MAT320 Spring 2015 Midterm 2

Name: ______________________________ SB ID number: ______________________

Problem 1: ______ /25 **Problem 2**: ______ /25 **Problem 3**: ______ /25

Problem 4: ______ /25 **Problem 5**: ______ /25

Total: ______ /100

Instructions: The exam is closed book, closed notes, calculators are not allowed, and all cellphones and other electronic devices must be turned off for the duration of the exam. You will have approximately 90 minutes for this exam. You may use either pencil or ink. If you have a question, need extra paper, need to use the restroom, etc., then **please raise your hand**.
You must attempt Problem 1. Of the remaining four problems, you should attempt three. If you choose to do all four, you will get the highest three scores. You may use results from the book, but you must give a clear reference, e.g., "the Cauchy criterion for convergence" or "equivalence of sequential continuity and the $\epsilon - \delta$ definition".

Name: ____ **Problem 1**: __________________ /25

Problem 1. Mandatory Problem.(25 points) Let (S, d) and (S^*, d^*) be metric spaces, and let $f : (S, d) \to (S^*, d^*)$ satisfy the $\epsilon - \delta$ definition of continuity (at all points of S). Do all of the following.

(a) Define the notion of **open subset** of S with respect to d.

(b) Define **compactness** of (S, d).

(c) Without quoting from the book, prove that for every subset U of S^* that is open (with respect to d^*), also the preimage subset $f^{-1}(U) \subset S$ is open (with respect to d).

Name: ______________________________________ **Problem 2**: ________________ /25

Problem 2(25 points) For each of the following series, determine whether the series converges or diverges with justification.
(a) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n!}}$.
(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.
(c) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$.
(d) $\sum_{n=1}^{\infty} \frac{1}{(1+(1/n))^n}$.

Name: ________________________________ **Problem 3**: ______________ /25

Problem 3(25 points) For each of the following subsets of $\mathbb{R}$, with respect to the usual metric, $d(x, y) = |y - x|$, find the interior and the closure. State whether the closure is compact. Justify your answers.

(a) $[0, \infty)$,

(b) $\mathbb{Q} \cap [0, \sqrt{2}]$,

(c) $\cup_{n=2}^{\infty}[1/n^2, 1/(n^2 - 1)]$

Name: ______________________________ **Problem 4**: ____________ /25

Problem 4(25 points) Let (S, d) be a compact metric space, let (S^*, d^*) be a metric space, and let $f : (S, d) \to (S^*, d^*)$ be a continuous function.

(a) If f is onto, prove that (S^*, d^*) is a compact metric space.

(b) Assume that (S^*, d^*) is $\mathbb{R}$ with the usual metric, and assume that S is nonempty. Prove that $f(S)$ has a maximum, and prove that $f(S)$ has a minimum. You may now assume (a).

(c) Now for (S^*, d^*) a general metric space, assume that f is not onto, and let y be an element in $S^* \setminus f(S)$. Prove that $\{d^*(f(x), y) | x \in S\}$ has a minimum that is positive. You may now assume both (a) and (b).

Name: ______________________________ **Problem 5**: ____________ /35

Problem 5(25 points) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence. Prove that if the series $\sum_{n=1}^{\infty} a_n$ converges absolutely, then also for every subsequence $(a_{n_k})_{k\in\mathbb{N}}$, the series $\sum_{k=1}^{\infty} a_{n_k}$ converges absolutely. Also give an example of (a_n) and a subsequence (a_{n_k}) such that the series $\sum_{n=1}^{\infty} a_n$ converges, yet the series $\sum_{k=1}^{\infty} a_{n_k}$ does not converge.