MAT 320 Foundations of Analysis **Jason Starr**
Final Exam **Spring 2015**
Wednesday 5/13 5:30pm – 8:00pm

MAT 320 Review Sheet for Final Exam

Remark. The Final Exam will be **cumulative**, although there will be an emphasis on material covered since Midterm 2. Please review Midterm 1 and Midterm 2. If you are comfortable with the material from Midterms 1 and 2, as well as the following, then you will be well prepared for the final exam.

Exam Policies. You must show up on time for all exams. Please bring your student ID card: ID cards may be checked, and students may be asked to sign a picture sheet when turning in exams. Other policies for exams will be announced / repeated at the beginning of the exam.

If you have a university-approved reason for taking an exam at a time different than the scheduled exam (because of a religious observance, a student-athlete event, etc.), please contact your instructor as soon as possible. Similarly, if you have a documented medical emergency which prevents you from showing up for an exam, again contact your instructor as soon as possible.

All exams are closed notes and closed book. Once the exam has begun, having notes or books on the desk or in view will be considered cheating and will be referred to the Academic Judiciary.

It is not permitted to use cell phones, calculators, laptops, MP3 players, Blackberries or other such electronic devices at any time during exams. If you use a hearing aid or other such device, you should make your instructor aware of this before the exam begins. You must turn off your cell phone, etc., prior to the beginning of the exam. If you need to leave the exam room for any reason before the end of the exam, it is still not permitted to use such devices. Once the exam has begun, use of such devices or having such devices in view will be considered cheating and will be referred to the Academic Judiciary. Similarly, once the exam has begun any communication with a person other than the instructor or proctor will be considered cheating and will be referred to the Academic Judiciary.

Review Topics.
Definitions. Please know all of the following definitions. **Connected Metric Space. Path Connected Metric Space. Uniform Metric. Uniform Continuity. Uniformly Cauchy Sequence of Functions. Uniform Convergence / Uniform Limit of Sequence of Functions. Limit of a Function at a Accumulation / Limit Point of the Domain. Differentiability. Derivative. Partition and Tagged (or Marked) Partition. Refinement of Partitions. Upper / Lower Darboux Sum. Darboux Integral. Riemann Sum. Mesh. Riemann Integral. Power Series. Taylor Polynomials and Taylor Series.**

Results. Please know all of the following lemmas, propositions, theorems and corollaries.

Connectedness of the Unit Interval. The unit interval with the Euclidean metric is a connected metric space.

Connectedness and Path Connectedness. Every path connected metric space is connected.

Uniform Continuity on Compact Domains. For every compact metric space (X, d_X), for every metric space (Y, d_Y), every continuous function f from (X, d_X) to (Y, d_Y) is uniformly continuous.

Completeness of $C(X)$ with Uniform Metric. For every metric space (X, d_X), for every complete metric space (Y, d_Y), the set $B((X, d_X), (Y, d_Y))$ of bounded functions from (X, d_X) to (Y, d_Y) endowed with the uniform metric is a complete metric space. The subset $C((X, d_X), (Y, d_Y))$ of bounded continuous functions is a closed subset. The subset $UC((X, d_X), (Y, d_Y))$ of uniformly continuous functions is also a closed subset.

Basic Properties of Differentiability. Differentiability is preserved by: scalar multiples, sums, pointwise products, pointwise quotients (where defined), composition. Moreover, $(cf)'(a) = c \cdot f'(a)$, $(f+g)'(a) = f'(a) + g'(a)$, $(fg)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a)$, $(f/g)'(a) = [g(a) \cdot f'(a) - f(a) \cdot g'(a)]/(g(a))^2$, $(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$.

Rolle's Theorem. A continuous (real-valued) function f on $[a, b]$ that is differentiable on (a, b) and has $f(a) = f(b)$ has a critical point in (a, b).

Mean Value Theorem. A continuous (real-valued) function f on $[a, b]$ that is differentiable on (a, b) has $f'(c) = [f(b) - f(a)]/(b - a)$ for some $c \in (a, b)$.

Increasing and Decreasing Functions. A continuous (real-valued) function f on $[a, b]$ that is differentiable on (a, b) is nondecreasing, resp. nonincreasing, if and only if $f'(c) \geq 0$, resp. $f'(c) \leq 0$, for every $c \in (a, b)$. If for every $c \in (a, b)$, $f'(c)$ is positive, resp. negative, then f is strictly increasing, resp. strictly decreasing.

Derivative of an Inverse Function. For a strictly increasing, surjective function $f : [a, b] \to [\widetilde{a}, \widetilde{b}]$, for $c \in (a, b)$ such that $f'(c)$ is defined and nonzero, then the inverse function f^{-1} is differentiable at $f(c)$ with $(f^{-1})'(f(c)) = 1/f'(c)$.

Darboux Sums of Refinements. For a bounded (real-valued) function f on $[a, b]$, for a partition P of $[a, b]$, for a refinement Q of P, the upper and lower Darboux sums satisfy

$$\inf(f) \cdot (b - a) \leq L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P) \leq \sup(f) \cdot (b - a).$$

Computation of Darboux Integral. For a bounded (real-valued) function f on $[a, b]$, f is (Darboux) integrable if and only if there exists a sequence of partitions $(P_n)_{n \in \mathbb{N}}$ such that $\lim_n U(f, P_n) - L(f, P_n) = 0$, in which case the (Darboux) integral equals $\lim_n L(f, P_n) = \lim_n U(f, P_n)$.

Riemann Integrals and Darboux Integrals. For a bounded (real-valued) function f on $[a, b]$, f is Darboux integrable if and only if it is Riemann integrable, in which case the Darboux integral equals the Riemann integral.

Properties of Integrals. For integrable functions f and g on $[a,b]$, every scalar multiple $c\cdot f$ is integrable with $\int_a^b c\cdot f(x)dx = c\cdot\int_a^b f(x)dx$, the sum $f+g$ is integrable with $\int_a^b (f(x)+g(x))dx = (\int_a^b f(x)dx)+(\int_a^b g(x)dx)$. If $f\leq g$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$. The function $|f|$ is integrable with $\int_a^b f(x)dx \leq \int_a^b |f(x)|dx$. In particular, the Euclidean distance between $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ is bounded by $d_{\text{unif}}(f,g)\cdot(b-a)$.

Integrability of Monotone and Piecewise Continuous Functions. Every bounded monontone function on $[a,b]$ is integrable. Every bounded, piecewise continuous function on $[a,b]$ is integrable.

Fundamental Theorem of Calculus. For a continuous function F on $[a,b]$ that is differentiable on (a,b) and whose derivative f is integrable, $\int_a^b f(x)dx$ equals $F(b)-F(a)$. For every continuous function f on $[a,b]$, the function $F(x)=\int_a^x f(t)dt$ is differentiable on (a,b) with derivative f.

Integration by Parts. For continuous function f, g on $[a,b]$ that are differentiable on (a,b) and whose derivatives are integrable,

$$\int_a^b f'(x)g(x)dx + \int_a^b f(x)g'(x)dx = f(b)g(g) - f(a)g(a).$$

Change of Variables Formula. For a strictly increasing, surjective function $u:[a,b]\to[\widetilde{a},\widetilde{b}]$ such that u is differentiable on (a,b), for every continuous, bounded function f on $[\widetilde{a},\widetilde{b}]$,

$$\int_a^b f(u(x))\cdot u'(x)dx = \int_{\widetilde{a}}^{\widetilde{b}} f(t)dt.$$

Re-expansion of Analytic Functions. For a power series $\sum_m a_m(x-x_0)^m$ with radius of convergence $R>0$, for every $0<R_1<R$, the partial sums $\sum_{m=1}^n a_m(x-a_0)^m$ converge uniformly to a uniformly continuous function $F(x)$ on $[x_0-R_1, x_0+R_1]$. For every $x_1\in(x_0-R,x_0+R)$, there is a power series $\sum_m b_m(x-x_1)^m$ with radius of convergence $R'\geq R-|x_0-x_1|>0$ that agrees with $f(x)$ on the common domain.

Infinite Differentiability of Analytic Functions. For a power series $F(x)=\sum_m a_m(x-x_0)^m$ with radius of convergence $R>0$, the power series $f(x)=\sum_m(m+1)a_{m+1}(x-x_0)^m$ has radius of convergence $R>0$ and $\int_{x_0}^x f(t)dt$ equals $F(x)-F(0)$. Thus F is differentiable on (x_0-R,x_0+R). It follows that F is infinitely differentiable on (x_0-R,x_0+R). There exists $0<R_1<R$ such that $(|F^{(n)}(x)|/n!)_{n=0,1,2,\ldots}$ is simultaneously uniformly bounded on (x_0-R_1,x_0+R_1). Conversely, every infinitely differentiable function $F(x)$ such that $(|F^{(n)}(x)|/n!)_n$ is simulteneously, uniformly bounded on (x_0-R,x_0+R) for some $R>0$ is analytic on some interval (x_0-R_1,x_0+R_1) with $0<R_1<R$.

Taylor's Theorem. For an n-times differentiable function f on (a,b), for $x_0\in(a,b)$, for the Taylor polynomial $T_n(f,x_0,x)=\sum_{m=0}^{n-1} f^{(m)}(x_0)/m!(x-x_0)^m$ and with remainder $R_n(f,x_0,x)=$

$f(x) - T_n(f, x_0, x)$, for every $x \in (a, b)$ there exists y between x_0 and x (inclusive) such that $R_n(f, x_0, x)$ equals $f^{(n)}(y)/n!(x - x_0)^n$. Also,

$$R_n(f, x_0, x) = \int_{x_0}^{x} \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t)dt.$$

Thus there exists y between x_0 and x (inclusive) such that $R_n(f, x_0, x)$ equals $(x - y)^{n-1}/(n - 1)!f^{(n)}(y)(x - x_0)$.

Please review all of the homework exercises. In addition the following theoretical problems are good practice for the new material.

Practice Problems.

(1) For a metric space (X, d_X), for a complete metric space (Y, d_Y), for a subset $A \subset X$, and for a uniformly continuous function $f : (A, d_X|_A) \to (Y, d_Y)$, prove that there exists a unique continuous function $\widetilde{f} : (\overline{A}, d_X|_{\overline{A}}) \to (Y, d_Y)$ whose restriction to $A \subset \overline{A}$ equals f. Moreover, prove that $\widetilde{f}$ is uniformly continuous.

(2) Inside $(C([a, b]), d_{\text{unif}})$, let A be the subset of continuous piecewise linear functions, i.e., the set of functions g such that there exists a partition P of $[a, b]$ such that the restriction of g to every P-interval is linear. Prove that A is dense, i.e., $\overline{A}$ equals the entire metric space. **Hint.** For a continuous f in $C([a, b])$, for every $\epsilon > 0$, since f is uniformly continuous, there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$. Let P be a partition with mesh $< \delta$, and let f_P be the piecewise function that agrees with f on the endpoints of P-intervals. Prove that $d_{\text{unif}}(f, f_P) < 3\epsilon$.

(3) With A as above, define $I : (A, d_{\text{unif}}) \to (\mathbb{R}, d_{\text{Eucl}})$ to be the integral of the piecewise linear function g by the "usual" formula, i.e., if g is linear on the P-intervals of a partition P, then

$$I(g) = \sum_{[x,y]\in P} \frac{(y-x)(f(y)+f(x))}{2}.$$

Prove that this is compatible with refinement of partitions. Use this to prove that for f and g in A, $|I(g) - I(f)| \leq d_{\text{unif}}(f, g)(b - a)$. Conclude that I is uniformly continuous, and hence extends to a uniformly continuous function

$$I : (C([a, b]), d_{\text{unif}}) \to (\mathbb{R}, d_{\text{Eucl}}).$$

Conclude that this extension agrees with the usual Darboux and Riemann integrals.

(4) Define $f(x)$ on $\mathbb{R}$ by $f(x) = e^{-1/x^2}$ for $x \neq 0$ and $f(0) = 0$. Prove that f is infinitely differentiable and all derivatives of f vanish at $x = 0$. Conclude that for every $R > 0$, the collection $(|f^{(n)}(x)|/n!)_{n=0,1,2,\ldots}$ is not uniformly bounded $[-R, R]$.

(5) Let $b > 1$ be a real number. For every $n \in \mathbb{N}$, define $q_n = \sqrt[n]{b}$, and define P_n to be the partition of $[1, b]$ with $x_k = q_n^k$. For $r \geq 0$, for $f(x) = x^r$, compute $L(f, P_n)$ and $U(f, P_n)$. Prove that $\lim_n L(f, P_n) = \lim_n U(f, P_n) = (b^{r+1} - 1)/(r + 1)$.

(6) Find an example of strictly increasing, differentiable function on $[-1,1]$ such that the derivative is not always positive. Can you find such an example where the derivative is 0 for infinitely many $x \in (-1,1)$? Can you find such an example where the derivative is 0 for uncountably many x?

(7) Define $\ell(x)$ on $(0,\infty)$ by $\ell(x) = \int_1^x (1/t)dt$. For $1 < a$ and $1 < b$, use the change of variables formula to prove that $\int_a^{ab}(1/t)dt = \int_1^b (1/t)dt$. Conclude that $\ell(ab)$ equals $\ell(a) + \ell(b)$. Similarly, prove that $\ell(1) = 0$, $\ell(1/a) = -l(a)$, and $\ell(a^r) = r\ell(a)$. For every $a > 0$ with $a \neq 1$, conclude that $\log_a(b) = \ell(b)/\ell(a)$. In particular, for the unique real number $e > 0$ with $\ell(e) = 1$, prove that $\ell(b) = \log_e(b)$ (assuming we have already made sense of exponentiation and logarithms for arbitrary real numbers).

(8) Combine the previous problem with the derivative of the inverse function to conclude that there exists a strictly increasing function $E(x)$ such that $E'(x) = E(x)$, $E(0) = 1$, $E(x+y) = E(x)\cdot E(y)$, $E(-x) = 1/E(x)$, and $E(rx) = E(x)^r$. Finally, assuming we have already made sense of exponentiation for real numbers, prove that $E(x)$ equals e^x with e as above.

(9) Repeate the previous two exercises with the function

$$T^{-1}(x) := \int_0^x \frac{1}{1+t^2}dt$$

to make sense of the arctangent function T^{-1} and thus the tangent function $T(x)$. Use the identities $S(x) = T(x/2)/(1+(T(x/2))^2)$ and $C(x) = (1-(T(x/2))^2)/(1+(T(x/2))^2)$ to make sense of the sine and cosine functions. Which trigonometric identities can you prove directly from these definitions?

(10) Give an example of a bounded function $f(x)$ that is not integrable, yet such that $|f(x)|$ is integrable.

(11) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of bounded, integrable functions on $[a,b]$, and let f be a bounded, integrable function on $[a,b]$. Assume that for every $x \in [a,b]$, $(f_n(x))_{n\in\mathbb{N}}$ is a nondecreasing sequence of real numbers that converges to $f(x)$. Prove that $(\int_a^b f_n(x)dx)_{n\in\mathbb{N}}$ is a nondecreasing sequence of real numbers that converges to $\int_a^b f(x)dx$.

(12) Find a sequence $(f_n)_{n\in\mathbb{N}}$ of bounded, integrable functions on $[a,b]$ that converges pointwise to a bounded, integrable function f on $[a,b]$ yet such that $(\int_a^b f_n(x)dx)_{n\in\mathbb{N}}$ does not converge to $\int_a^b f(x)dx$. (This is a challenging exercise, more challenging than would be asked on the final exam.)

(13) Let F be a bounded, continuous function on $[a,b]$ such that for some partition P, the restriction of F to every P-interval is differentiable on the interior of the interval and the derivative f is uniformly continuous. Apply the usual integration by parts to each P-interval to prove that for every continuously differentiable function u on $[a,b]$,

$$\int_a^b f(x)u(x)dx + \int_a^b F(x)u'(x)dx = F(b)u(b) - F(a)u(a).$$

Thus, as far as integration by parts is concerned, it is as if F is everywhere differentiable with derivative f, i.e., f is a "weak derivative" of F.

(13) Let (F,f) be a pair as in the previous exercise, and let (G,g) be another such pair. Prove that (cF,cf), $(F+G,f+g)$ and $(FG,fG+Fg)$ are also such pairs.

(14) Let $(F_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions on $[a,b]$ that converges uniformly to F. Assume that every F_n is differentiable on (a,b) and the sequence $(F_n')_{n\in\mathbb{N}}$ is a sequence of continuous functions that converges uniformly to a continuous function f. Prove that F is differentiable and the derivative equals f.

(15) Find a uniformly convergent sequence $(F_n)_{n\in\mathbb{N}}$ of continuous functions that are differentiable and such that $(F_n')_{n\in\mathbb{N}}$ are continuous functions that converges pointwise to a function f, yet such that the limit F is not everywhere differentiable. Is your (F,f) a pair as in (13)? (As with (12), this is a challenging problem.)

(16) For every subset A of $[a,b]$, define

$$\mu(A) = \inf\{\int_a^b f(x)dx | f \text{ integrable }, f \geq 0, f|_A \geq 1\}.$$

Prove that for every finite union of intervals, $\mu(A)$ is the sum of the lengths of the intervals. Prove that $\mu([a,b]\cap\mathbb{Q})$ equals $\mu([a,b])$.