Name: ______________________________ Problem 1: ______________ /25

Problem 1. Mandatory Problem.(25 points) For each of the following sequences $(s_n)_{n\in\mathbb{N}}$, say whether or not the limit exists. If the limit **does** exist, evaluate the limit, and then **PROVE** that the sequence converges to the limit you evaluated (in terms of the definition with ϵ and N). You may either prove this directly, or you may use limit theorems. If you use limit theorems, please explain clearly what limit theorems you use. If the limit does not converge, it is enough to say "does not exist" (you need not prove it).

(a)(5 points) $s_n = \frac{1+(-1)^n n^2}{1+n^2}$.

n even	n odd
$S_n = \frac{1+n^2}{1+n^2} = 1$, so $\lim_{m\to\infty} S_{2m} = 1.$	$S_n = \frac{-n^2+1}{n^2+1} = -1 + \frac{2}{n^2+1}$ so $\lim_{m\to\infty} S_{2m+1} = -1$

Since $(S_n)_{n\in\mathbb{N}}$ has two subsequences converging to different limits - both 1 and -1 - the sequence does not converge.

(b)(10 points) $s_n = \frac{1-n^2}{1+n^2}$.

$$S_n = \frac{1-n^2}{1+n^2} = -1 + \frac{2}{1+n^2}$$

For every real $\varepsilon > 0$, for $N =$ integer $> \sqrt{2/\varepsilon}$ (which exists by the Archimedean property), for every $n \in \mathbb{N}$ with $n \geq N$, also $n^2 \geq N^2 > \frac{2}{\varepsilon}$ (by axioms of ordered fields), so $n^2+1 > \frac{2}{\varepsilon}$. Thus, $|S_n - (-1)| = \frac{2}{1+n^2} < \varepsilon$. So $(S_n)_{n\in\mathbb{N}}$ converges to -1.

(c)(10 points) $s_n = \frac{2^n}{n!}$

For every $n \in \mathbb{N}$, S_n is positive. Moreover, for every $n \geq 2$,

$$\frac{S_{n+1}}{S_n} = \frac{2^{n+1}/(n+1)!}{2^n/n!} = \frac{2}{n+1} \leq \frac{2}{3} < 1.$$

Thus, by ~~the~~ induction on n, $0 \leq |S_n| \leq \left(\frac{2}{3}\right)^{n-2} S_2$, which converges to 0 geometrically.

Thus $\lim_{n\to\infty} S_n$ equals 0.

Name: ______________________________ Problem 2: ____________ /30

Problem 2. Mandatory Problem.(30 points) For each of the following series, determine whether the series converges or diverges with justification.

(a)(5 points) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{2^n+1}}$.

For every n, $\sqrt{2^n+1} > (\sqrt{2})^n$, so that $\frac{1}{\sqrt{2^n+1}} < \left(\frac{1}{\sqrt{2}}\right)^n$. Since $\sum_{n=2}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n$ converges absolutely to $\frac{1}{2}\frac{\sqrt{2}}{\sqrt{2}-1}$, by the Comparison Test, also $\sum_{n=2}^{\infty} \frac{1}{\sqrt{2^n+1}}$ converges absolutely.

(b)(10 points) $\sum_{n=1}^{\infty} \ln(1-(1/n))$. $n=2$ Observe that $\ln\left(1-\frac{1}{n}\right) = \ln\left(\frac{n-1}{n}\right) = \ln(n-1) - \ln(n)$.

Thus, by induction on $N \in \mathbb{N}$ with $N \geq 2$, $\sum_{n=2}^{N} \ln\left(1-\frac{1}{n}\right)$ equals $(\ln(1)-\ln(2)) + (\ln(2)-\ln(3)) + \cdots + (\ln(N-1)-\ln(N)) = -\ln(N)$. Since $\lim_{N\to\infty} -\ln(N) = -\infty$, the series diverges.

(c)(5 points) $\sum_{n=1}^{\infty} \left(\frac{1}{2n} - \frac{1}{2n+1}\right)$. Observe that $\frac{1}{2n} - \frac{1}{2n+1} = \frac{1}{4n^2+2n}$.

Since $\sum_{n=1}^{\infty} \frac{1}{4n^2}$ converges absolutely (to $\frac{\pi^2}{24}$, using Fourier analysis) by the Power Test, and since $0 \leq \frac{1}{4n^2+2n} \leq \frac{1}{4n^2}$, also $\sum_{n=1}^{\infty} \left(\frac{1}{2n} - \frac{1}{2n+1}\right)$ converges absolutely, by the Comparison Test.

(d)(10 points) $\sum_{n=1}^{\infty} (n^2 - \sqrt{n^4+1})$. Observe that $n^2 - \sqrt{n^4+1} = \frac{-1}{n^2+\sqrt{n^4+1}}$

$= -\frac{1}{2n^2\left(\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{1}{n^4}}\right)}$. Thus $|n^2 - \sqrt{n^4+1}| = \frac{1}{2n^2} \cdot \frac{1}{\left(\frac{1}{2}+\frac{1}{2}\sqrt{1+n^{-4}}\right)} < \frac{1}{2n^2}$.

As in (c), by the Power Test and by the Comparison Test, $\sum_{n=1}^{\infty} (n^2 - \sqrt{n^4+1})$ converges absolutely.

Name: ______________________________ **Problem 3:** ____________ /30

Problem 3. Mandatory Problem(30 points) Let $b > 1$ be a real number. Let $r > 1$ be a real number and let $f(x)$ be the decreasing function x^{-r} on $[1, b]$. For every integer $n \in \mathbb{N}$, define $q_n = \sqrt[n]{b}$. Let P_n be the partition of $[1, b]$ given by

$$1 = x_0 < x_1 < \cdots < x_k < \cdots < x_{n-1} < x_n = b, \quad x_k = q_n^k.$$

Define I_k to be the subinterval $[x_{k-1}, x_k] = [q_n^{k-1}, q_n^k]$ of width $\Delta x_k = x_k - x_{k-1}$.

(a)(3 points) Find each length Δx_k.

$$\Delta x_k = x_k - x_{k-1}$$
$$= q_n^k - q_n^{k-1}$$
$$= \boxed{(q_n - 1)\, q_n^{k-1}}.$$

(b)(7 points) Find $\sup_{x \in I_k} f(x)$ and $\inf_{x \in I_k} f(x)$.

Since f is decreasing, $\sup_{x \in [x_{k-1}, x_k]} f(x)$ equals $f(x_{k-1}) = \underline{q_n^{-r(k-1)}}$.

Similarly, $\inf_{x \in [x_{k-1}, x_k]} f(x)$ equals $f(x_k) = \underline{q_n^{-r} \cdot q_n^{-r(k-1)}}$.

Name: ______________________________ **Problem 3 continued.**

(c)(10 points) Find the upper Darboux sum $U(f, P_n)$ and the lower Darboux sum $L(f, P_n)$. Please use a summation identity to reduce your answer to a fraction whose numerator and denominator are polynomials in q_n.

By definition, $U(f, P_n)$ equals $\sum_{k=1}^{n} \sup_{x\in I_k} f(x) \cdot \Delta x_k = \sum_{k=1}^{n} q_n^{-r(k-1)} \cdot (q_n-1) q_n^{k-1}$

$= \sum_{k=1}^{n} (q_n-1)\, q_n^{(-r+1)(k-1)} = (q_n-1) \cdot \sum_{k=1}^{n} \left(q_n^{(-r+1)}\right)^{k-1}$. By the formula for a geometric series, namely $\sum_{k=1}^{n} a^{k-1} = \frac{1-a^n}{1-a}$, $a \neq 1$, the sum is

$U(f,P_n) = (q_n-1) \cdot \frac{1-(q_n^n)^{-r+1}}{1-q_n^{-r+1}} = \left(\frac{q_n-1}{q_n^{-r+1}-1}\right) \cdot (b^{-r+1}-1)$, since q_n^n equals b. Similarly,

$L(f,P_n) = q_n^{-r}\left(\frac{q_n-1}{q_n^{-r+1}-1}\right)(b^{-r+1}-1)$.

(d)(10 points) Prove that both of the following limits exist and are equal,

$$\lim_{n\to\infty} U(f, P_n), \quad \lim_{n\to\infty} L(f, P_n).$$

Conclude that $f(x)$ is Darboux integrable on $[1, b]$.

When r is an integer, $\frac{q_n-1}{q_n^{-r+1}-1} = \frac{-1}{q_n^{-r+1}} \cdot \frac{1}{1+q_n+q_n^2+\cdots+q_n^{r-2}}$, so that

$\lim_{n\to\infty} \frac{q_n-1}{q_n^{-r+1}-1} = \frac{-1}{1} \cdot \frac{1}{\underbrace{1+1+1+\cdots+1}_{r-1 \text{ times}}} = \frac{-1}{r-1}$. Similarly, $\lim_{n\to\infty} q_n^{-r}\left(\frac{q_n-1}{q_n^{-r+1}-1}\right) =$

$1^{-r} \cdot \left(\frac{-1}{r-1}\right) = \frac{-1}{r-1}$. Thus, $\lim_{n\to\infty} U(f,P_n) = \frac{b^{-r+1}-1}{-r+1} = \lim_{n\to\infty} L(f,P_n)$.

Even when r is not an integer, by L'Hôpital's rule,

$\lim_{q\to 1} \frac{q-1}{q^{-r+1}-1} = \lim_{q\to 1} \frac{1}{(-r+1)q^{-r+2}} = \frac{1}{-r+1}$. Thus, again,

$\lim_{n\to\infty} U(f,P_n) = \frac{b^{-r+1}-1}{-r+1} = \lim_{n\to\infty} L(f,P_n)$.

Name: ______________________________ Problem 4: ____________ /30

Problem 4. Mandatory Problem.(30 points) Let $f(x)$ be the function

$$f(x) = \begin{cases} x^2\sin(1/x^2), & x \neq 0, \\ 0, & x = 0 \end{cases}$$

(a)(5 points) Using derivative rules and the known derivative $d\sin(x)/dx = \cos(x)$, prove that for every $a \neq 0$, $f(x)$ is differentiable at $x = a$, and compute the derivative. Please show all steps.

For $a \neq 0$, $f'(a) = 2a \cdot \sin\left(\frac{1}{a^2}\right) + a^2 \cdot \cos\left(\frac{1}{a^2}\right) \cdot \left(\frac{-2}{a^3}\right) = \underline{2a\sin\left(\frac{1}{a^2}\right) - \frac{2}{a}\cos\left(\frac{1}{a^2}\right)}$.

(b)(10 points) Prove that $f(x)$ is differentiable at $x = 0$, and find the derivative. Justify your answer.

By definition, f is differentiable at 0 if $\lim_{x\to 0} \frac{f(x)-f(0)}{x}$ exists.
This is $\lim_{x\to 0} \frac{x^2\sin\left(\frac{1}{x^2}\right)}{x} = \lim_{x\to 0} x\sin\left(\frac{1}{x^2}\right)$. Since $0 \le \left|\sin\left(\frac{1}{x^2}\right)\right| \le 1$, also $0 \le \left|x\sin\left(\frac{1}{x^2}\right)\right| \le |x|$. Thus, since $\lim_{x\to 0}|x|$ equals 0, also $\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}$ exists and $\underline{f'(0)=0}$.

(c)(10 points) Consider the function $g(x) = x + f(x)$. Prove that $g'(0)$ exists and is positive, yet there exists no $\delta > 0$ such that $g(x)$ is increasing on $(-\delta, +\delta)$.

Since "$g'(x) = 1 + f'(x)$," in particular g is differentiable at 0 and $g'(0)$ equals 1, which is positive. On the other hand, for every real $\delta > 0$, there exists a with $0 < a < \delta$ and
$1 + 2a\sin\left(\frac{1}{a^2}\right) - \frac{2}{a}\cos\left(\frac{1}{a^2}\right) \approx -\frac{2}{a}\cos\left(\frac{1}{a^2}\right)$ is negative. Since increasing g has $g' \geq 0$ everywhere, g is not increasing.

(d)(5 points) Why does the answer to (c) not contradict the Increasing Function Theorem (corollary of the Mean Value Theorem) regarding increasing functions and positive derivatives?

The hypothesis of the Increasing Function Theorem is that g be continuous on $[-\delta, +\delta]$ (which is valid) and that $g'(a) \geq 0$ for every $a \in (-\delta, \delta)$. Although $g'(0) > 0$, there exist $a \in (-\delta, \delta)$ with $g'(a) < 0$. So the hypothesis is violated.

Name: ______________________ **Problem 5:** __________ /60

Problem 5. Additional Problems.(60 points) Solve (at least) two of the following. You will receive the highest two scores among the scores you earn on any of the following. For each solution, make clear which part you are solving.

(a)(30 points) Let $(s_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers that converges to a positive limit s. Prove directly (without citing general theorems) that also $(\sqrt{s_n})_{n\in\mathbb{N}}$ converges to $\sqrt{s}$.

(b)(30 points) Let (S, d) be a compact metric space, let (S', d') be a metric space, and let $f : (S, d) \to (S', d')$ be a function that is continuous. You may assume the "Extreme Value Theorem": the continuous image of every compact metric space is a compact metric space. Prove that whenever f is bijective, the inverse function $f^{-1} : (S', d') \to (S, d)$ is also continuous.

(c)(30 points) Let $f : [0, 1] \to [a, b]$ be a continuous, increasing function. Prove that for every $x \in (0, 1)$, there exists a unique $y \in [0, x]$ such that

$$f(y) = \frac{1}{x}\int_{t=0}^{x} f(t)dt.$$

(d)(30 points) Let $f : [0, 1] \to [-R, R]$ be an integrable function such that for every $0 < a < b < 1$, there exists $x_- \in (a, b)$ such that $f(x_-) \leq 0$ and there exists $x_+ \in (a, b)$ such that $f(x_+) \geq 0$. Prove that for every $0 < a < b < 1$, $\int_{x=a}^{b} f(x)dx$ equals 0.

(e)(30 points) On $[0, 1]$, let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous, real-valued functions that converge uniformly to f. State the definition of "uniform convergence", and then prove that for every convergent sequence $(x_n)_{n\in\mathbb{N}} \to x$ in $[0, 1]$, also $(f_n(x_n))_{n\in\mathbb{N}}$ converges to $f(x)$.

(a) Since $(s_n) \to s$, for every real $\varepsilon > 0$, since also $\varepsilon\sqrt{s} > 0$, there exists $N \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ with $n \geq N$, $|s_n - s| < \varepsilon\sqrt{s}$. Thus, also $\frac{|s_n - s|}{\sqrt{s}} < \varepsilon$. Thus $0 \leq \frac{|s_n - s|}{\sqrt{s_n} + \sqrt{s}} \leq \frac{|s_n - s|}{\sqrt{s}} < \varepsilon$. Finally, $\frac{s_n - s}{\sqrt{s_n} + \sqrt{s}}$ equals $\sqrt{s_n} - \sqrt{s}$. Thus, for every $n \in \mathbb{N}$ with $n \geq N$, $|\sqrt{s_n} - \sqrt{s}| < \varepsilon$.

(b) Proof 1. Let $\mathcal{U} \subset S$ be any open subset. Then $(f^{-1})^{\text{preimage}}(\mathcal{U})$ equals $f(\mathcal{U})$. Since f is bijective, $S^* \setminus f(\mathcal{U})$ equals $f(S \setminus \mathcal{U})$. Thus, to prove $(f^{-1})^{\text{preimage}}(\mathcal{U})$ is open in S^*, it suffices to prove that $f(S \setminus \mathcal{U})$ is closed in S^*. Since $S \setminus \mathcal{U}$ is closed in S, and since S is compact, $S \setminus \mathcal{U}$ is compact. Thus $f(S \setminus \mathcal{U})$ is compact. Thus $f(S \setminus \mathcal{U})$ is closed in S^*.

(b) Proof 2. Let $(s_n^*)_{n\in\mathbb{N}} \to s^*$ be a convergent ~~sub~~sequence in S^*. By way of contradiction, assume that $(f^{-1}(s_n^*))_{n\in\mathbb{N}}$ does not converge to $f^{-1}(s^*)$. Then $\exists$ real $\varepsilon>0$ and $(n_k)_{k\in\mathbb{N}}$ infinite such that $\forall k\in\mathbb{N}$, $d(f^{-1}(s_{n_k}^*), f^{-1}(s^*))>\varepsilon$. Since S is compact, the sequence $(f^{-1}(s_{n_k}^*))_{k\in\mathbb{N}}$ has a subsequence $(f^{-1}(s_{n_{k_\ell}}^*))_{\ell\in\mathbb{N}}$ converging to some

Extra Page.

element $s\in S$. And $d(s, f^{-1}(s^*))\geq\varepsilon$, so $s\neq f^{-1}(s^*)$. Since f is continuous, $(s_{n_{k_\ell}}^*)_{\ell\in\mathbb{N}}$ converges to $f(s)$. As a subsubsequence of (s_n^*), also $(s_{n_{k_\ell}}^*)$ converges to s^*. Hence $f(s)$ equals s^*, contradicting that $s\neq f^{-1}(s^*)$.

(c) On $[0,x]$, $f(0)\leq f(t)\leq f(x)$, since f is increasing. Hence $\int_{t=0}^{x} f(0)\,dt \leq \int_{t=0}^{x} f(t)\,dt \leq \int_{t=0}^{x} f(x)\,dx$, i.e.

$$f(0)\cdot x \leq \int_{t=0}^{x} f(t)\,dt \leq f(x)\cdot x$$

Dividing by $x>0$, also

$$f(0) \leq \frac{1}{x}\int_{t=0}^{x} f(t)\,dt \leq f(x).$$

By the Intermediate Value Theorem, since ~~f~~ f is continuous, there exists $y\in(0,x)$ such that $f(y)=\frac{1}{x}\int_{t=0}^{x} f(t)\,dt$. If also $z\in(0,x)$ satisfies $f(z)=\frac{1}{x}\int_{t=0}^{x} f(t)\,dt$, since f is increasing, also $z=y$. Hence y is unique.

(d). Let P be any partition of $[0,1]$. For every P-interval $I=(a,b)$, by hypothesis there exists $x_{I,+}^*\in I$ such that $f(x_{I,+}^*)\geq 0$, and there exists $x_{I,-}^*\in I$ such that $f(x_{I,-}^*)\leq 0$. This gives Riemann sums $R(f,P,\{x_{I,+}^*\})=\sum_I f(x_{I,+}^*)\cdot \text{length}(I)\geq 0$ and also

$$R(f,P,\{x_{I,-}^*\})=\sum_I f(x_{I,-}^*)\cdot\text{length}(I)\leq 0.$$

Thus,

$$\int_{x=0}^{1} f(x)\,dx = \lim_{\text{mesh}\to 0} R(f,P,\{x_{I,+}^*\})\geq 0$$

and also

$$\int_{x=0}^{1} f(x)\,dx = \lim_{\text{mesh}\to 0} R(f,P,\{x_{I,-}^*\})\leq 0.$$

So $\int_{x=0}^{1} f(x)\,dx$ equals 0.

(e) Since the functions f_n are continuous, and since $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f, also f is continuous (by the "$\frac{\varepsilon}{3}$ proof").
Thus, for every real $\varepsilon>0$, there exists real $\delta>0$ such that for every $x'\in[0,1]$ with $|x'-x|<\delta$, also $|f(x')-f(x)|<\frac{\varepsilon}{2}$.
Since $(x_n)_{n\in\mathbb{N}}\to x$, there exists $L\in\mathbb{N}$ such that for every $n\in\mathbb{N}$ with $n\geq L$, $|x_n-x|<\delta$, so that $|f(x_n)-f(x)|<\frac{\varepsilon}{2}$.
Since $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f, there exists $M\in\mathbb{N}$ such that for every $n\in\mathbb{N}$ with $n\geq M$, for every $x'\in[0,1]$, $|f_n(x')-f(x')|<\frac{\varepsilon}{2}$.
Thus, setting $N=\max\{L,M\}$, for every $n\in\mathbb{N}$ with $n\geq N$,
$|f_n(x_n)-f(x)|\leq|f_n(x_n)-f(x_n)|+|f(x_n)-f(x)|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$.
Therefore $(f_n(x_n))_{n\in\mathbb{N}}$ converges to $f(x)$.