathrm{Ker}(v')$ & $\mathrm{Coker}(u'')$ are zero. Thus also $\mathrm{Ker}(v)/\mathrm{Im}(u)$ is zero, i.e., $\mathrm{Ker}(v)$ equals $\mathrm{Im}(u)$.

Name: ______________________________ Problem 4: ____________ /20

Problem 4(20 points) Let A be a finite abelian group. Let p be a prime integer. Let the order of the Sylow p-subgroup of A be p^n. Prove that the natural map

$$A \to A \otimes_{\mathbb{Z}} (\mathbb{Z}/p^n\mathbb{Z})$$

maps the Sylow p-subgroup of A isomorphically onto $A \otimes_{\mathbb{Z}} (\mathbb{Z}/p^n\mathbb{Z})$.

The exact sequence $0 \to \mathbb{Z} \xrightarrow{p^n} \mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z} \to 0$ gives an exact sequence $A \otimes_{\mathbb{Z}} \mathbb{Z} \xrightarrow{p^n} A \otimes_{\mathbb{Z}} \mathbb{Z} \to A \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z} \to 0$. Since $A \otimes_{\mathbb{Z}} \mathbb{Z}$ equals A, $A \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$ equals A/p^nA. And the natural map $A \to A \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$ is just the quotient map $A \to A/p^nA$.

<u>1. $\mathrm{Syl}_p(A) \to A/p^nA$ is injective</u>. Let $a \in \mathrm{Syl}_p(A)$ also lie in p^nA, say $a = p^n b$. Then a has order p^m for some integer m, since $\mathrm{order}(a)$ divides $|\mathrm{Syl}_p(A)| = p^n$. So $\mathrm{order}(b)$ equals $p^n \cdot \mathrm{order}(a)$, which is larger than p^n unless $\mathrm{order}(a) = 1$, i.e., a is zero. And if $\mathrm{order}(b) = p^{n+m}$ is bigger than p^n, then the cyclic group generated by b is a <u>larger</u> p-group than $\mathrm{Syl}_p(A)$, a contradiction. So a is zero. Thus $\mathrm{Syl}_p(A) \to A/p^nA$ is injective

<u>2. $\mathrm{Syl}_p(A) \to A/p^nA$ is surjective</u>. By the structure theorem for finitely generated groups, A is the direct product of $\mathrm{Syl}_p(A)$ and $A_m := \{a \in A \mid \mathrm{order}(a) \text{ divides } m\}$, where $m = |A|/p^n$. By the Sylow theorem, m is prime to p. So m is prime to p^n, i.e. $u \cdot p^n + v \cdot m = 1$ for integers u, v. Thus for a in A_m, $a = p^n \cdot (u \cdot a)$ is in p^nA. So $A = \mathrm{Syl}_p(A) \times p^nA$.