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MAT 322 Problem Set 2

Homework Policy. Please read through all the problems. Please write up solutions of the required problems. Please also read and attempt the extra problems, but please do not write up those solutions for grading. I will be happy to discuss the extra problems during office hours.

Each student is encouraged to work on problem sets with other students, but each submitted problem set must be in the student's own words and based on the student's own understanding. It is against university policy to copy answers from other students or from any other resource (such as a webpage).

Required Problems.

Problem 1. For every $n \times n$ matrix $A = [a_{i,j}]_{1 \leq i,j \leq n}$ in $\text{Mat}_{n \times n}$, define the $n \times n$ matrix, $\widetilde{A} = [(-1)^{i+j}\text{det}(A_{j,i})]$, the transpose matrix of cofactors, so that $A \cdot \widetilde{A} = \widetilde{A} \cdot A = \text{det}(A)I_n$ by Cramer's Rule. Since the entries of $\widetilde{A}$ are polynomials in the entries of A, this defines a continuous function (for the norm / Euclidean topology),

$$\widetilde{\bullet} : \text{Mat}_{n \times n} \to \text{Mat}_{n \times n}.$$

Recall that the **characteristic polynomial** of A is defined to be the degree n polynomial $c_A(x)$ with real coefficients such that for every real number λ,

$$c_A(\lambda) = \text{det}(\lambda I_n - A), \; c_A(x) = x^n - \text{tr}(A)x^{n-1} + \cdots + (-1)^m b_m(A)x^{n-m} + \cdots + (-1)^n \text{det}(A),$$

where $b_0(A), b_1(A), \ldots, b_n(A)$ are real numbers with $b_0(A) = 1$, $b_1(A) = \text{tr}(A)$ and $b_n(A) = \text{det}(A)$.

(a) Prove that for every integer $m = 0, \ldots, n$, $b_m(A)$ is a polynomial in the entries of A. Thus,

$$b_m : \text{Mat}_{n \times n} \to \mathbb{R},$$

is a continuous function (for the Euclidean topologies).

(b) Prove that for every real number r, and every $n \times n$ matrix S, for every integer $m = 0, 1, \ldots, n$, $b_m(r \cdot S)$ equals $r^m b_m(S)$, i.e., b_m is a **homogeneous** polynomial of degree m.

(c) Assume that A is an invertible $n \times n$ matrix, so that $\widetilde{A}$ equals $\text{det}(A) \cdot A^{-1}$. **Find** a nontrivial linear relation between $b_{n-1}(A)$ and $\text{tr}(\widetilde{A})$. Does your relation still hold when A is not invertible? **Hint.** What is the relationship between $\lambda I_n - A$ and $(1/\lambda)I_n - A^{-1}$?

Problem 2. Let d be the usual Euclidean metric on $\mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be the function,

$$x = f(y) = \frac{\sqrt{1+4y^2}-1}{2y}, y \neq 0,\ f(0) = 0.$$

Prove that f is a continuous, bounded function (you may use standard results from single variable calculus). On the other hand, prove that f achieves neither a minimum nor a maximum value. Conclude that in the Extreme Value Theorem, it does not suffice to replace the compactness hypothesis by a completeness hypothesis.

Problem 3. Let $C \subset \mathbb{R}^n$ be a nonempty subset that is closed (with respect to the Euclidean metric d), and let $K \subset \mathbb{R}^n$ be a nonempty subset that is compact (with the subspace topology induced by d). Assume that C and K are disjoint. Prove that there exists $x_0 \in K$ and $y_0 \in C$ such that for every $x \in K$ and for every $y \in C$, $d(x_0, y_0) \leq d(x, y)$. Also find a counterexample if K is assumed to be closed, but not necessarily compact.

Problem 4. Exercise 1, p. 48. Let $A \subset \mathbb{R}^m$ be an open subset, let $\vec{a} \in A$ be an element, and let $f : A \to \mathbb{R}^n$ be a function. Let $\vec{u} \in \mathbb{R}^m$ be a nonzero vector, and let $c \in \mathbb{R}$ be a nonzero scalar. Prove that if the directional derivative, $f'(\vec{a}; \vec{u})$ exists, then also the directional derivative $f'(\vec{a}; c \cdot \vec{u})$ exists, and $f'(\vec{a}; c \cdot \vec{u})$ equals $c \cdot f'(\vec{a}; \vec{u})$. Hence directional derivatives are homogeneous in $\vec{u}$ (but they are not always linear in $\vec{u}$).

Problem 5. Exercise 2, p. 48. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(0, 0) = 0$ and for all $(x, y) \neq (0, 0)$,

$$f(x, y) = \frac{xy}{x^2 + y^2}.$$

(a) For which nonzero $\vec{u} \in \mathbb{R}^2$ does $f'(\vec{0}; \vec{u})$ exist? Find the value when it exists.
(b) Do $D_1 f$ and $D_2 f$ exist at $\vec{0}$?
(c) Is f differentiable at $\vec{0}$?
(d) Is f continuous at $\vec{0}$?

Problem 6. Repeat the previous problem with $f(x, y) = x^2y^2/(x^2 + y^2)$. **Not to be turned in:** For a homogeneous polynomial $P(x, y)$ of degree $d \geq 2$, for which values of the integer d would you guess that $P(x, y)/(x^2 + y^2)$ is differentiable at $\vec{0}$?
Extra Problems. p. 40, Exercise 4; p. 49, Exercises 4, 5, 6.