MAT 322 Problem Set 11

Homework Policy. Please read through all the problems. Please write up solutions of the required problems. Please also read and attempt the extra problems, but please do not write up those solutions for grading. I will be happy to discuss the extra problems during office hours.

Each student is encouraged to work on problem sets with other students, but each submitted problem set must be in the student's own words and based on the student's own understanding. It is against university policy to copy answers from other students or from any other resource (such as a webpage).

Required Problems.

Problem 1.(Problem 4, p. 280) Let $U \subset \mathbb{R}^k$ be an open subset. Let $\eta \in \Omega^k(U)$ be an integrable k-form on U. Let $\mathcal{B} = (\mathbf{x}_1, \dots, \mathbf{x}_k)$ be an ordered basis for $\mathbb{R}^k$, and let $A = [\mathbf{x}_1|\dots|\mathbf{x}_k]$, the associated invertible $k \times k$ matrix. Define η_A to be the function $\eta_A : U \to \mathbb{R}$ whose value at every point $p \in A$ equals $\eta_p(\mathbf{x}_1, \dots, \mathbf{x}_k)$. Find a nontrivial linear relation between $\int_U \eta$, the integral of the k-form η, and $\int_U \eta_A$, the integral of the function η_A (with respect to the standard volume form on $\mathbb{R}^k$ induced by the Euclidean inner product). In particular, if A is a special orthogonal matrix, prove that these two integrals are equal.

Problem 2.(Problem 1, p. 291) Let $M \subset \mathbb{R}^m$ be an embedded m-dimensional manifold of class C^r with $r \geq 1$ (the boundary may be nonempty). For every relatively open subset $V \cap M \subset M$, for every connected, open subset $U \subset \mathbb{R}^m$, resp. for every connected, relatively open subset $U \cap \mathbb{H}_m \subset \mathbb{H}_m$, for every diffeomorphism $\alpha : U \to V \cap M$, define α to be *positive* if $\det(D\alpha)$ is everywhere positive on U. First, prove that for every connected open subset $U' \subset U$, for the restriction $\alpha' = \alpha|U'$, the sign of $\det(D\alpha')$ equals the sign of $\det(D\alpha)$. Second, for every pair of positive coordinate patches $\alpha : U \to V \cap M$, $\alpha'' : U'' \to V \cap M$, for the unique diffeomorphism $\beta : U \to U''$ with $\alpha'' \circ \beta = \alpha$, prove that also $\det(D\beta)$ is everywhere positive. Conclude that this definition of positive coordinate patches of M is a well-defined orientation.

Problem 3.(Problem 8, p. 292, Problem 3, p. 296, and Problem 4, p. 321) Let m be an integer, $m \geq 2$. Denote by $\mathbf{H}^m \subset \mathbb{R}^m$ the upper halfspace $\{(x_1, \dots, x_m) \in \mathbb{R}^m : x_m \geq 0\}$, and denote by $\mathbf{H}^m_o \subset \mathbf{H}^m$ the interior, i.e., $x_m > 0$. Let $\langle \bullet, \bullet \rangle$ be the standard Euclidean inner product on $\mathbb{R}^m$, and let $\| \bullet \|$ be the associated norm, $\|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x} \rangle$. For every positive real r, denote by $\overline{B}^m_r(0)$, resp. $B^m_r(0)$, $\mathbf{S}^{m-1}_r$, the subset of all $\mathbf{x} \in \mathbb{R}^m$ that have norm $\leq r$, resp. $< r$, resp. equal to r. The *upper hemisphere* orientation of $\mathbf{S}^{m-1}_r$ is the unique orientation on $\mathbf{S}^{m-1}_r$ such that the following coordinate patch is positively oriented,

$$\alpha_{r,+} : B_r^{m-1}(0) \to \mathbf{S}_r^{m-1} \cap \mathbf{H}_o^m,\ \alpha_+(\mathbf{y}) = (\mathbf{y}, (r^2 - \|\mathbf{y}\|^2)^{1/2}).$$

Let $0 < q < r$ be real numbers. Let $A_{q,r} \subset \mathbb{R}^m$ be the subset

$$A_{q,r} = \{\mathbf{x} \in \mathbb{R}^m : q^2 \leq \|\mathbf{x}\|^2 \leq r^2\} = \overline{B}_r^m(0) \setminus B_q^m(0).$$

(a) Prove that $A_{q,r}$ is an embedded m-dimensional submanifold of $\mathbb{R}^m$ of class C^∞, and prove that the boundary of M is the disjoint union $\partial M = \mathbf{S}_q^{m-1} \sqcup \mathbf{S}_r^{m-1}$.

(b) Give $M_{q,r}$ the natural orientation from Problem 2. Prove that for the induced orientation on ∂M, for the induced orientation on $\partial M_{q,r}$, the orientation on $\mathbf{S}_r^{m-1}$ equals the upper hemisphere orientation if m is odd, and it is the opposite of the upper hemisphere orientation if m is even. Similarly, the induced orientation on $\mathbf{S}_q^{m-1}$ equals the upper hemisphere orientation if m is even, and it is the opposite of the upper hemisphere orientation if m is odd. In particular, independent of the parity of m, the orientations on the two components $\mathbf{S}_r^{m-1}$ and $\mathbf{S}_q^{m-1}$ are opposite (via the standard diffeomorphism between the two by radial scaling).

(c) For $m = 2$ and for $m = 3$, sketch the manifolds with an indication of the induced orientation on the boundary (by indicating the positively oriented normal to the boundary). Also, without writing up a response, consider what happens when m equals 1.

Problem 4. This problem continues the previous problem. Let $(x_1, \ldots, x_m)$ be the standard ordered basis of coordinates on $\mathbb{R}^m$ with the corresponding standard ordered basis $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ for $\Omega^p(\mathbb{R}^m)$ (allowing smooth functions ω_I as coefficients). Define the following $(m-1)$-form on $\mathbb{R}^m \setminus \{0\}$.

$$\eta_{\mathbf{x}} = \frac{1}{\|\mathbf{x}\|^m} \sum_{\ell=1}^{m} (-1)^{\ell-1} x_\ell dx_1 \wedge \cdots \wedge dx_{\ell-1} \wedge dx_{\ell+1} \wedge \cdots \wedge dx_m.$$

(a) For every real number e, prove that $\eta_e = \|\mathbf{x}\|^e \eta$ is an $(m-1)$-form on $\mathbb{R}^m \setminus \{0\}$ of class C^∞.

(b) Let $\alpha = d(\|\mathbf{x}\|^2) = 2\sum_{\ell=1}^m x_\ell dx_\ell$. Prove that η_e is the unique $(m-1)$-form on $\mathbb{R}^m \setminus \{0\}$ such that for every $1 \leq i < j \leq m$,

$$(x_i dx_j - x_j dx_i) \wedge \eta_e = 0,$$

and also

$$\alpha \wedge \eta_e = 2\|\mathbf{x}\|^{e+2-m} d\mathrm{vol}_{\mathbb{R}^m,m},$$

where, as usual, $d\mathrm{vol}_{\mathbb{R}^m,m}$ denotes $dx_1 \wedge \cdots \wedge dx_m$.

(c) Let A be an orthogonal matrix, and let $F : \mathbb{R}^m \to \mathbb{R}^m$ be $F(\mathbf{x}) = A\mathbf{x}$. Prove that $F^*(\|\mathbf{x}\|^2)$ equals $\|\mathbf{x}\|^2$ as differential 0-forms. Conclude that $F^*(\alpha)$ equals α. Prove that $F^*(d\mathrm{vol}_{\mathbb{R}^m,m})$ equals $\det(A) d\mathrm{vol}_{\mathbb{R}^m,m}$ (in fact this is true for any linear transformation). Since $\det(A)$ equals ± 1, conclude that also $F^*(\eta_e)$ equals $\det(A)\eta_e$. In particular, if A is special orthogonal, conclude that $F^*(\eta_e)$ equals η_e.

(d) Now let c be a positive real number, and let $G : \mathbb{R}^m \to \mathbb{R}^m$ be the linear transformation $G(\mathbf{x}) = c\mathbf{x}$. Prove that $G^*(\eta_e)$ equals $c^e\eta_e$. In particular, precisely for $e = 0$, i.e., for $\eta_e = \eta$, does $G^*(\eta_e)$ equal η_e for every positive real c. Thus, the symmetry group of η contains $\mathbf{SO}_{m-1}(\mathbb{R}) \times \mathbb{R}^*_{>0}$.

(e) Finally, compute that the exterior derivative equals the following,

$$d(\eta_e)_{\mathbf{x}} = e\|\mathbf{x}\|^{e-m} d\mathrm{vol}_{\mathbb{R}^m,m}.$$

In particular, conclude that η is closed.

Problem 5. This problem continues the previous two problems.

(a) Use the method of Problem 6, p. 168 or Problem 5, p. 218 to compute that for $e \neq 0$,

$$\int_{A_{q,r}} d(\eta_e) = m\mathrm{Vol}_{\mathbb{R}^m,m}(\overline{B}_1^m(0))(r^e - q^e).$$

Also, since $d\eta$ equals 0, compute that $\int_{A_{q,r}} d(\eta)$ equals 0.

(b) For every $e \neq 0$, use Stokes's Theorem to conclude that

$$f(r) = m\mathrm{Vol}_{\mathbb{R}^m,m}(\overline{B}_1^m(0))r^e - \int_{\mathbf{S}_r^{m-1}} \eta_e$$

is constant. Next, use Problem 4(d) to conclude that $f(cr)$ equals $c^e f(r)$, so that the constant $f(r)$ equals 0. Conclude that

$$\int_{\mathbf{S}_r^{m-1}} \eta_e = m\mathrm{Vol}_{\mathbb{R}^m,m}(\overline{B}_1^m(0))r^e.$$

Use Problem 5, p. 218, to rewrite this as

$$\int_{\mathbf{S}_r^{m-1}} \eta_e = \mathrm{Vol}_{\mathbb{R}^m,m-1}(\mathbf{S}_1^{m-1})r^e = \mathrm{Vol}_{\mathbb{R}^m,m-1}(\mathbf{S}_r^{m-1})r^{e+1-m}$$

(c) Check that $\eta_e|\mathbf{S}_r^{m-1}(0)$ equals $r^e \cdot \eta|\mathbf{S}_r^{m-1}(0)$. Conclude that

$$\int_{\mathbf{S}_r^{m-1}} \eta = \mathrm{Vol}_{\mathbb{R}^m,m-1}(\mathbf{S}_1^{m-1}) = \mathrm{Vol}_{\mathbb{R}^m,m-1}(\mathbf{S}_r^{m-1})/(r^{m-1}).$$

In fact, η_{m-1} is a volume element on $\mathbf{S}_r^{m-1}$; for every relatively open subset $W \cap \mathbf{S}_r^{m-1} \subset \mathbf{S}_r^{m-1}$, $\mathrm{Vol}_{\mathbb{R}^m,m-1}(W \cap \mathbf{S}_r^{m-1})$ equals $\int_{W\cap\mathbf{S}_r^{m-1}} \eta_{m-1}$.

(d) In particular, since $\int_{\mathbf{S}_r^{m-1}} \eta$ is nonzero, explain why $\eta|\mathbf{S}_r^{m-1}$ is not an exact form. Explain why η is not an exact form on $\mathbb{R}^m \setminus \{0\}$. Thus η is a closed $(m-1)$-form on $\mathbb{R}^m \setminus \{0\}$ that is not exact.

Problem 6.(Problem 6, p. 309) Let $M \subset \mathbb{R}^m$ be an embedded k-dimensional manifold of class C^r, $r \geq 1$, that is compact and that has empty boundary. Let ω, resp. η, be a differential p-form, resp. differential $(k-p-1)$-form, of class C^∞ on M. Prove that there exists a nonzero real number $a = a_{k,p}$ such that for every (M, ω, η) as above,

$$\int_M (d\omega) \wedge \eta = a \int_M \omega \wedge (d\eta),$$

and compute a. Not to be written up: for a p-form ω that is merely continuous but not necessarily differentiable, would it be consistent to formally define $\int_M (d\omega) \wedge \bullet$ as an $\mathbb{R}$-linear functional on $\Omega^{k-p-1}(M)$ to equal $a \int_M \omega \wedge (d\eta)$?