SOLUTIONS TO PRACTICE MIDTERM

# Practice Midterm 2

MAT 118

March 30, 2009

| Name: (please print) | ID #: |
| --- | --- |

**Directions:** There are 5 problems on 6 pages (including this one) in this exam. Please make sure you have all the pages.

Do all of your work in this exam booklet, and cross out any work that should be ignored. **Show your reasoning and computations — not just the answer.**

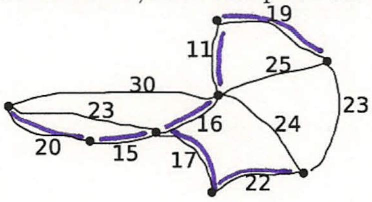
You will need a scientific calculator. You can also use a single letter size sheet of paper with the most important formuals you prepared at home. You can not use any other materials, including books.

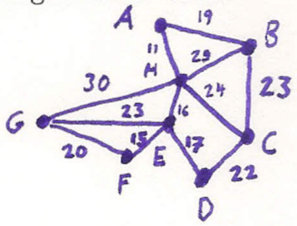
You have 1 hr 20 minutes.

Good luck!

**DO NOT OPEN THE EXAM UNTIL INSTRUCTED BY THE PROCTOR!**

**1.** The figure below shows a map with several cities and roads between them; the numbers shown are distances in miles. You need to connect these cities using fiber optics cable (for high speed internet connection); to save money, the cable must follow existing roads. What is the most economical way of doing this? Mark the roads along which you will bury the cable in bold, and compute the total length of the cable.



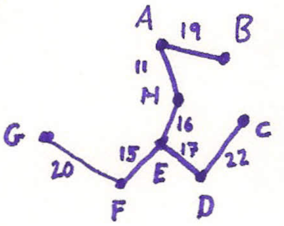




Edges by Cost (increasing)

| | Edge | Cost |
|---|---|---|
| | A-H | 11 |
| | E-F | 15 |
| | E-H | 16 |
| | D-E | 17 |
| | A-B | 19 |
| | F-G | 20 |
| | C-D | 22 |
| Tie | B-C | 23 |
| | E-G | 23 |
| | C-H | 24 |
| | B-H | 25 |
| | G-H | 30 |

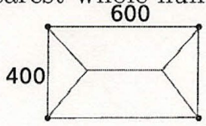
Minimal Spanning Tree

| | |
|---|---|
| A-H | 11 |
| E-F | 15 |
| E-H | 16 |
| D-E | 17 |
| A-B | 19 |
| F-G | 20 |
| C-D | 22 |
| | 120 |

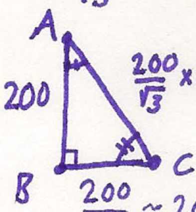



$\boxed{\text{Total length: 120}}$

**2.** The figure below shows four points in the plane forming a rectangle with sides 400 km and 600 km. Compute the length of the shortest network that can be used to connect these points, shown in red (the marked points are the Steiner points). Round the answer to the nearest whole number.

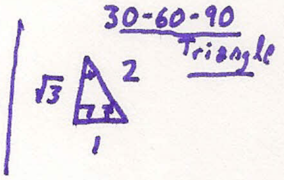


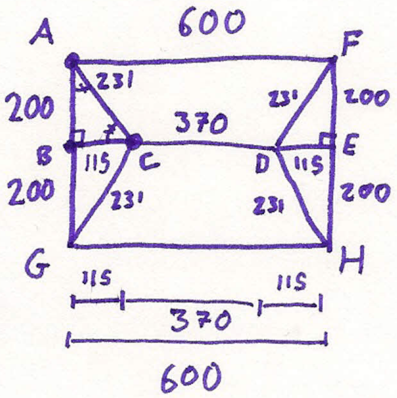

$\sqrt{3} \approx 1.732, \quad \frac{1}{\sqrt{3}} \approx \frac{1000}{1732} \approx 0.577$

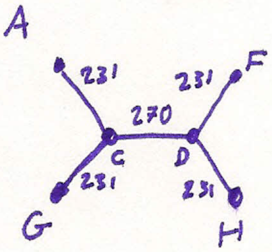



$\frac{200}{\sqrt{3}} \times 2 \approx 400 \times 0.577 = 230.8$

$\frac{200}{\sqrt{3}} \approx 200 \times 0.577 = 115.4$




① The triangle $\triangle ABC$ is a 30-60-90 triangle, thus has legs in proportion to $1:\sqrt{3}$ with hypotenuse in proportion to the shorter leg $1:2$. Since the longer leg is $\frac{400}{2} = 200$, the short leg is $\frac{200}{\sqrt{3}} \approx 115$ and the hypotenuse is $\frac{2 \times 200}{\sqrt{3}} \approx 231$.

② Since $|BC| + |CD| + |DE|$ equals 600, and since $|BC| = |DE| = 115$, it follows that $|CD| = 600 - 115 - 115 = 370$.

③ The total length of the shortest network is $|AC| + |CG| + |CD| + |DF| + |DH| =$

$231 + 231 + 270 + 231 + 231 = \boxed{1194 \text{ km}}$

**3.** The half-life of radioactive element bismuth-210 is about 5 days. Assuming that we begin with 100 grams of bismuth-210, answer the following questions:

(a) Find how much bismuth-210 will be left after 10 days

(b) Write the formula for the amount left after $n$ half-lifes, i.e. $5n$ days

(c) Find how much will be left after 30 days.

| Days | 0 | 5 | 10 | 15 | 20 | 25 | 30 |
|---|---|---|---|---|---|---|---|
| Mass remaining (grams) | $200\times(\frac{1}{2})^0 = 200$ | $200\times(\frac{1}{2})^1 = 100$ | $200\times(\frac{1}{2})^2 = 50$ | $200\times(\frac{1}{2})^3 = 25$ | $200\times(\frac{1}{2})^4 = 12.5$ | $200\times(\frac{1}{2})^5 = 6.25$ | $200\times(\frac{1}{2})^6 = 3.125$ |

(a) $200 \times \frac{1}{2} \times \frac{1}{2} = \underline{50 \text{ grams}}$

Number of half-lives

(b) Mass remaining after $5n$ days $= \underbrace{200g}_{\text{Initial Mass}} \times \left(\frac{1}{2}\right)^n$

(c) $30 = 5 \times 6$ , 6 half-lives

Mass remaining $= 200_{\text{grams}} \times \left(\frac{1}{2}\right)^6 = \underline{3.125 \text{ grams}}$

**4.** The Bank of Batavia offers a savings account with annual interest rate of 6%, compounded monthly. A student puts \$500 into this account on Jan 1, 2000.

(a) Find the monthly interest rate (assuming for simplicity that all months have same duration).

(b) Write a formula for the amount of money in the account after $n$ years.

(c) How much money will there be in the account on Jan 1, 2001? on Jan 1, 2100?

(d) What is the annual yield?

APR = 6%, $r = \frac{6}{100} = 0.06$, Principal $P = \$500$, Number of compounding periods per year $k = 12$.

Future value after n years $\underline{F_n = P\left(1+\frac{r}{k}\right)^{kn}}$

(a) Monthly Interest Rate $= \frac{r}{k} = \frac{0.06}{12} = 0.005 = \underline{0.5\% \text{ per month}}$

(b) $F_n = P\left(1+\frac{r}{k}\right)^{kn} = \underline{\$500 \times (1.005)^{12n}}$

(c) $(1+x)^{12} = \left(\left(\left(1+x\right)^3\right)^2\right)^2$, $\left(1+\frac{5}{10^3}\right)^3 = 1 + 3\times\left(\frac{5}{10^3}\right)^1 + 3\times\left(\frac{5}{10^3}\right)^2 + \left(\frac{5}{10^3}\right)^3$

$= 1 + \frac{15}{10^3} + \frac{75}{10^6} + \frac{125}{10^9} = 1.015075625$

$\approx 1.015$

$\left(1+\frac{15}{10^3}\right)^2 = 1 + 2\times\left(\frac{15}{10^3}\right) + \left(\frac{15}{10^3}\right)^2 = 1 + \frac{30}{10^3} + \frac{225}{10^6}$

$= 1.030225 \approx 1.030$

$(1.030)^2 = 1 + 2\times\left(\frac{3}{10^2}\right) + \left(\frac{3}{10^2}\right)^2 = 1 + \frac{6}{10^2} + \frac{9}{10^4} = 1.0609 \approx \underline{1.061}$

$F_1 = \$500 \times (1.005)^{12} \approx \$500 \times 1.061 = \underline{\$530.50}$ (actual = \$530.85)

$F_{100} = \$500 \times (1.005)^{12\cdot 100} \approx \$500 \times 397.4 = \underline{\$198,721.16}$

(d) Annual yield $= \left(1+\frac{r}{k}\right)^k - 1 \approx 1.061 - 1 = 0.061 = \underline{6.1\% \text{ APY}}$

(Actual = 6.16% APY)

**5.** The population of rabbits on a certain island is decribed by the logistic model, with carrying capacity $C = 10,000$ and annual growth factor $r = 2.5$. If the initial population is $P_0 = 2,000$ rabbits, answer the following questions:

(a) Compute $P_1$, $P_2$,$P_3$ (rounding to the nearest whole number).

(b) Which of the following CAN NOT happen in the long run:

(i) The population varies wildly, with no obvious pattern

(ii) The population oscillates, with 2 year cycle

(iii) The population steadily increases, growing larger and larger with no limit

$p_n := \frac{P_n}{C}$. Logistic Equation : $\underline{p_{n+1} = r(1-p_n)p_n}$ , $r = 5/2 = 2.5$

$C = 10,000$, $P_0 = 2000$

$p_0 = \frac{P_0}{C} = \frac{2000}{10000} = \frac{1}{5} = 0.2$

(a). $p_1 = r(1-p_0)p_0 = \frac{5}{2}\left(1-\frac{1}{5}\right)\frac{1}{5} = \frac{5}{2} \times \frac{4}{5} \times \frac{1}{5} = \frac{2}{5}$ , $P_1 = p_1 C = \frac{2}{5} \times 10,000 = \underline{\underline{4,000}}$

$p_2 = r(1-p_1)p_1 = \frac{5}{2}\left(1-\frac{2}{5}\right)\frac{2}{5} = \frac{5}{2} \times \frac{3}{5} \times \frac{2}{5} = \frac{3}{5}$ , $P_2 = p_2 C = \frac{3}{5} \times 10,000 = \underline{\underline{6,000}}$

$p_3 = r(1-p_2)p_2 = \frac{5}{2}\left(1-\frac{3}{5}\right)\frac{3}{5} = \frac{5}{2} \times \frac{2}{5} \times \frac{3}{5} = \frac{3}{5}$ , $P_3 = p_3 C = \frac{3}{5} \times 10,000 = \underline{\underline{6,000}}$

(b) As the computation above shows, for $p_n = \frac{3}{5}$ $\left(= \frac{r-1}{r}\right)$ also $p_{n+1} = \frac{5}{2}\left(1-\frac{3}{5}\right)\frac{3}{5} = \frac{3}{5}$. So the population achieves equilibrium of $P_n = \underline{6,000}$.

So for this particular population, none of (i), (ii) or (iii) occur.

However for some choices of $r$ with $0 \leq r \leq 4$ and $p_0$, the values $p_n$ vary chaotically between $0$ & $1$, i.e. (i) holds. For other choices (ii) holds. For no choice of $r$ with $0 \leq r \leq 4$ does (iii) hold: if the population steadily increases, then it approaches the equilibrium value $p = \frac{r-1}{r}$.