

A local–global principle for weak approximation for varieties over function fields

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This is joint work with Jason Starr.

Setup: B smooth projective curve over \mathbb{C} . $\pi: X \rightarrow B$ a family of projective varieties with general fibre rationally connected.

Fix $b_1, \dots, b_k \in B$, and formal sections \hat{t}_i over $\text{Spec}(\hat{\mathcal{O}}_{b_i})$.

Q: Is it possible to find a section s of π agreeing with each \hat{t}_i to order $\geq N$?

We say that weak approximation holds for π and b_1, \dots, b_k if we can find s for any $\hat{t}_1, \dots, \hat{t}_k$. We say that weak approximation holds for π if it holds for all b_1, \dots, b_k .

Why WA? Number theory: X/\mathbb{Q} a projective variety. Then $X(\mathbb{Q}) = X(\mathbb{Z})$. We have reduction maps $X(\mathbb{Z}) \rightarrow X(\mathbb{F}_p), X(\hat{\mathbb{Z}}_p), X(\mathbb{R})$, etc. Number theoretic question: can we reverse these arrows?

Rational connectedness. Suppose Y is a projective variety. Say Y is *rationally chain connected* if given any two points p, q there exists a chain of rational curves connecting p and q . Say Y is *rationally connected* if given any 2 points (equivalently, any finite set of points) then there is a single rational curve containing all of them. Equivalently (if Y is smooth) there is a single map $f: \mathbb{P}^1 \rightarrow Y$ such that $f^*T_Y = \bigoplus \mathcal{O}_{\mathbb{P}^1}(a_i)$ with $a_i \geq 1$.

Theorem 0.1. (*Kollár–Miyaoka–Mori*) *If Y is smooth, $RCC \Rightarrow RC$.*

Why RC? Suppose $\pi: X \rightarrow B$ satisfies WA.

Fix $b_0 \in B$ such that fibre is smooth and $p, q \in X_{b_0}$.

Pick points b_1, b_2, \dots in the base converging to b_0 and points $q_i \in X_{b_i}$ converging to q .

Look for sections passing through p and q_i , $i = 1, 2, \dots$. In the limit, the section has to split, so there exists a chain of rational curves connecting p and q . (Actually, we need to take uncountably many points in the base, and

pass to a subsequence such that the class of the section is fixed.) So WA implies that the general fibre is RC.

Theorem 0.2. (*Graber–Harris–Starr*) *General fibre is RC \Rightarrow there are lots of sections.*

Conjecture 0.3. (*Hassett–Tschinkel*) *General fibre RC \Rightarrow WA holds.*

Some results and the method of Hassett–Tschinkel. Suppose $\pi: X \rightarrow B$ is as in the setup and $b_1, \dots, b_k \in B$.

Theorem 0.4. *WA holds at b_1, \dots, b_k if the fibres b_i are (a) smooth, or (b) cubic surfaces with at most RDP’s.*

Also known if fibres are del Pezzo surfaces of degree ≥ 4 , of degree ≥ 2 with at worst ?? singularities (A. Knecht), or del Pezzo surfaces with quotient singularities (C. Xu).

Reduction step: By blowing up (repeatedly) on $X_{b_i} \subset X$ we can translate problem of “matching to order N ” to “matching to order 0”.

Method of HT. Start with some section s (we know it exists by GHS). Connect $S \cap X_b$ with q by a creatively chosen chain of rational curves. Add other general rational curves in general fibres such that the resulting curve can be smoothed to pass through the correct point.

Q: Are there only local obstructions to WA? For example, if $\pi: X \rightarrow B$, $\pi': X' \rightarrow B'$, $X_{b_0} \simeq X'_{b'_0}$, does WA at $b_0 \in B$ imply WA at $b'_0 \in B'$? Results:

Definition 0.5. *Say the method of HT holds for $X_{\hat{b}}$ if given any two sections \hat{t} (target) and \hat{s} (starting section) it is possible to add rational curves in fibre $X_{\hat{b}}$ to \hat{s} and smooth to \hat{t} .*

Theorem 0.6. (A) *If method of HT holds for $X_{\hat{b}}$ then WA holds at all $b' \in B'$ such that $X'_{b'} \simeq X_{\hat{b}}$.*

Theorem 0.7. (B) *There is a purely local criterion in terms of the Laurent fibre $X_{\hat{K}} = X_{\hat{b}} \otimes_{\mathcal{O}_{\hat{b}}} \hat{K}$, $\hat{K} = \text{Frac}(\mathcal{O}_{\hat{b}})$ that guarantees WA, namely, $X_{\hat{K}}(\hat{K})$ is “ R -connected” in the sense of Manin.*

How to prove Theorem A: Look for mystery moduli space MMS parametrising intersection data of curves in X with X_{b_0} , satisfying

- (1) Given family of curves, get map to moduli space of restriction data, in particular, have map

$$\text{Hilb}_s \rightarrow \text{MMS}$$

where Hilb_s is the Hilbert scheme of “section-like” curves in X .

- (2) Given $[C] \in \text{MMS}$ it's possible to attach rational tails in fibres away from b_0 to get a curve C' such that the map $\text{Hilb}_s \rightarrow \text{MMS}$ is smooth at C' .

Given this, prove A: First blowup so that agreeing with t is the same as passing through a point.

- (a) Start with some section s and restrict to get \hat{s} over $\mathcal{O}_{\hat{t}}$.
- (b) Use HT method: add rational curves to \hat{s} and smooth to \hat{t} .
- (c) Add same rational curves to s .
- (d) Add more tails away from b_0 to get C' such that the map to MMS is smooth at C' .
- (e) Follow deformation to get section.

What is MMS? The moduli space of pseudo ideal sheaves on X_b . A pseudo ideal sheaf is a finitely presented quasi coherent sheaf \mathcal{F} together with a map $u: \mathcal{F} \rightarrow \mathcal{O}_{X_b}$ such that the map

$$\mathcal{F} \otimes \mathcal{F} \rightarrow \mathcal{F}, \quad f_1 \otimes f_2 \mapsto u(f_1)f_2 - f_1u(f_2)$$

is identically zero. We have a map

$$\text{Hilb}_s \rightarrow \text{MMS}, \quad I_C \mapsto (I_C \otimes \mathcal{O}_{X_b} \rightarrow \mathcal{O}_{X_b}).$$